2024 National Astronomy Competition

1 Instructions (Please Read Carefully)

The top 5 eligible scorers on the NAC will be invited to represent USA at the next IOAA. In order to qualify for the national team, you must be a high school student with US citizenship or permanent residency.

This exam consists of 3 parts: Short Questions, Medium Questions and Long Questions.

The maximum number of points is **225 points**.

The test must be completed within 2.5 hours (150 minutes).

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution.

This is a written exam. Contestants can only use a scientific or graphing calculator for this exam. A table of physical constants will be provided. **Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on April 13th.** Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books or notes are allowed during the exam. Exam is proctored and recorded. You are expected to have your video on at all times.

We acknowledge the following people for their contributions to this year's exam:

Abhay Bestrapalli, Orion Foo, Hagan Hensley, David Lee, Sandesh Kalantre, Andrew Liu, Joe McCarty, Leo Yao

2 Short Questions

1. **(15 points)** The Boomerang Nebula is the coldest known place in the universe (outside of labs on Earth), with a temperature of just 1 K. The nebula is a rapidly expanding cloud of gas ejected from a red giant star, with a radius of 1 light year and an unusually high expansion velocity of 165 km/s. Let's make a highly simplified model to understand why it's so cold.

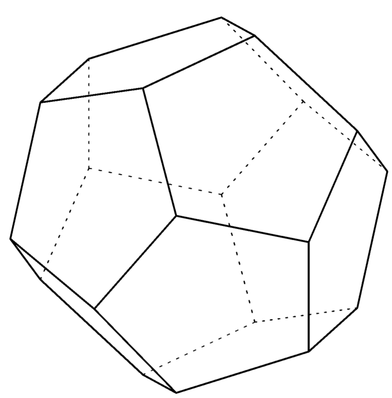
   (a) **(2 points)** Observations show that the nebula is gradually warming up near the very edge where it's in thermal contact with the surrounding interstellar medium, but it has not yet had time to reach thermal equilibrium with its environment by exchanging heat. (This is due to the rapid expansion - most such nebulae would take much longer to expand to this size.) However, it has done work on its environment, by pushing the interstellar medium out of the way as it expands. Let's approximate that no heat has been exchanged at all. In thermodynamics, what is the name for this kind of process?

   (b) **(6 points)** Assume that the nebula is an ideal gas, expanding as a spherical shell which maintains a constant thickness. In this kind of process, an ideal gas follows a relation $PV^\gamma = \text{constant}$, where $\gamma = \frac{5}{3}$.
   Based on this relation, how does the temperature T of the gas scale with the radius r of the gas as it expands?

   (c) **(7 points)** At very high temperature (right after ejection from the star) hydrogen is a plasma and so cannot be modeled as an ideal gas. For simplicity, take the initial temperature and radius of the cloud (once it has cooled down enough to no longer be a plasma) to be 10 000 K and 50 AU. What should the temperature be now, after the rapid expansion?

2. **(10 points)** David the astronomy enthusiast loves looking at stars! Specifically, he particularly enjoys looking at stars on the ecliptic. One day, he is out stargazing at midnight (local solar time) and looks at the antisolar point (the point on the celestial sphere exactly opposite to the Sun). He notices a faint glow of magnitude 12 mag/arcsec^2, and after some research he concludes that this is caused by a phenomenon known as *gegenschien*, where Solar System dust is lit up by the Sun and reflects some light back towards Earth. These particles are in an orbit of 2.06 AU around the Sun. Assuming the radii of these particles are around 1 cm and their albedo is 0.14, estimate the density of these particles. Express your answer in particles per square arcsecond.

3. **(15 points)** Astronomer Evan noticed that there is a group of stars $\{X_1, X_2, \dots, X_{12}\}$ which forms a perfect dodecahedron around the Earth.



Evan went outside to observe some of these stars. At a particular moment in time, he computed the angle

$$\delta_Z = \max(ZX_{a_1}, ZX_{a_2}, ZX_{a_3}, ZX_{a_4}, ZX_{a_5}),$$

where Z is his zenith and $\{a_1, a_2, \ldots, a_5\} \subset \{1, \ldots, 12\}$ are the indices of the five closest stars in angular distance to Z. Compute the minimum possible value of δ_Z that he could have observed.

Note that δ_Z is not a declination.

