2024 National Astronomy Competition

1 Instructions (Please Read Carefully)

The top 5 eligible scorers on the NAC will be invited to represent USA at the next IOAA. In order to qualify for the national team, you must be a high school student with US citizenship or permanent residency.

This exam consists of 3 parts: Short Questions, Medium Questions and Long Questions.

The maximum number of points is **225 points**.

The test must be completed within 2.5 hours (150 minutes).

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution.

This is a written exam. Contestants can only use a scientific or graphing calculator for this exam. A table of physical constants will be provided. **Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on April 13th.** Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books or notes are allowed during the exam. Exam is proctored and recorded. You are expected to have your video on at all times.

We acknowledge the following people for their contributions to this year's exam:

Abhay Bestrapalli, Orion Foo, Hagan Hensley, David Lee, Sandesh Kalantre, Andrew Liu, Joe McCarty, Leo Yao

2 Short Questions

1. **(15 points)** The Boomerang Nebula is the coldest known place in the universe (outside of labs on Earth), with a temperature of just 1 K. The nebula is a rapidly expanding cloud of gas ejected from a red giant star, with a radius of 1 light year and an unusually high expansion velocity of 165 km/s. Let's make a highly simplified model to understand why it's so cold.

   (a) **(2 points)** Observations show that the nebula is gradually warming up near the very edge where it's in thermal contact with the surrounding interstellar medium, but it has not yet had time to reach thermal equilibrium with its environment by exchanging heat. (This is due to the rapid expansion - most such nebulae would take much longer to expand to this size.) However, it has done work on its environment, by pushing the interstellar medium out of the way as it expands. Let's approximate that no heat has been exchanged at all. In thermodynamics, what is the name for this kind of process?

   (b) **(6 points)** Assume that the nebula is an ideal gas, expanding as a spherical shell which maintains a constant thickness. In this kind of process, an ideal gas follows a relation $PV^\gamma = \text{constant}$, where $\gamma = \frac{5}{3}$.
   Based on this relation, how does the temperature T of the gas scale with the radius r of the gas as it expands?

   (c) **(7 points)** At very high temperature (right after ejection from the star) hydrogen is a plasma and so cannot be modeled as an ideal gas. For simplicity, take the initial temperature and radius of the cloud (once it has cooled down enough to no longer be a plasma) to be 10 000 K and 50 AU. What should the temperature be now, after the rapid expansion?

Solution:

(a) A process in which a system does work on its environment but does not exchange heat with its environment is known as an **adiabatic** process.

(b) We can write the ideal gas law as $PV = Nk_BT$. Since N and k_B are constants, the relevant proportionality relation is $PV \propto T$. If we factor out the constant $PV^\gamma = C$, then we get $V^{1-\gamma} \propto T$. For a spherical shell V scales like r^2 (rather than the usual r^3 for a solid sphere), so $r^{2(1-\gamma)} \propto T$.

$$\boxed{T \propto r^{2(1-\gamma)} = r^{-4/3}}$$

(c) Just plugging in the given numbers to the proportionality relation, we get

$$T = T_0 \left(\frac{r}{r_0}\right)^{-4/3} = (10000^\circ\text{K}) \left(\frac{1\text{ly}}{50\text{AU}}\right)^{-4/3}$$

$$\boxed{= 0.7\,\text{K}}$$

2. **(10 points)** David the astronomy enthusiast loves looking at stars! Specifically, he particularly enjoys looking at stars on the ecliptic. One day, he is out stargazing at midnight (local solar time) and looks at the antisolar point (the point on the celestial sphere exactly opposite to the Sun). He notices a faint glow of magnitude 12 mag/arcsec^2, and after some research he concludes that this is caused by a phenomenon known as *gegenschien*, where Solar System dust is lit up by the Sun and reflects some light back towards Earth. These particles are in an orbit of 2.06 AU around the Sun. Assuming the radii of these particles are around 1 cm and their albedo is 0.14, estimate the density of these particles. Express your answer in particles per square arcsecond.

Solution: We can find the total luminosity from one square arcsecond of this source by using

$$M - M_\odot = -2.5\log_{10}\left(\frac{L}{L_\odot}\right) \qquad \text{and} \qquad m - M = 5\log_{10}\left(\frac{r}{10\,\text{pc}}\right)$$

From this we get

$$2.5\log_{10}\left(\frac{L}{L_\odot}\right) = M_\odot + 5\log_{10}\left(\frac{r}{10\,\text{pc}}\right) - m$$

$$\implies L = L_\odot\left(\frac{r}{10\,\text{pc}}\right)^2 \cdot 10^{(M_\odot - m)/2.5} = 1.37\times 10^{11}\,\text{W}$$

where we used $r = 1.06\,\text{AU}$ since the particles are directly opposite to the Sun.

Now, let's calculate how much luminosity one particle reflects. To do this, we simply calculate the flux from the Sun and multiply by the cross sectional area of the particle and it's albedo. This gives

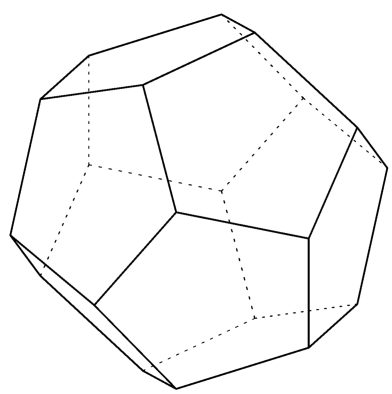
$$L_p = \frac{L_\odot}{4\pi R^2}\cdot a\pi r^2 = 0.0141\,\text{W}$$

Finally, all that is left to do is divide these two numbers to get an answer of

$$\frac{1.37\times 10^{11}\,\text{W}}{0.0141\,\text{W}} = \boxed{9.71\times 10^{12}\,\text{particles/arcsec}^2}$$

Note: one reasonable thing to do is to divide this number by two because the light emitted by these particles is not isotropic, it is mostly scattered back towards the direction of the Earth. This and other reasonable assumptions should be accepted for full credit.

3. **(15 points)** Astronomer Evan noticed that there is a group of stars $\{X_1, X_2, \ldots, X_{12}\}$ which forms a perfect dodecahedron around the Earth.



Evan went outside to observe some of these stars. At a particular moment in time, he computed the angle

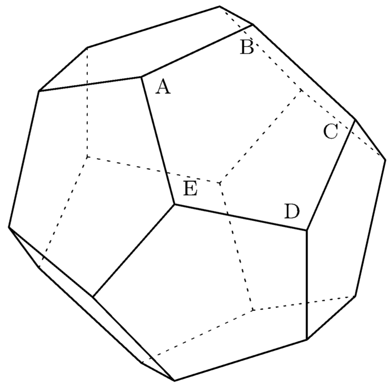
$$\delta_Z = \max(ZX_{a_1}, ZX_{a_2}, ZX_{a_3}, ZX_{a_4}, ZX_{a_5}),$$

where Z is his zenith and $\{a_1, a_2, \ldots, a_5\} \subset \{1, \ldots, 12\}$ are the indices of the five closest stars in angular distance to Z. Compute the minimum possible value of δ_Z that he could have observed.

Note that δ_Z is not a declination.

Solution: The answer is $\boxed{37.38^\circ}$.

δ_Z is minimized when Z points in the center of any polygonal face, call $ABCDE$. When this is the case, $\delta_Z = ZA = ZB = ZC = ZD = ZE := X$. If Z is not located at the center of a face, then the distance of the 5th farthest star from Z is necessarily larger than X.