3 Medium Questions

1. **(30 points)** *Solar eclipse*

   Throughout this question, you can use that the eccentricity of the Moon's orbit is $e_{☾} = 0.055$. We will use the simplifying assumptions that the Earth's orbit is circular with a radius of 1 AU, the Earth has no axial tilt (i.e., the equator and ecliptic coincide), and the Moon's orbit lies exactly on the ecliptic.

   (a) **(7 points)** Calculate the maximum distance of the Moon from the center of the Earth for a total solar eclipse to happen. Also calculate the distance of the Moon at perigee and apogee, and compare these three numbers.

   (b) **(3 points)** The Moon is slowly moving away from the Earth due to tidal effects, with its semimajor axis increasing at 38 mm per year. Assuming its eccentricity remains the same, how many years in the future will total solar eclipses become impossible?

   (c) **(5 points)** Assume there is a new moon at **perigee**. What is the Moon's speed relative to the Earth and the fixed background stars? You may neglect the mass of the Moon in comparison with the Earth.

   For the remainder of this problem, consider an observer on the equator for whom the Sun is eclipsed at noon, and the Moon is at perigee.

   (d) **(3 points)** The answer from (c) is the speed of the Moon in the geocentric frame in which the Earth and the background stars are fixed. However, for an eclipse, we are interested in the Moon's motion relative to the Sun instead of the background stars. What is the Moon's speed in a geocentric frame in which the Earth and the Sun are fixed? Hint: rotate the previous frame by the apparent angular velocity of the Sun.

   (e) **(6 points)** What is the diameter of the Moon's shadow on the observer?

   (f) **(6 points)** What is the duration of totality for the observer?

2. **(25 points)** *Neutrino decoupling*

   At the beginning of the Universe, immediately following inflation, neutrinos were in thermal equilibrium with other radiation and matter, maintained through weak interactions. As the Universe expanded and the temperature of the Universe dropped, the rate of these weak interactions decreased until thermal equilibrium was no longer maintained, and neutrinos began to propagate freely through the Universe. This event is called neutrino decoupling, and occurred at about $t_n = 1\text{sec}$ after the Big Bang. Initially, the energy density of radiation was greater than that of matter, but as energy of radiation decreased with cosmological expansion, eventually the energy density of radiation fell below that of matter. This earlier period is called the radiation-dominated era, while the later period is called the matter-dominated era, with a transition time of about $t_r = 50000\text{yr}$.

   (a) **(4 points)** Write proportionalities for the energy density of matter $\rho_m(a)$ and the energy density of radiation $\rho_r(a)$ as a function of scale factor. Your answer should look like $\rho_m(a) \propto f(a)$ and $\rho_r(a) \propto g(a)$ for some single variable functions f, g.
   Hint: radiation is redshifted with cosmological expansion, while matter is not.

   (b) **(6 points)** Using the Friedmann equation:

   $$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

   Derive proportionalities for the evolution of a matter-dominated and a radiation-dominated Universe, $a_m(t)$ and $a_r(t)$. Take t to be the time since the start of the Universe in each case, assuming the Universe was always made of only matter or only radiation. Assume no curvature or cosmological constant. Derivation required for credit.
   Hint: guess a form $a(t) = bt^c$ with b, c constants, which has derivative $\dot{a}(t) = c \cdot bt^{c-1}$, and equate exponents.

From this point forwards, take the simple (and unrealistic) assumption that in the radiation-dominated era, our Universe was entirely made up of radiation from the start up to some time t_r, then was entirely made up by matter afterwards in the matter-dominated era.

(c) **(10 points)** Using the conditions that the scale factor $a(t)$ and expansion rate $H(t) = \frac{\dot{a}(t)}{a(t)}$ of our Universe must be continuous, write expressions for the scale factor $a(t)$ of our Universe for $t < t_r$ and $t > t_r$, up to a single scaling constant. Do not substitute in the value of t_r yet.

Hint: naively trying to equate the expressions of the previous part will not allow both $a(t)$ and $H(t)$ to be continuous. What assumption holds in those expressions that no longer holds here, giving us an extra degree of freedom?

(d) **(2 points)** Solve for the scaling constant in the previous part in terms of t_r and t_0, the current age of our Universe.

Hint: what is the current value of $a_0 = a(t_0)$

(e) **(3 points)** For a boundary between matter and radiation-dominated eras of $t_r = 50000$yr, estimate the temperature T_n of the Universe at the time of neutrino decoupling. The current age of the Universe is $t_0 = 13.8$ billion years, and the current temperature of the Universe is $T_0 = 2.73$K.

3. **(30 points)** *Spiff's adventures*

You may attempt subsections without attempting the previous parts by making appropriate assumptions.

Spaceman Spiff is exploring the distant reaches of the Milky Way. He ends up crashing on a very cold planet. He calls the corresponding star Burner and the planet Ash. Though there aren't signs of life nearby, Spiff wants to determine if Ash is habitable.

(a) **(5 points)** The first thing Spiff observes is that Burner (seen through safe equipment!) is not uniformly bright over its disc in the sky. Indeed, the edges seem to be less bright than the center. Spiff opens up an astrophysics book (his computer is not working) and find the following equation:

$$I(\tau, \mu) = \frac{3}{4\pi}\left(\tau + \mu + \frac{2}{3}\right) F(0)$$

Here, $I(\tau, \mu)$ gives the intensity of light at optical depth τ and $\mu = \cos\theta$ where the angle θ is the angle between the direction of the ray and the radial direction. $F(0)$ is the net flux at the surface. Help Spiff find the ratio of intensity of the center of Burner to the edge (as it appears to Spiff). What is this phenomenon called?

(b) **(15 points)** Spiff walks for a few days but doesn't find any life nearby. Over the many nights and days, Spiff makes several observations about Burner and Ash:

- The peak wavelength of Burner appears to be 500 nm.
- Before the ship crashed, Spiff had found the planet has a radius twice that of Earth and that it has a circular orbit around Burner.
- Precisely calculation yields that each solar (Burnar?) day is 192 hours long and that Burner moved 1 degree against the background stars over a day.
- The mass and radius of Burner are similar to that of the Sun.
- Comparing the rotation of Burner (found by tracking a Burnerspot) to the sky, Spiff finds out that Ash has an axial tilt of 60 degrees! Spiff hypothesises that, because of a thin atmosphere and other effects, the heat received from the star never redistributes to the part of Ash that is not exposed to the star, and is only emitted by the exposed part. Also somewhat peculiarly, the unexposed part always remains unexposed (Ash is always titled towards Burner). Assume that Ash is at thermal equilibrium.

What is the equilibrium temperature of Ash? Is it theoretically possible to have life on this planet?

(c) **(4 points)** Spiff, being the genius he is, manages to fix the spaceship on his own, weaves through a dangerous meteor field and escape Burner's system. When he is around 10 parsecs away, he stops and looks back at Burner. He realises that he is still on Ash's plane of revolution when it starts transiting across Burner's disc. How long will the transit of Ash across Burner take in seconds?

(d) **(6 points)** Now Spiff has moved away further to 100 parsecs and has finally reached home. He looks back at Burner one last time. What is the apparent magnitude of the star (Assume that the transit of Ash is not happening) for Spiff? Can he see it with his naked eye (Spiff is indeed human)? Assume that extinction is significant, and is equal to 2 mag/kpc.

4 Long Questions

1. **(50 points)** *TESS*

   David the observation master just launched his new telescope into a geostationary orbit, which he calls the Terrestrial Earth Stalking Satellite (TESS)! To test his telescope, he aims it straight down at his friend Matthew.

   (a) **(9 points)** Assume Matthew is a perfect gray body radiator with an albedo of $a = 0.60$. Matthew has a surface area of $1.95\,\mathrm{m}^2$ and a temperature of $37\,^\circ\mathrm{C}$. Estimate his apparent bolometric magnitude as seen by TESS. You may make the slightly unrealistic assumption that Matthew is in full sunlight (i.e. his full surface area is reflecting light coming solely from the Sun). Also assume that the Earth's albedo is 0.30, that Matthew is directly below the telescope, and that he radiates isotropically. Hint: the Stefan-Boltzmann Law can be modified for gray bodies with albedo a by multiplying the power by a factor of $(1-a)$. The absolute bolometric magnitude of the Sun is 4.74.

   Having captured images of Matthew, he is eager to analyze the data! However, due to various sources of noise, the picture isn't as clear as he expected. The below table gives some of the properties of the telescope's CCD camera.