To compute X, consider the spherical triangle $\triangle ZAE$. When we look at the configuration from a bird's eye perspective centered at Z, we see that $\angle AZE = 72^\circ$ and $ZA = ZE = X$. Moreover, $\angle ZEA = \angle DEA/2$. When we look at the configuration from a bird's eye perspective centered at E, we see that $\angle DEA = 120^\circ$, which implies $\angle ZEA = 60^\circ = \angle ZAE$.

Let $Y = AE$. Using the spherical law of cosines and spherical law of sines in $\triangle ZEA$, we have

$$\cos Y = \sin^2 X \cos 72^\circ + \cos^2 X \quad \text{and} \quad \frac{\sin X}{\sin 60^\circ} = \frac{\sin Y}{\sin 72^\circ},$$

respectively. From here, there are many ways to solve for X. One way is to substitute $\cos Y = \sqrt{1 - \sin^2 Y}$ into the first equation. Then, plugging in the expression for $\sin Y$ given by the second equation into the first gives a quadratic in $\sin^2 X$, which can be bashed using the quadratic formula.

If we instead first solve for Y, a slightly cleaner approach is possible. We have

$$\cos Y = \left(\frac{\sin 60^\circ}{\sin 72^\circ} \sin Y\right)^2 \cos 72^\circ + 1 - \left(\frac{\sin 60^\circ}{\sin 72^\circ} \sin Y\right)^2,$$

which implies

$$1 - \cos Y = 2 \sin^2 Y \frac{\sin^2 60^\circ \sin^2 36^\circ}{\sin^2 72^\circ} \implies \cos Y = \frac{1}{2}\left(\frac{\sin 72}{\sin 36 \sin 60}\right)^2 - 1.$$

Solving for Y and plugging back into our original equations gives $X \approx 37.38^\circ$, as desired.

3 Medium Questions

1. **(30 points)** *Solar eclipse*

   Throughout this question, you can use that the eccentricity of the Moon's orbit is $e_{☾} = 0.055$. We will use the simplifying assumptions that the Earth's orbit is circular with a radius of 1 AU, the Earth has no axial tilt (i.e., the equator and ecliptic coincide), and the Moon's orbit lies exactly on the ecliptic.

   (a) **(7 points)** Calculate the maximum distance of the Moon from the center of the Earth for a total solar eclipse to happen. Also calculate the distance of the Moon at perigee and apogee, and compare these three numbers.

   (b) **(3 points)** The Moon is slowly moving away from the Earth due to tidal effects, with its semimajor axis increasing at 38 mm per year. Assuming its eccentricity remains the same, how many years in the future will total solar eclipses become impossible?

   (c) **(5 points)** Assume there is a new moon at **perigee**. What is the Moon's speed relative to the Earth and the fixed background stars? You may neglect the mass of the Moon in comparison with the Earth.

   For the remainder of this problem, consider an observer on the equator for whom the Sun is eclipsed at noon, and the Moon is at perigee.

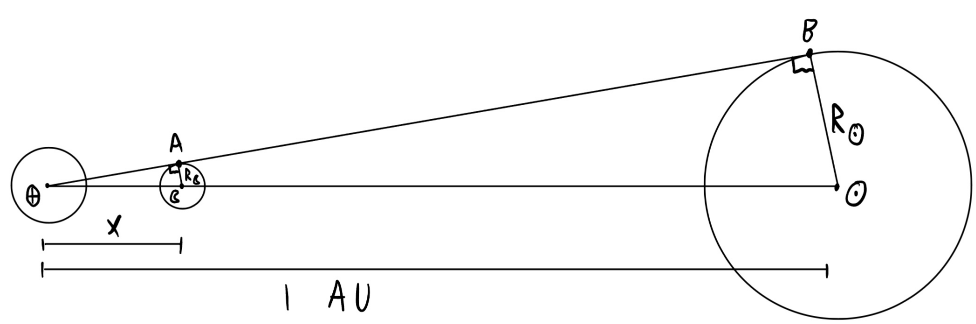
   (d) **(3 points)** The answer from (c) is the speed of the Moon in the geocentric frame in which the Earth and the background stars are fixed. However, for an eclipse, we are interested in the Moon's motion relative to the Sun instead of the background stars. What is the Moon's speed in a geocentric frame in which the Earth and the Sun are fixed? Hint: rotate the previous frame by the apparent angular velocity of the Sun.

   (e) **(6 points)** What is the diameter of the Moon's shadow on the observer?

   (f) **(6 points)** What is the duration of totality for the observer?

Solution:

(a) We can use the similarity of the triangles $\oplus ☾ A$ and $\oplus \odot B$. **(2 points)**



The Moon's full shadow ends at 1 AU when the distance of the Moon from the center of the Earth is

$$x = 1\,\mathrm{AU} \cdot \frac{R_☾}{R_\odot} = 1.496 \times 10^{11}\,\mathrm{m} \cdot \frac{1.737 \times 10^6}{6.96 \times 10^8} = 3.7336 \times 10^8\,\mathrm{m} \rightarrow \boxed{3.73 \times 10^8\,\mathrm{m}}. \textbf{ (2 points)}$$

The Moon can actually be slightly further away for an eclipse to occur at noon, because the shadow can converge at the Earth's surface instead of its center. Subtracting $R_\oplus$ from 1 AU in the above expression and adding it back at the end (to give the distance from the Moon to the center of the Earth) gives 3.80×10^8 m, which should be accepted as well.

Perigee: $r_p = (1 - e_☾)a_☾ = (1 - 0.055) \cdot 3.84399 \times 10^8\,\text{m} = 3.6326 \times 10^8\,\text{m} \rightarrow \boxed{3.63 \times 10^8\,\text{m}}$. **(1 point)**

Apogee: $r_a = (1 + e_☾)a_☾ = 1.055 \cdot 3.84399 \times 10^8\,\text{m} = 4.0554 \times 10^8\,\text{m} \rightarrow \boxed{4.06 \times 10^8\,\text{m}}$. **(1 point)**

So the marginal distance for a total solar eclipse to occur is between the perigee and apogee distances, corroborating the fact that there are both total and annular solar eclipses. **(1 point)**

(b) Total solar eclipses will become impossible when the Moon's perigee is further than the maximum distance found in part (a). Using the value of 3.7336×10^8 m gives a semimajor axis of

$$a_☾ = \frac{1}{1 - e_☾} \cdot 3.7336 \times 10^8\,\text{m} = 3.9509 \times 10^8\,\text{m}. \textbf{ (1 point)}$$

Subtracting the current value of the Moon's semimajor axis, 3.84399×10^8 m, means that the Moon needs to move further away by 1.0691×10^7 m. At a rate of 38 mm per year this will take $\boxed{2.8 \times 10^8\,\text{years}}$. **(2 points)**

Instead using the value for the maximum perigee of 3.80×10^8 m gives an answer of $4.6 - 4.7 \times 10^8$ years, which should also be accepted.