Dimensions	Pixel Size	Focal Length	Read Noise	Dark Current	Saturation	m_0
4096×4096	$15\,\mu\mathrm{m}$	$3.14\,\mathrm{m}$	$9\,\mathrm{e}^-$	$1.1\,\mathrm{e}^-/\mathrm{pix/s}$	$200\,000\,\mathrm{e}^-$	20.44

Table 1: TESS Camera Parameters

   (b) **(6 points)** What is the maximum signal to noise ratio (SNR) David can get from a single exposure before his pixels saturate? What exposure time would this correspond to? Hint: The value of m_0 corresponds to the magnitude at which the detector registers one count/second. The signal to noise ratio for a single CCD pixel with photon flux P can be calculated using the following formula (where t is the exposure time, D is the dark current and N_r is the read noise):

$$\mathrm{SNR} = \frac{Pt}{\sqrt{Pt + Dt + N_r^2}}$$

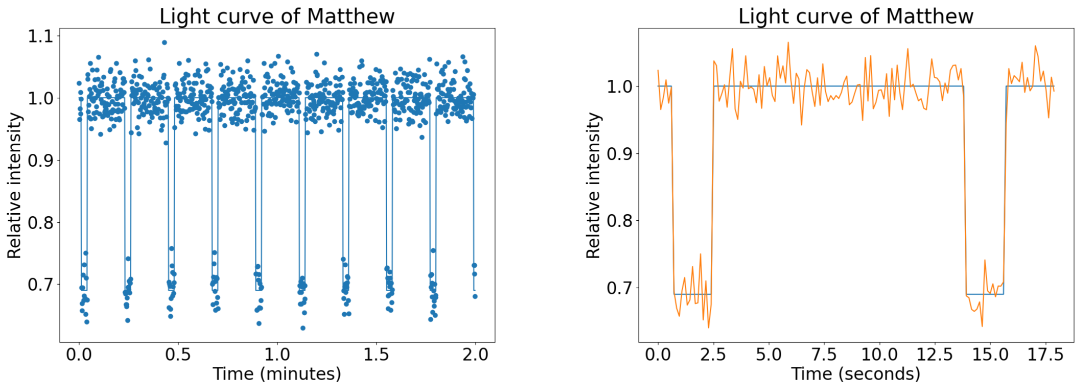
   (c) **(10 points)** Looking closer at the data, David realizes that there are photon sources other than Matthew. Specifically, the ground produces a lot of background noise. David realizes he cannot include these counts as signal, but they still contribute to the $\sqrt{n}$ photon (shot) noise. You can use the albedo of the Earth given in part (a) for your estimate, and assume that the temperature of the ground is $20\,^\circ\mathrm{C}$. Calculate the new maximum signal to noise ratio and corresponding exposure time.

   While analyzing his data, David notices something interesting. He notices that every 45 seconds, the location of Matthew changes by 1 pixel (assume the direction of the motion is exactly parallel to the columns of the detector array). He wonders if this variation could be explained by small vibrations in his satellite.

   (d) **(3 points)** What would the minimum angular velocity of the satellite have to be in order to explain this movement? Express your answer in arcseconds per minute.

   Being the observational master that he is, David knows that his control of the satellite is far too precise for camera jitter to be a viable explanation. Furthermore, as more data comes in, he see that the movement continues to be in a straight line, something unlikely to be observed if random shaking of the satellite was at fault. Knowing Matthew, he comes up with an alternative explanation: Matthew is in the middle of a cross country race!

   (e) **(4 points)** What would the minimum velocity Matthew would have to run to produce this movement?



(a) (b)

Figure 1: Light Curves of Matthew

Although Matthew is too dim for David to achieve a reasonable SNR, as an observational master, David uses his knowledge of the dark arts to remove enough noise to do basic photometry experiments. Performing photometry on Matthew, David once again notices something interesting. Matthew's light curve has periodic dips! Figure 1 shows plots of the light curve.