(c) By conservation of energy for the Moon's orbit,

$$-\frac{GM_\oplus m}{2a_☾} = \frac{1}{2}mv^2 - \frac{GM_\oplus m}{r},$$

so at perigee,

$$\begin{aligned} v_p &= \sqrt{GM_\oplus \left(\frac{2}{r_p} - \frac{1}{a}\right)} \textbf{ (3 points)} \\ &= \sqrt{(6.674 \times 10^{-11}\,\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2})(5.976 \times 10^{24}\,\text{kg})\left(\frac{2}{1 - 0.055} - 1\right)/(3.84399 \times 10^8\,\text{m})} \\ &= 1.0763 \times 10^3\,\text{m/s} \rightarrow \boxed{1.08 \times 10^3\,\text{m/s}}. \textbf{ (2 points)} \end{aligned}$$

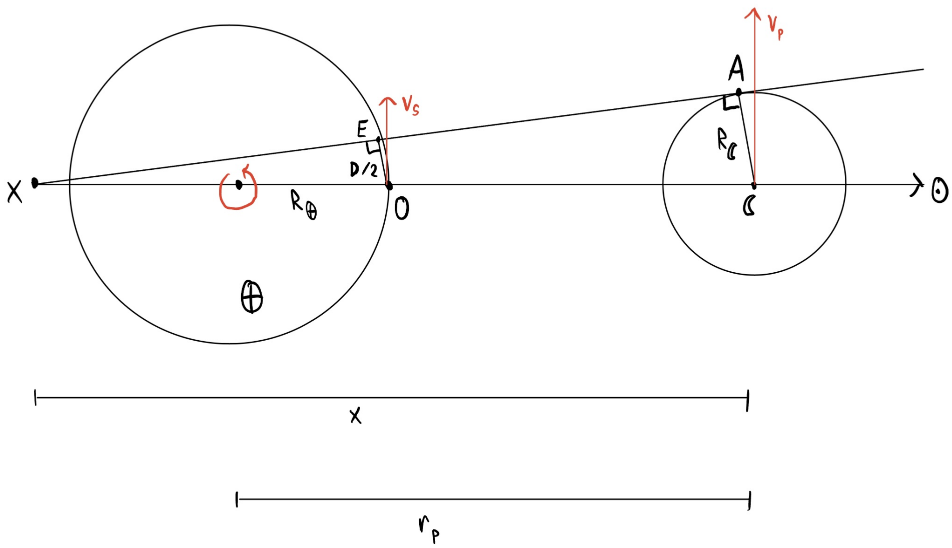
(d) The apparent angular velocity of the Sun is $(2\pi\,\text{rad})/(1\,\text{yr})$. **(1 point)**

At new moon, this is in the same direction as the motion of the Moon, so we need to subtract this angular velocity times the distance to the Moon **(1 point)**:

$$1.0763 \times 10^3\,\text{m/s} - \frac{2\pi}{365.2564 \cdot 24 \cdot 3600\,\text{s}} \cdot 3.6326 \times 10^8\,\text{m} = 1004\,\text{m/s} \rightarrow \boxed{1.00 \times 10^3\,\text{m/s}}.$$

(1 point)

(e) We can use the similarity of the triangles $X☾A$ and XOE. **(2 points)**



The diameter of the shadow (i.e., twice the length of OE) is

$$\begin{aligned} D &= (x - r_p + R_\oplus) \cdot \frac{2R_☾}{x} \textbf{ (2 points)} \\ &= (3.7336 \times 10^8 - 3.6326 \times 10^8 + 6.371 \times 10^6)\,\mathrm{m} \cdot \frac{2 \cdot 1.737 \times 10^6}{3.7336 \times 10^8} \\ &= 153.24\,\mathrm{km} \rightarrow \boxed{150\,\mathrm{km}}. \textbf{ (2 points)} \end{aligned}$$

We can neglect the curvature of the Earth in this calculation because this distance is much less than the radius of the Earth. Note also that the segment OE is at an angle so we could actually multiply this by the cosine of the angle $\angle EXO$ (i.e., roughly the angular size of the Moon and Sun), but the angle is small enough that the cosine is essentially 1.

(f) We can simply divide the diameter of the shadow by the relative speed of the Moon and the observer on the surface. **(2 points)**

The observer on the surface moves at speed $v_s = 2\pi R_\oplus/(24\,\mathrm{hr})$. **(1 point)**

Approximating the motion of both the Moon and the observer as straight lines as shown in the previous drawing (which we can do because the timescales involved are much shorter than a day) **(1 point)**, this gives

$$\frac{D}{v_p - v_s} = \frac{153.24 \times 10^3\,\mathrm{m}}{1004\,\mathrm{m/s} - 2\pi \cdot 6.371 \times 10^6\,\mathrm{m}/(24 \cdot 3600\,\mathrm{s})} = 283\,\mathrm{s} = \boxed{4.7\,\mathrm{min}}. \textbf{ (2 points)}$$

This is similar to the length of the recent April 8th, 2024 total solar eclipse through North America! It is possible for eclipses to be longer than this because the Earth can be further than 1 AU from the Sun, which would increase the size of the shadow.

2. **(25 points)** *Neutrino decoupling*

At the beginning of the Universe, immediately following inflation, neutrinos were in thermal equilibrium with other radiation and matter, maintained through weak interactions. As the Universe expanded and the temperature of the Universe dropped, the rate of these weak interactions decreased until thermal equilibrium was no longer maintained, and neutrinos began to propagate freely through the Universe. This event is called neutrino decoupling, and occurred at about $t_n = 1$sec after the Big Bang. Initially, the energy density of radiation was greater than that of matter, but as energy of radiation decreased with cosmological expansion, eventually the energy density of radiation fell below that of matter. This earlier period is called the radiation-dominated era, while the later period is called the matter-dominated era, with a transition time of about $t_r = 50000$yr.

(a) **(4 points)** Write proportionalities for the energy density of matter $\rho_m(a)$ and the energy density of radiation $\rho_r(a)$ as a function of scale factor. Your answer should look like $\rho_m(a) \propto f(a)$ and $\rho_r(a) \propto g(a)$ for some single variable functions f, g.

Hint: radiation is redshifted with cosmological expansion, while matter is not.

(b) **(6 points)** Using the Friedmann equation:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2},$$

where a is the scale factor and k is the curvature. Derive proportionalities for the evolution of a matter-dominated and a radiation-dominated Universe, $a_m(t)$ and $a_r(t)$. Take t to be the time since the start of the Universe in each case, assuming the Universe was always made of only matter or only radiation. Assume no curvature or cosmological constant. Derivation required for credit.

Hint: guess a form $a(t) = bt^c$ with b, c constants, which has derivative $\dot{a}(t) = c \cdot bt^{c-1}$, and equate exponents.

From this point forwards, take the simple (and unrealistic) assumption that in the radiation-dominated era, our Universe was entirely made up of radiation from the start up to some time t_r, then was entirely made up by matter afterwards in the matter-dominated era.

(c) **(10 points)** Using the conditions that the scale factor $a(t)$ and expansion rate $H(t) = \frac{\dot{a}(t)}{a(t)}$ of our Universe must be continuous, write expressions for the scale factor $a(t)$ of our Universe for $t < t_r$ and $t > t_r$, up to a single scaling constant. Do not substitute in the value of t_r yet.

Hint: naively trying to equate the expressions of the previous part will not allow both $a(t)$ and $H(t)$ to be continuous. What assumption holds in those expressions that no longer holds here, giving us an extra degree of freedom?

(d) **(2 points)** Solve for the scaling constant in the previous part in terms of t_r and t_0, the current age of our Universe.

Hint: what is the current value of $a_0 = a(t_0)$?

(e) **(3 points)** For a boundary between matter and radiation-dominated eras of $t_r = 50000$yr, estimate the temperature T_n of the Universe at the time of neutrino decoupling. The current age of the Universe is $t_0 = 13.8$ billion years, and the current temperature of the Universe is $T_0 = 2.73$K.

Solution:

(a) Consider matter in the Universe as made up of numerous tiny dust particles. As the Universe expands, the dust particles get further away from each other, but the total number of dust particles remains unchanged. Expansion also does not cause dust particles to be created or to disappear, or their masses to change.

Therefore, the total energy density of matter in the Universe remains unchanged with time, so matter density is inversely proportional to volume, which goes as the cube of the scale factor:

$$\rho_m(a) \propto a^{-3}$$

Expansion also does not create or destroy radiation. However, the energy of photons is affected due to cosmological redshift: a photon of wavelength $a_1\lambda$ is redshifted to a wavelength $a_2\lambda$, leading to an energy per particle inversely proportional to a. Then, energy density of radiation goes as the fourth power of the scale factor (3 for volume, 1 for redshift):

$$\rho_r(a) \propto a^{-4}$$

(b) As we are assuming a flat Universe, we have no curvature, $k = 0$, and can discard the second term. We guess a solution of the form $a(t) = bt^c$, for b, c constants, and can drop non-exponential constants as we are solving for a proportionality.