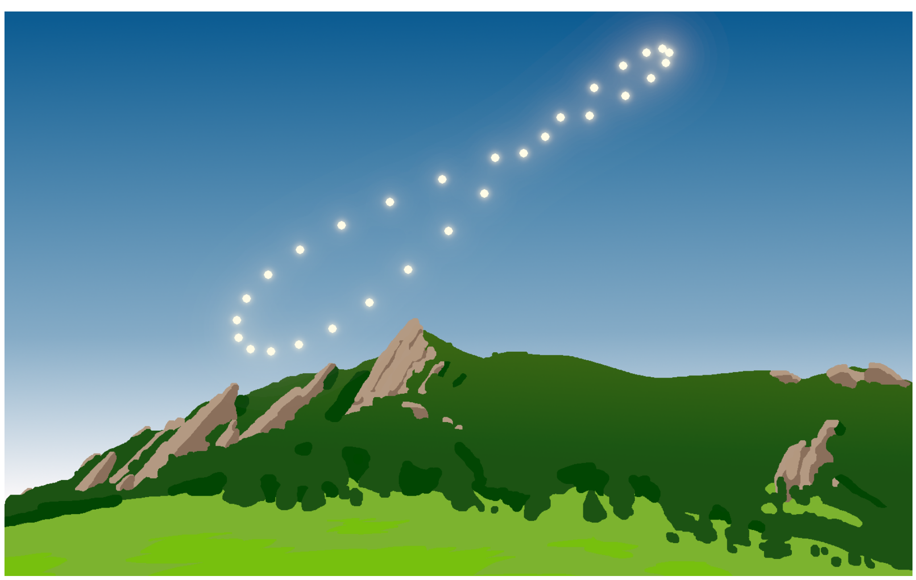
(f) David hypothesizes that these dips are due to trees being planted at regular intervals along the route, blocking some fraction of the sunlight incident on Matthew (you may assume that the trees are not directly over Matthew and so they do not block any of the light reflected off him).

   (i) **(5 points)** How far apart are these trees planted?

   (ii) **(5 points)** Estimate the width of each tree.

   (iii) **(8 points)** By what factor does each tree reduce the incoming sunlight?

2. **(50 points)** *The Analemma*



If you take a picture of the Sun at the same time every day over the course of the year, you will see it traverse a famous figure-8 pattern known as the analemma. You can think of this as happening because the Sun's right ascension and declination vary over the course of the year relative to a "hypothetical sun" that travels at a constant speed along the celestial equator, and would appear to stay in place if observed at the same time each day.

In this problem, we will derive an expression for the shape of the analemma. Assume throughout this problem that $e \ll |\epsilon| \ll 1$, where e is the Earth's eccentricity and ϵ is the Earth's axial tilt. Use the small-angle approximation wherever it is justified - you'll need it often (sometimes in conjunction with other trig identities).

(a) **(20 points)** We conventionally parameterize the time of year in terms of the "mean anomaly" M, which starts at 0 at Earth's perihelion, and increases from 0 to 2π (radians) at a constant rate over the course of a year.

Since perihelion does not occur at the same time as the vernal equinox, we define the angle ϕ to be equal to the mean anomaly at the time of the vernal equinox. For convenience define another variable: the "mean longitude" $L = M - \phi$, which is equal to the longitude of the "hypothetical sun" mentioned above.

Write down an expression (valid for $e \ll 1$) for the ecliptic longitude l of the Sun, as a function of L and ϕ. Hint: you can apply Kepler's second law to calculate l at a few specific points along Earth's orbit, and then make an educated guess of the general form from there.

We can describe the analemma as a parametric function of L, in terms of its right-ascension and declination components:

$$(\Delta\alpha(L), \delta(L))$$

where $\delta(L)$ is the declination of the Sun, and $\Delta\alpha(L) = \alpha(L) - L$ is the *equation of time*, defined as the difference in right ascension between the real Sun and the mean longitude.

(b) **(10 points)** Using spherical trigonometry and your expression for $l(L)$, write down an expression for $\delta(L)$. Remember to use the small-angle approximation!

(c) **(15 points)** Making the same assumptions as before, find an expression for $\Delta\alpha(L)$. Be careful: there will be a term that depends on e, and a term that depends on ϵ - they're approximately equal in magnitude (for Earth), so you need to include them both.

You'll need couple of useful approximations, both of which hold for any $x \ll 1$:

$$\cos(x) \approx 1 - \frac{x^2}{2}$$

$$\frac{1}{1-x} \approx 1 + x$$

(d) **(5 points)** Based on the results of parts (b) and (c), explain qualitatively why the analemma looks like a figure-8 curve.

(If you're curious, after the exam, you can try to graph or sketch a parametric plot of $\delta(L)$ against $\Delta\alpha(L)$. For Earth in 2024, $\phi = 77°$, $\epsilon = 23.44°$ and $e = 0.0167$. The result should look familiar.)