For matter density:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &\propto a^{-3} \\ \frac{t^{2(c-1)}}{t^{2c}} &\propto t^{-3c} \\ t^{-2} &\propto t^{-3c} \end{aligned}$$

$$c = \frac{2}{3} \implies a_m(t) \propto t^{2/3}$$

For radiation density:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &\propto a^{-4} \\ \frac{t^{2(c-1)}}{t^{2c}} &\propto t^{-4c} \\ t^{-2} &\propto t^{-4c} \end{aligned}$$

$$c = \frac{1}{2} \implies a_m(t) \propto t^{1/2}$$

(c) Adding constants back to the proportionalities:

$$\begin{aligned} a_r(t) &= b_r t^{1/2} \\ a_m(t) &= b_m t^{2/3} \end{aligned}$$

And the corresponding Hubble constants:

$$\begin{aligned} H_r(t) &= \frac{\frac{1}{2} b_r t^{-1/2}}{b_r t^{1/2}} = \frac{1}{2t} \\ H_m(t) &= \frac{\frac{2}{3} b_m t^{-1/3}}{b_m t^{2/3}} = \frac{2}{3t} \end{aligned}$$

Where t is defined to be the time since the start of the Universe in the case where the Universe has always been made of only radiation or only matter.

We need to enforce the conditions that $a(t)$ and $H(t)$ are continuous at the transition point $t = t_r$. If we naively try to equate the proportionalities, for the $a(t)$ condition we get:

$$\begin{aligned} b_r t_r^{1/2} &= b_m t_r^{2/3} \\ b_r &= b_m t_r^{1/6} \end{aligned}$$

However, there is no way to get the Hubble constants $H_r(t)$ and $H_m(t)$ to be equal! This suggests that there is an assumption we have made in the expressions for $a(t)$ that do not hold in this case.

The assumptions in our scale factor expressions are that $t = 0$ is the start of the Universe, and the Universe has always been made up of the specific substance since $t = 0$. This is true for our radiation-dominated phase, but is not the case for our matter-dominated phase, as it comes after the radiation-dominated phase.

If we evolve our matter-dominated Universe back in time, the time at which $a(t) = 0$ does not necessarily match our coordinate choice $t = 0$, as it is defined to be the zero time for the radiation-dominated phase. Therefore, we must introduce an offset t^* to our matter-dominated Universe expression:

$$a_r(t) = b_r t^{1/2}$$
$$a_m(t) = b_m (t - t^*)^{2/3}$$

We can now equate Hubble constants:

$$\frac{1}{2t_r} = \frac{2}{3(t_r - t^*)}$$

$$t^* = -\frac{1}{3} t_r$$

Our matter-dominated phase therefore is the same as a Universe that was always made up of matter, but started at time $t = -\frac{1}{3}t_r$ instead of $t = 0$.

Substituting back in:

$$a_r(t) = b_r t^{1/2}$$
$$a_m(t) = b_m \left(t + \frac{t_r}{3}\right)^{2/3}$$

Equating scale factor $a(t)$ at $t = t_r$:

$$b_r t_r^{1/2} = b_m \left(\frac{4}{3} t_r\right)^{2/3}$$

$$b_r = b_m \left(\frac{4}{3}\right)^{2/3} t_r^{1/6}$$

Then expressions for scale factor $a(t)$ before and after turning point:

$$a(t) = \begin{cases} b_m \left(\frac{4}{3}\right)^{2/3} t_r^{1/6} t^{1/2} & t < t_r \\ b_m \left(t + \frac{t_r}{3}\right)^{2/3} & t > t_r \end{cases}$$

(d) The current value of the scale factor is $a_0 = 1$. Enforcing $a(t_0) = 1$:

$$b_m \left(t_0 + \frac{t_r}{3}\right)^{2/3} = 1$$

$$b_m = \left(t_0 + \frac{t_r}{3}\right)^{-2/3}$$

(e)

$$a(1\text{sec}) = \left(t_0 + \frac{t_r}{3}\right)^{-2/3} \left(\frac{4}{3}\right)^{2/3} t_r^{1/6} t^{1/2}$$

Substituting $t_0 = 1.38 \times 10^{10} \cdot 3.15 \times 10^7$, $t_r = 5 \times 10^4 \cdot 3.15 \times 10^7$, and $t = 1$ gives scale factor:

$$a(1\text{sec}) = 2.276 \times 10^{-10}$$

And as temperature is inversely proportional to scale factor:

$$T_n = \frac{a_0}{a(1\text{sec})} T_0 = 1.2 \times 10^{10}\text{K}$$

Adapted and extended from a problem posed by Alan Guth.

3. **(30 points)** *Spiff's adventures*

You may attempt subsections without attempting the previous parts by making appropriate assumptions.

Spaceman Spiff is exploring the distant reaches of the Milky Way. He ends up crashing on a very cold planet. He calls the corresponding star Burner and the planet Ash. Though there aren't signs of life nearby, Spiff wants to determine if Ash is habitable. **In addition to constants from the TOC, you can assume the apparent magnitude of the Sun is -26.74.**

(a) **(5 points)** The first thing Spiff observes is that Burner (seen through safe equipment!) is not uniformly bright over its disc in the sky. Indeed, the edges seem to be less bright than the center. Spiff opens up an astrophysics book (his computer is not working) and find the following equation:

$$I(\tau, \mu) = \frac{3}{4\pi}\left(\tau + \mu + \frac{2}{3}\right) F(0)$$

Here, $I(\tau, \mu)$ gives the intensity of light at optical depth τ and $\mu = \cos\theta$ where the angle θ is the angle between the direction of the ray and the radial direction. $F(0)$ is the net flux at the surface. Help Spiff find the ratio of intensity of the center of Burner to the edge (as it appears to Spiff). What is this phenomenon called?

(b) **(15 points)** Spiff walks for a few days but doesn't find any life nearby. Over the many nights and days, Spiff makes several observations about Burner and Ash:

- The peak wavelength of Burner appears to be 500 nm.
- Before the ship crashed, Spiff had found the planet has a radius twice that of Earth and that it has a circular orbit around Burner.
- Precise calculation yields that each solar (Burnar?) day is 192 hours long and that Burner moved 1 degree against the background stars over a day.
- The mass and radius of Burner are similar to that of the Sun.
- Comparing the rotation of Burner (found by tracking a Burnerspot) to the sky, Spiff finds out that Ash has an axial tilt of 60 degrees! Spiff hypothesises that, because of a thin atmosphere and other effects, the heat received from the star never redistributes to the part of Ash that is not exposed to the star, and is only emitted by the exposed part. Also somewhat peculiarly, the unexposed part always remains unexposed (Ash is always titled towards Burner). Assume that Ash is at thermal equilibrium.

What is the equilibrium temperature of Ash? Is it theoretically possible to have life on this planet?

(c) **(4 points)** Spiff, being the genius he is, manages to fix the spaceship on his own, weaves through a dangerous meteor field and escape Burner's system. When he is around 10 parsecs away, he stops and looks back at Burner. He realises that he is still on Ash's plane of revolution when it starts transiting across Burner's disc. How long will the transit of Ash across Burner take in seconds?

(d) **(6 points)** Now Spiff has moved away further to 100 parsecs and has finally reached home. He looks back at Burner one last time. What is the apparent magnitude of the star (Assume that the transit of Ash is not happening) for Spiff? Can he see it with his naked eye (Spiff is indeed human)? Assume that extinction is significant, and is equal to 2 mag/kpc.

Solution:

(a) Let's first consider a point on the edge of Burner. The optical depth is by definition 0 (as it is on the surface) while $\mu = \cos(90°)$. Thus, the intensity/brightness is given by the radiative transfer equation above: $I(0, \cos(90°)) = \frac{F(0)}{2\pi}$. **(1 point)**

Next, for a point at the center of Burner, we have $\mu = \cos(0)$ with $\tau = 0$. It follows that $I(0, \cos(0)) = \frac{5F(0)}{4\pi}$. **(1 point)**

This gives a ratio of

$$\boxed{\frac{5}{2}}$$

(2 points)

Notably, the ratio between the brightness at the center of Burner and the brightness at the surface is > 1; this is a phenomenon called **limb darkening**. **(1 point)**

(b) The temperature of the star, from Wien's law is: $T_B = \frac{b}{\lambda_{\max}} = 5795.54K$ **(1 point)**

Since the radius of the star is the same as the Sun's radius, we can find the net luminosity of Burner with Stephen Boltzmann as: $L_B = 4\pi R_B^2 \sigma T_B^4 = 3.891 \times 10^{26}$ W. **(1 point)**

Burner moves 1 degree in the sky every day and it needs to move 360 degrees to come back to the same place in the sky. There are thus 360 days in a year. Each such day is 192 hours. So each year is $\frac{192\times 360}{24\times 365.25} = 7.885$ Earth years. **(3 point)**

Since the mass of Burner is the same as Sun, we can use Kepler's second law in earth years and A.U. as $a^3 \propto T^2$. Thus $a = (7.885)^{2/3} = 3.961 A.U.$.

Thus we find that radius $a = d$ of the orbit is: 5.92557×10^{11} meters. **(1 point)**

Using the fact that Ash's radius R_A is known (in terms of Earth's radius), we can write that the net radiant power absorbed by the planet is: $\frac{\pi R_A^2}{4\pi d^2} L_B = 4.508 \times 10^{16}$ W. **(2 points)**

Now, Ash will also emit some radiation in order to be in thermal equilibrium. However, it does not emit in all directions: only in the area exposed to the star. The solid angle not exposed to the star is the cone of angle 60 degrees. The solid angle is given by $2\pi(1 - \cos\theta) = \pi$. This $\frac{\pi}{4\pi} = \frac{1}{4}$ of the entire sphere that does not emit. **(3 points)**

Thus if the temperature of Ash is T, then the net flux emitted by Ash by Stephen Boltzmann's law can be written as: $\frac{3}{4} \times 4\pi R_A^2 \sigma T^4 = 8.696 \times 10^7 T^4$. **(1 point)**

Equating the two expressions we can get: $4.508 \times 10^{16} = 8.696 \times 10^7 T^4$

So, $\boxed{\text{T} = 150.892\ \text{K.}}$ **(2 points)**

Water freezes at $273K$, much higher than the temperature above. Thus, this planet is probably too cold for life. Note that the unexposed part is probably even colder. **(1 point)**

(c) Note: Some of the calculations in the previous part may be done here if the previous one is skipped. Appropriate points should be given in the previous part.

Ash has to cover the diameter of Burner as well as the the diameter of itself. The total distance that Ash must cover is $2(R_B + R_A)$. **(1 point)**

The angle that the star subtends is given by $2 * \arcsin(\frac{R_A + R_B}{d}) = 0.137$ degrees. **(1 point)**

Thus the time it takes to cover this is $\frac{0.137}{360} \times 7.885$ Earth years or $\boxed{9.46 \times 10^4 \text{ seconds}}$. **(2 points)**

(d) Now that Spiff is 100 pc away from the system, we note that his line of sight must travel through $\frac{2}{1000} * 100 = 0.2$ magnitudes of ISM extinction. **(1 point)**

To find the apparent magnitude of Burner, we use the Sun-Earth system as a reference:

$$m_1 + 26.74 = -2.5 \log \frac{F_{Burner \to Spiff}}{F_{\odot \to Earth}} + 0.2$$

(1 point)

$$\implies m_1 = -26.74 - 2.5 \log \frac{\frac{L_B}{4\pi(100pc)^2}}{\frac{L_\odot}{4\pi(1AU)^2}} + 0.2$$

(1 point)

$$= -26.74 + 36.55 + 0.2 = \boxed{10.01}$$

(2 points)

The limit magnitude of the human eye is 6, so he cannot see it. **(1 point)**

4 Long Questions

1. **(50 points)** *TESS*

   David the observation master just launched his new telescope into a geostationary orbit, which he calls the Terrestrial Earth Stalking Satellite (TESS)! To test his telescope, he aims it straight down at his friend Matthew.

   (a) **(9 points)** Assume Matthew is a perfect gray body radiator with an albedo of $a = 0.60$. Matthew has a surface area of $1.95\,\mathrm{m}^2$ and a temperature of $37\,^\circ\mathrm{C}$. Estimate his apparent bolometric magnitude as seen by TESS. You may make the slightly unrealistic assumption that Matthew is in full sunlight (i.e. his full surface area is reflecting light coming solely from the Sun). Also assume that the Earth's albedo is 0.30, that Matthew is directly below the telescope, and that he radiates isotropically. Hint: the Stefan-Boltzmann Law can be modified for gray bodies with albedo a by multiplying the power by a factor of $(1 - a)$. The absolute bolometric magnitude of the Sun is 4.74.

   Having captured images of Matthew, he is eager to analyze the data! However, due to various sources of noise, the picture isn't as clear as he expected. The below table gives some of the properties of the telescope's CCD camera.

Dimensions	Pixel Size	Focal Length	Read Noise	Dark Current	Saturation	m_0
4096×4096	$15\,\mu\mathrm{m}$	$3.14\,\mathrm{m}$	$9\,\mathrm{e}^-$	$1.1\,\mathrm{e}^-/\mathrm{pix/s}$	$200\,000\,\mathrm{e}^-$	20.44

Table 1: TESS Camera Parameters

   (b) **(6 points)** What is the maximum signal to noise ratio (SNR) David can get from a single exposure before his pixels saturate? What exposure time would this correspond to? Hint: The value of m_0 corresponds to the magnitude at which the detector registers one count/second. The signal to noise ratio for a single CCD pixel with photon flux P can be calculated using the following formula (where t is the exposure time, D is the dark current and N_r is the read noise):

$$\mathrm{SNR} = \frac{Pt}{\sqrt{Pt + Dt + N_r^2}}$$

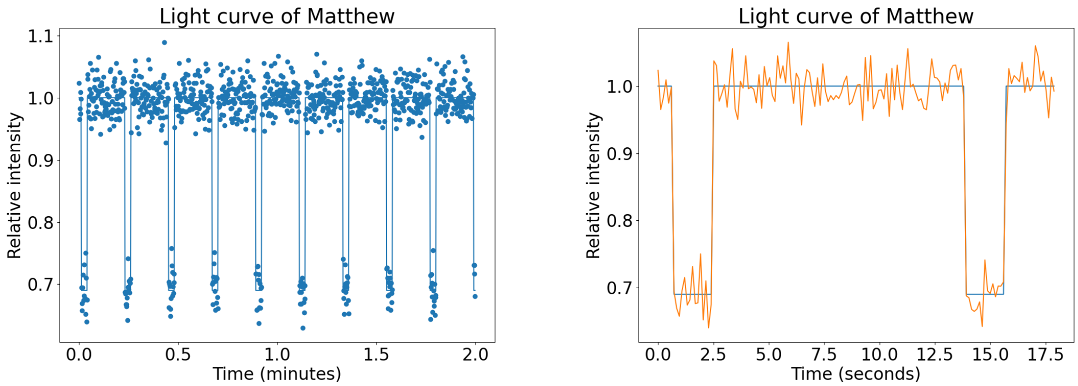
   (c) **(10 points)** Looking closer at the data, David realizes that there are photon sources other than Matthew. Specifically, the ground produces a lot of background noise. David realizes he cannot include these counts as signal, but they still contribute to the $\sqrt{n}$ photon (shot) noise. You can use the albedo of the Earth given in part (a) for your estimate, and assume that the temperature of the ground is $20\,^\circ\mathrm{C}$. Calculate the new maximum signal to noise ratio and corresponding exposure time.

   While analyzing his data, David notices something interesting. He notices that every 45 seconds, the location of Matthew changes by 1 pixel (assume the direction of the motion is exactly parallel to the columns of the detector array). He wonders if this variation could be explained by small vibrations in his satellite.

   (d) **(3 points)** What would the minimum angular velocity of the satellite have to be in order to explain this movement? Express your answer in arcseconds per minute.

   Being the observational master that he is, David knows that his control of the satellite is far too precise for camera jitter to be a viable explanation. Furthermore, as more data comes in, he see that the movement continues to be in a straight line, something unlikely to be observed if random shaking of the satellite was at fault. Knowing Matthew, he comes up with an alternative explanation: Matthew is in the middle of a cross country race!

   (e) **(4 points)** What would the minimum velocity Matthew would have to run to produce this movement?



(a) (b)

Figure 1: Light Curves of Matthew

Although Matthew is too dim for David to achieve a reasonable SNR, as an observational master, David uses his knowledge of the dark arts to remove enough noise to do basic photometry experiments. Performing photometry on Matthew, David once again notices something interesting. Matthew's light curve has periodic dips! Figure 1 shows plots of the light curve.

(f) David hypothesizes that these dips are due to trees being planted at regular intervals along the route, blocking some fraction of the sunlight incident on Matthew (you may assume that the trees are not directly over Matthew and so they do not block any of the light reflected off him).

(i) **(5 points)** How far apart are these trees planted?

(ii) **(5 points)** Estimate the width of each tree.

(iii) **(8 points)** What percentage of the incoming sunlight is blocked by the tree?

Solution:

(a) There are two sources of luminosity emitted by Matthew: gray body radiation and reflected sunlight.

The gray body radiation is simply given by our modified Stefan-Boltzmann Law, which gives $(1-a)A\sigma T^4 = 409.2\,\mathrm{W}$.

The contribution from reflected sunlight can be calculated by taking the flux from the Sun, multiplying by $1-a_\oplus$ to take into account the portion of sunlight reflected by the Earth's atmosphere, and then multiplying by the albedo of Matthew and the area of Matthew. This gives

$$(1-a_\oplus)a\frac{L_\odot}{4\pi(1\,\mathrm{AU})^2}A = 1114.5\,\mathrm{W}.$$

The total luminosity of Matthew is therefore $L = 1524\,\mathrm{W}$.

To get his apparent magnitude, we also need how far he is away from the telescope. We know that the telescope is in geostationary orbit, so we can calculate it's orbital radius:

$$r = \sqrt{\frac{GM_\oplus}{\omega^3}} = 4.22\times10^7\,\mathrm{m}$$

Note: here we used $\omega = 2\pi/T_{\text{sidereal}}$. Solutions using $T = 86\,400\,\text{s}$ should also be accepted for full credit because the difference is very small.

To get the distance between Matthew and TESS, we simply subtract the radius of the Earth.

Finally, if we let M and m be the absolute and apparent magnitudes of Matthew, respectively, then we can use the equations

$$M - M_\odot = -2.5\log_{10}\left(\frac{L}{L_\odot}\right) \qquad \text{and} \qquad m - M = 5\log_{10}\left(\frac{r - R_\oplus}{10\,\text{pc}}\right)$$

We get that

$$m = M_\odot - 2.5\log_{10}\left(\frac{L}{L_\odot}\right) + 5\log_{10}\left(\frac{r - R_\oplus}{10\,\text{pc}}\right) = \boxed{13.56}$$

(b) We can get the photon flux from Matthew by using

$$m - m_0 = -2.5\log_{10}\left(\frac{P}{1\,\text{e}^-/\text{s}}\right)$$

We get

$$P = 10^{(m_0 - m)/2.5}\text{e}^-/\text{s} = 10^{(20.44 - 13.56)/2.5}\text{e}^-/\text{s} = 563.7\,\text{e}^-/\text{s}$$

Now the maximum SNR is obtained by the longest exposure possible before the pixel saturates. (We can prove this, but solutions that just state this intuition should be given full credit).

We will get an exposure time t equal to

$$t = \frac{200\,000\,\text{e}^-}{P} = \boxed{354.8\,\text{s}}$$

We can plug this exposure time into the formula given to get the SNR:

$$\text{SNR} = \frac{Pt}{\sqrt{Pt + Dt + N_r^2}} = \boxed{446.7}$$

(c) We can calculate the luminosity from the ground. First, we want the area of the ground that is captured by 1 pixel. To do this, we can use the plate scale formula to calculate the angular size of each pixel:

$$\theta = \frac{y}{f} = \frac{15\,\mu\text{m}}{3.14\,\text{m}} = 4.78 \times 10^{-6}\,\text{rad}$$

Multiplying this by the distance to ground, we get that the pixel is a square with side length

$$l = \theta(r - R_\oplus) = 171\,\text{m} \implies A = l^2 = 2.93 \times 10^4\,\text{m}^2$$

Now we can break up the luminosity from the ground into a reflected component an a graybody component. Similar to part (a), our graybody component is $(1 - a_\oplus)\sigma A T^4 = 8.57 \times 10^6\,\text{W}$.

For the reflected component, we can simply take the flux from the Sun and multiply it by the area of the ground and the albedo of the ground:

$$a_\oplus \frac{L_\odot}{4\pi(1\,\text{AU})^2} A = 1.19 \times 10^7\,\text{W}$$

Adding these together gives a total luminosity of $2.05 \times 10^7\,\text{W}$. Using the same formula in part (a), we get that

$$m = M_\odot - 2.5\log_{10}\left(\frac{L}{L_\odot}\right) + 5\log_{10}\left(\frac{r - R_\oplus}{10\,\text{pc}}\right) = 3.24$$

Now by the same logic as part (b), we get $P_b = 10^{(m_0-m)/2.5}\,\mathrm{e^-/s} = 7.56 \times 10^6\,\mathrm{e^-/s}$.

Now all we need to do is recalculate the time:

$$t = \frac{200\,000\,\mathrm{e^-}}{P + P_b} = \boxed{0.0263\,\mathrm{s}}$$

and then plug this into a modified SNR formula:

$$\mathrm{SNR} = \frac{Pt}{\sqrt{Pt + P_b t + Dt + N_r^2}} = \boxed{0.0332}$$

(d) We calculated the angular size of each pixel in part (c). Now all we have to do is divide this by 45 seconds and convert to arcseconds per minute:

$$\omega = \frac{\theta}{t} = 3.33 \times 10^{-7}\,\mathrm{rad/s} = \boxed{1.31\,\mathrm{arcsec/min}}$$

(e) For this part, we simply take the angular velocity in part (d) and multiply it by the distance from the satellite to the ground:

$$v = \frac{\theta \cdot (r - R_\oplus)}{t} = \boxed{3.80\,\mathrm{m/s}}$$

(f) Note: graders should be lenient about values read off the graphs.

(i) We can see in the first graph that 9 intervals happen in just under 2 minutes. Therefore, we can use a value of $\frac{120\,\mathrm{s}}{9} = 13.33\,\mathrm{s}$ as the time between two trees (the actual value used to generate the graphs was 13.20 s).

Multiplying this value by the velocity in part (e), we get

$$l = 13.2\,\mathrm{s} \cdot 3.80\,\mathrm{m/s} = \boxed{50.2\,\mathrm{m}}$$

(ii) From the second graph, we can tell a dip lasts for approximately 3/4 of a x-axis tick, or 1.875 s (the actual value used to generate the graphs was 1.80 s).

Again, we simply need to multiply by the velocity in part (e) to get a tree width of

$$w = 1.80\,\mathrm{s} \cdot 3.80\,\mathrm{m/s} = \boxed{6.84\,\mathrm{m}}$$

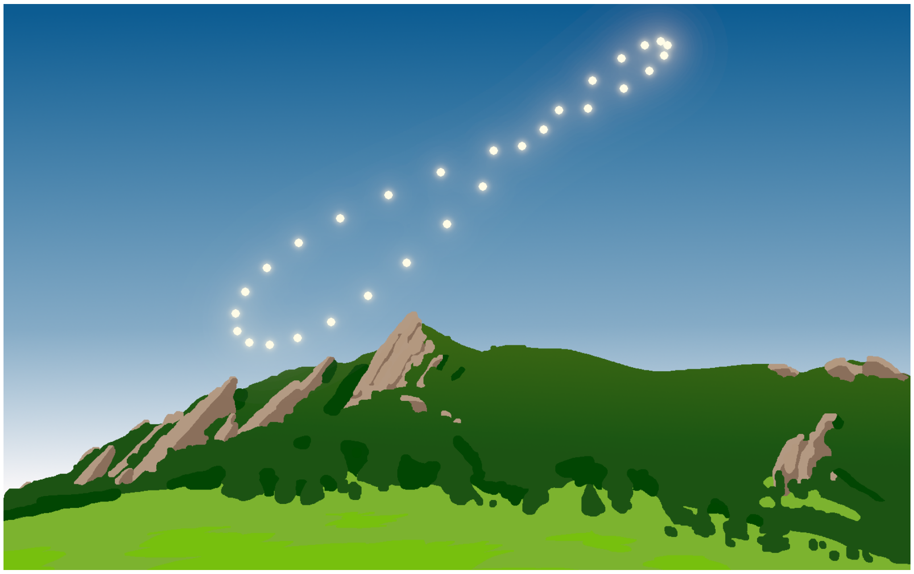
(iii) From the second graph we can read a transit depth of approximately 0.3 (the actual value used to generate the graphs was 0.31).

From part (a), we got that the luminosity of Matthew was 1524 W, with 409.2 W coming from graybody radiation and 1114.5 W coming from reflected sunlight.

We know that the tree blocked $0.31 \cdot 1524\,\mathrm{W} = 472.3\,\mathrm{W}$. Since we are told to assume that the tree only blocks sunlight and not any of his graybody radiation, the percent of incoming sunlight blocked is simply

$$\frac{472.3\,\mathrm{W}}{1114.5\,\mathrm{W}} = \boxed{42.4\%}$$

2. **(50 points)** *The Analemma*



If you take a picture of the Sun at the same time every day over the course of the year, you will see it traverse a famous figure-8 pattern known as the analemma. You can think of this as happening because the Sun's right ascension and declination vary over the course of the year relative to a "hypothetical sun" that travels at a constant speed along the celestial equator, and would appear to stay in place if observed at the same time each day.

In this problem, we will derive an expression for the shape of the analemma. Assume throughout this problem that $e \ll |\epsilon| \ll 1$, where e is the Earth's eccentricity and ϵ is the Earth's axial tilt. Use the small-angle approximation wherever it is justified - you'll need it often (sometimes in conjunction with other trig identities).

(a) **(20 points)** We conventionally parameterize the time of year in terms of the "mean anomaly" M, which starts at 0 at Earth's perihelion, and increases from 0 to 2π (radians) at a constant rate over the course of a year.

Since perihelion does not occur at the same time as the vernal equinox, we define the angle ϕ to be equal to the mean anomaly at the time of the vernal equinox. For convenience define another variable: the "mean longitude" $L = M - \phi$, which is equal to the longitude of the "hypothetical sun" mentioned above.

Write down an expression (valid for $e \ll 1$) for the ecliptic longitude l of the Sun, as a function of L and ϕ. Hint: you can apply Kepler's second law to calculate l at a few specific points along Earth's orbit, and then make an educated guess of the general form from there.

We can describe the analemma as a parametric function of L, in terms of its right-ascension and declination components:

$$(\Delta\alpha(L), \delta(L))$$

where $\delta(L)$ is the declination of the Sun, and $\Delta\alpha(L) = \alpha(L) - L$ is the *equation of time*, defined as the difference in right ascension between the real Sun and the mean longitude.

(b) **(10 points)** Using spherical trigonometry and your expression for $l(L)$, write down an expression for $\delta(L)$. Remember to use the small-angle approximation!

(c) **(15 points)** Making the same assumptions as before, find an expression for $\Delta\alpha(L)$. Be careful: there will be a term that depends on e, and a term that depends on ϵ - they're approximately equal in magnitude (for Earth), so you need to include them both.

You'll need couple of useful approximations, both of which hold for any $x \ll 1$:

$$\cos(x) \approx 1 - \frac{x^2}{2}$$

$$\frac{1}{1-x} \approx 1 + x$$

(d) **(5 points)** Based on the results of parts (b) and (c), explain qualitatively why the analemma looks like a figure-8 curve.

(If you're curious, after the exam, you can try to graph or sketch a parametric plot of $\delta(L)$ against $\Delta\alpha(L)$. For Earth in 2024, $\phi = 77°$, $\epsilon = 23.44°$ and $e = 0.0167$. The result should look familiar.)

Solution:

(a) A convenient variable to define (though you don't have to) is the true anomaly $\nu = l + \phi$, such that we'd expect $\nu = M$ for a circular orbit. The quantity we want to know is the difference between ν and M, or $\Delta\nu$, which should be small in the limit where the orbit is nearly circular.

We can find it carefully by expanding Kepler's equation in small e, but that's an unnecessarily complex approach if we just want the lowest order term of the expansion. I'll adopt the strategy of just making an educated guess based on a few points in the orbit. We can think of $\Delta\nu$ as a function of M, and guess the form of this function $\Delta\nu(M)$.

First we should come up with a qualitative expectation/"ansatz" of what the answer will be. $\Delta\nu$ must be periodic in M with period 2π, so the answer should be built out of trig functions. As the Earth is passing perihelion, its angular velocity about the sun is a little higher than its average angular velocity over the full orbit, so we'd expect $\Delta\nu$ to be increasing. As the Earth is passing aphelion, by the same logic, we'd expect $\Delta\nu$ to be decreasing. Thus, a reasonable guess is $\Delta\nu = C \sin M$, where C is some coefficient to be determined.

At perihelion ($M = 0$), by definition $\nu = M = 0$, so $\Delta\nu(0) = 0$.

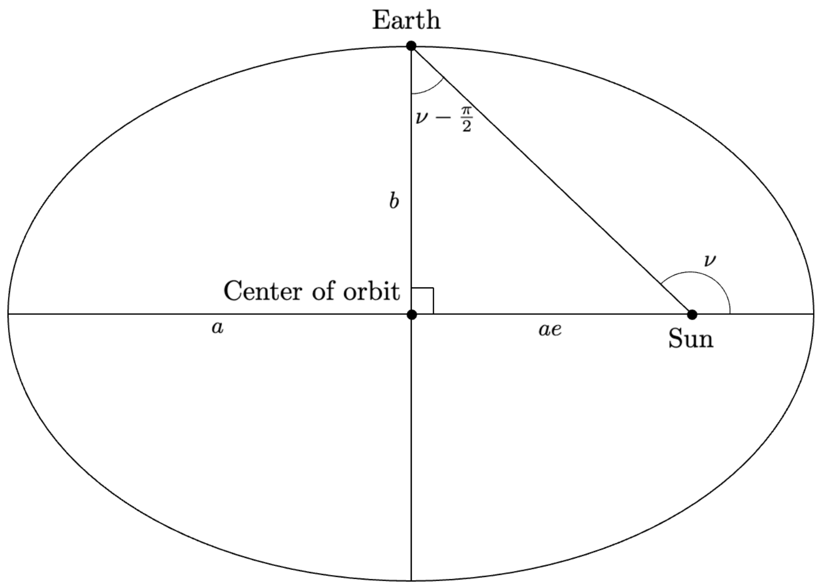
At aphelion $\left(M = \frac{\pi}{2}\right)$, by the symmetry of the orbit we must have $\nu = M = \pi$, so $\Delta\nu(\pi) = 0$ again.

The other two points to consider are along the semi-minor axis. Specifically, let's consider the point along the semi-minor axis where Earth is moving from perihelion to aphelion. By Kepler's second law, the mean anomaly is

$$M = 2\pi \frac{\text{(area swept out by Earth)}}{\text{(area of ellipse)}}$$

If we draw a triangle through the Earth, the center of the orbit, and the Sun (see the figure), it is a right triangle with base ae and height $b = a\sqrt{1-e^2}$. The area swept out by Earth is $\frac{1}{4}$ the area of the ellipse, minus this triangle, which is $\frac{1}{4}\pi ab - \frac{1}{2}abe$. Thus

$$M = 2\pi \frac{\frac{1}{4}\pi ab - \frac{1}{2}abe}{\pi ab} = \frac{\pi}{2} - e$$



The true anomaly ν is given by the Earth-Sun-perihelion angle, which using the small-angle approximation is $\frac{\pi}{2} + e\frac{a}{b}$. For a nearly-circular orbit we can say $\frac{a}{b}$ is close to 1, so

$$\nu \approx \frac{\pi}{2} + e$$

This gives $\Delta\nu(\pi/2) \approx 2e$. At the opposite point in the orbit along the semi-minor axis, we can carry out a very similar argument to show $\Delta\nu(3\pi/2) \approx -2e$.

Considering all four of these points, as well as our guess for the form of $\Delta\nu$, we get

$$\Delta\nu(M) \approx 2e \sin M$$

which turns out to be correct in the small e limit. (For non-small e there is no closed-form solution; we would have to write it as an infinite series with higher order terms like $e^2 \sin 2M$, etc.) We can rewrite this as

$$\nu \approx M + 2e \sin M$$

Finally plugging in the definitions of l, L in terms of ν, M:

$$\boxed{l(L) \approx L + 2e \sin(L + \phi)}$$

(b) We can get δ as a function of l from spherical trigonometry. Draw a spherical triangle with the ecliptic and celestial equator as two sides, and the projection of the sun's position onto the celestial equator giving a right angle. Then the spherical law of sines gives

$$\frac{\sin\delta}{\sin\epsilon} = \sin l$$

$$\delta = \arcsin(\sin\epsilon \sin l)$$

Since ϵ is small we can apply the small angle approximation:

$$\delta \approx \epsilon \sin l$$

But we want δ as a function of L. To get this, plug in the expression for $l(L)$ from part (a):

$$\delta \approx \epsilon \sin(L + 2e \sin(L + \phi))$$

Since e is much smaller than 1, we might want to use the small angle approximation here. However, this isn't quite the right form for it. The trick is to first apply the trig identity for the sine of a sum of two angles:

$$\delta \approx \epsilon(\sin(L)\cos(2e\sin(L+\phi)) + \cos(L)\sin(2e\sin(L+\phi)))$$

and we can now apply a small-angle approximation in e on both of the resulting terms:

$$\delta \approx \epsilon \sin(L) + 2e\epsilon \sin(L+\phi)\cos(L)$$

(If you're familiar with calculus, you may also recognize that you can get this from a first-order Taylor series expansion.)

The second term here is proportional to $e\epsilon$, while the first term is just proportional to ϵ. Since we assumed both e and ϵ are much smaller than 1, we know the $e\epsilon$ term is much smaller than the ϵ term, so we can ignore it.

$$\boxed{\delta \approx \epsilon \sin(L)}$$

(c) This is the hardest part of the problem. It really just comes down to expanding everything out with trig identities and carefully determining which terms we are allowed to ignore.

We can derive α in terms of l by drawing the same spherical triangle as before. The spherical law of cosines gives:

$$\cos l = \cos\alpha \cos\delta$$

Since ϵ is small, we can write

$$\cos\delta = \cos(\epsilon \sin L) \approx 1 - \frac{\epsilon^2 \sin^2 L}{2}$$

(Unlike in the usual small-angle approximation, we have to include the second-order term here, otherwise we'd lose all dependence on ϵ!)

Write $\alpha = L + \Delta\alpha$:

$$\cos l \approx \cos(L + \Delta\alpha)\left(1 - \frac{\epsilon^2 \sin^2 L}{2}\right)$$

We can assume $\Delta\alpha$ will be small. Therefore we can do a similar trick as in part (b). Applying the cosine angle addition identity and the small-angle approximation:

$$\cos l \approx (\cos L - \Delta\alpha \sin L)\left(1 - \frac{\epsilon^2 \sin^2 L}{2}\right)$$

Rearranging, and using the other approximation given in the problem:

$$\Delta\alpha \approx -\frac{\cos l}{\sin L}\frac{1}{1 - \frac{\epsilon^2 \sin^2 L}{2}} + \cot L$$

$$\approx -\frac{\cos l}{\sin L}\left(1+\frac{\epsilon^2\sin^2 L}{2}\right)+\cot L$$

Plugging in the expression for $l(L)$:

$$\Delta\alpha \approx -\frac{\cos(L+2e\sin(L+\phi))}{\sin L}\left(1+\frac{\epsilon^2\sin^2 L}{2}\right)+\cot L$$

Again using the small-angle approximation trick:

$$\cos(L+2e\sin(L+\phi)) \approx \cos L - 2e\sin(L+\phi)\sin M$$

Plugging this back in:

$$\begin{aligned}\Delta\alpha &\approx -\frac{\cos L - 2e\sin(L+\phi)\sin L}{\sin L}\left(1+\frac{\epsilon^2\sin^2 L}{2}\right)+\cot L\\ &= -\frac{\epsilon^2}{4}\sin(2L)+2e\sin(L+\phi)+e\epsilon^2\sin(L+\phi)\sin^2(M)\end{aligned}$$

As before, we can ignore the term of order $e\epsilon^2$, since we assumed both e and ϵ were small. However, we shouldn't ignore the term proportional to ϵ^2 since we didn't assume anything about how e and ϵ^2 compare! For Earth, it turns out that they are roughly the same size.

$$\boxed{\Delta\alpha \approx 2e\sin(L+\phi)-\frac{\epsilon^2}{4}\sin(2L)}$$

(d) The analemma looks like a figure-8 because of the presence of the $\sin(2L)$ term, which gives a contribution to $\Delta\alpha$ that oscillates with a period of 6 months rather than 1 year. Without this term, the analemma would just be an ellipse as both $\Delta\alpha$ and δ are oscillating at the same frequency.

If we plug in the parameters for Earth:

$$e = 0.0167$$

$$\epsilon = 23.44^\circ$$

$$\phi = 77^\circ$$

we can make a parametric plot of δ against $\Delta\alpha$: