2020 National Astronomy Competition

1 Instructions (Please Read Carefully)

The top 5 eligible scorers on the NAC will be invited to represent USA at the next IOAA. The next 5 eligible scorers will be invited to join the USA guest team at the next IOAA. In order to qualify for the national team, you must be a high school student with US citizenship or permanent residency.

This exam consists of 3 parts: Short Questions, Medium Questions and Long Questions.

The test must be completed within 2.5 hours (150 minutes).

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution.

This is a written exam. Contestants can only use a scientific calculator for this exam. A table of physical constants will be provided. **Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on April 5th.** Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a scientific calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books or notes are allowed during the exam.

2 Short Questions

1. (**5 points**) The sidereal period of Mars is 687 days, while the sidereal period of Earth is 365.26 days. The most recent opposition of Mars occurred on July 27, 2018. Predict all dates in the year 2020 when Mars will be in quadrature. You may use the fact that the orbital radius of Mars is 1.52 AU and that Earth and Mars have circular orbits. Why might your answer be inaccurate?

Solution: The synodic period is computed as $\frac{1}{P_{syn}} = \frac{1}{P_{Earth}} - \frac{1}{P_{Mars}}$. Using the values given in the problem, we obtain $P_{syn} = 779.9$ days. Thus, oppositions happen every 779.9 days, so the next one will be around 9/15/2020. Quadrature occurs when Mars is approximately $\arccos 1/1.52 = 48.9$ degrees behind or in front of its position at opposition, or $779.9(\frac{48.9}{360}) = 106$ days before or after opposition. The only such date in 2020 is around 6/2/2020 (accept answers within two days of this date).

Inaccuracies could be due to the nonzero eccentricities of Earth and Mars, as well as the nonzero relative inclination between the two planets' orbits. The actual date is nearer 6/7/2020.

2. (**10 points**) The star Betelgeuse has recently made news for its abnormal dimming. Although the dimming has now been attributed to dust, we consider in this problem that it was due to radial pulsations. Suppose that Betelgeuse's mass is 11 solar masses and its radius is 887 solar radii. Furthermore, Betelgeuse is currently rotating such that the tangential velocity of a point on its equator is 5000 m/s (assume Betelgeuse is perfectly spherical). The dimming has increased Betelgeuse's apparent magnitude by 1.05. You may neglect the contribution of pulsation to the surface velocity.

(a) Assuming contraction and expansion are isothermal, find the (new) radius of the star (in solar radii) needed to account for the dimming.

(b) Assuming no mass loss, find the new angular rotation velocity of the star.

Solution: An increase in apparent magnitude of 1.05 corresponds to a decrease in brightness by a factor of $100^{1.05/5} = 0.38$. This is the ratio of the final to the initial luminosity. Applying the Stefan-Boltzmann law, $\frac{L_f}{L_i} = \frac{R_f^2}{R_i^2}$, so the final radius is 547 solar radii.

Apply conservation of angular momentum to find the final angular velocity. The initial angular velocity is $\omega_i = \frac{v_T}{R_i} = 8.10 \times 10^{-9}$ rad/s. The initial moment of inertia is $I_i = \frac{2}{5}MR_i^2 = 3.34 \times 10^{54}$ kg-m^2 and the final moment of inertia is $I_i = \frac{2}{5}MR_f^2 = 1.27 \times 10^{54}$ kg-m^2. $I_i\omega_i = I_f\omega_f$, so $\omega_f = 2.13 \times 10^{-8}$ rad/s.

3. (**10 points**) The Lyman-break galaxy selection technique makes use of the fact that any light from galaxies with wavelength shorter than the Lyman limit (the shortest wavelength in the Lyman series) is essentially totally absorbed by neutral gas surrounding the galaxies. The ionization energy of hydrogen is 13.6 eV. Suppose that we are observing galaxies in the V band, whose effective midpoint is 551 nm and bandwidth is 99 nm.

(a) At what range of redshifts would we begin to see galaxies "disappear" (break) from images in the V band?

(b) What range of recessional velocities (km/s) and distances (Mpc) does this correspond to? Assume only Hubble expansion contributes to the radial velocity and redshift.

Solution: The Lyman limit is obtained via $E = \frac{hc}{\lambda_L} = \frac{1}{1^2} - \frac{1}{\infty^2}$. We have $E = 13.6$ eV, so $\lambda_L = 91.2$ nm. For the V band, $\lambda_{max} = 600.5$ nm and $\lambda_{min} = 501.5$ nm. Applying $z = \frac{\delta\lambda}{\lambda_L}$, the desired range of redshifts is $z = 4.50 - 5.58$. Using relativistic Doppler shift, $v_{rel} = \frac{(1+z)^2-1}{(1+z)^2+1}c$, the range of recessional velocities is $v_{rel} = 281,000 - 286,000$ km/s. Applying Hubble's law, the range of distances is 4010-4090 Mpc.

4. (**5 points**) TRAPPIST-1d is a temperate exoplanet that orbits the ultra-cool M dwarf star TRAPPIST-1 with a semi-major axis of 0.022 AU. TRAPPIST-1 has a mass of 0.089 Solar masses and an effective temperature of 2511 K. Through transit timing variations induced by other planets in the TRAPPIST-1 system, TRAPPIST-1d is estimated to have a mass of 0.297 Earth masses. Assuming that TRAPPIST-1d has a circular orbit (which is a good approximation because the measured eccentricity is only 0.008), what is the radial velocity semi-amplitude of TRAPPIST-1 due to the orbital motion of TRAPPIST-1d, in m/s?

Solution: $M_p/M_\star = V_\star/V_p \to V_\star = M_pV_p/M_\star$. $V_p = 2\pi a/T$.
$T = ([a(AU)]^3/[M_\star(M_{\text{Sun}})])^{1/2} = (0.022AU^3/0.089M_{\text{Sun}})^{1/2} \times 365.25\, days/year = 4.00$ days.

$$V_p = \frac{2\pi(0.022AU \times 1.496 \times 10^{11}m}{4.00days \times 86400sec/day} = 59.8\text{km/s}$$

.

$$V_\star = \frac{0.297 \times 5.97 \times 10^{24}kg \times 59.8 \times 10^3 m/s}{0.089 \times 1.99 \times 10^{30}kg} = 0.597 \text{ m/s}$$

.
The radial velocity semi-amplitude of TRAPPIST-1 due to TRAPPIST-1d is 0.597 m/s.

5. (**5 points**) HD 209458b is a hot Jupiter exoplanet with a mass of 0.69 Jupiter masses. However, HD 209458b has an anomalous radius of 1.38 Jupiter radii that is inflated relative to Jupiter. Jupiter has an interior that is comprised of metallic hydrogen at pressures greater than 1 Mbar. Estimate the pressure, in Mbar, at the center of HD 209458b, and determine whether or not the interior of HD 209458b will also be comprised of metallic hydrogen.

Solution: Estimate: $P_c \sim \rho g R \sim 3GM^2/(4\pi R^4)$. The actual solution (if a student knows the equation or can derive it) is $P_c = 3GM^2/(8\pi R^4)$.
Plugging in, $P_c = 3 \times 6.67 \times 10^{-11} m^3/kg/s^2 \times (0.69 * 1.898 \times 10^{27}kg)^2/(8\pi(1.38 \times 69.91 \times 10^6 \text{ m})^4 =$ 1.57 Mbar. Estimate would give twice the value (because of ignoring the integral, which is acceptable because no calculus is required), which is 3.15 Mbar.
Accept answers between 1 and 5 Mbar. Yes, the interior of HD 209458b will be composed of metallic hydrogen.

3 Medium Questions

1. (**15 points**) An astronomer used his f/5 telescope with a diameter of 130 mm to observe a binary system. He is using an eyepiece with a field of view of 45° and a focal length of 25 mm. In this system, star A has a mass of 18.9 solar masses, and an apparent magnitude in the V filter of 9.14. Star B has a mass of 16.2 solar masses, and an apparent magnitude in the V filter of 9.60. The period of the system is 108

days, and the distance between the binary stars and the Solar System is 2.29 kpc. The binary system has an edge-on orbit relative to the Solar System.

(a) What is the field of view of the telescope?
(b) What is the limiting magnitude of the telescope?
(c) What is the angular resolution of the telescope?
(d) What is the angular separation between the stars?
(e) Is the astronomer able to observe both stars as distinct points in the telescope? Answer as YES or NO.

The limiting magnitude for the human eye is 6.0, and the diameter of the pupil is equal to 7.0 mm. Also consider that visible light has a wavelength of 550 nm.

Solution:

(a) It is possible to divide the field of view of the eyepiece by the magnification of the telescope in order to obtain an approximation for the field of view of the telescope:

$$
\begin{aligned}
FOV_{telescope} &= \frac{FOV_{eyepiece}}{m} \\
FOV_{telescope} &= \frac{FOV_{eyepiece}}{f_{telescope}/f_{eyepiece}} \\
FOV_{telescope} &= \frac{45^\circ}{(130 \times 5)/(25)} \\
FOV_{telescope} &= 1.7^\circ
\end{aligned}
$$

(b)

$$
\begin{aligned}
m_{eye} - m_{telescope} &= 2.5 \log \frac{D^2_{eye}}{D^2_{telescope}} \\
6.0 - m_{telescope} &= 2.5 \log \frac{7^2}{130^2} \\
m_{telescope} &= 12.3
\end{aligned}
$$

Since both of stars are brighter than the limiting magnitude, they both can be detected by the telescope.

(c) It is possible to use the following expression to calculate the angular resolution of the telescope:

$$
\begin{aligned}
\theta &= \frac{1.22\lambda}{D_{telescope}} \\
\theta &= \frac{1.22 \times 550 \times 10^{-9}}{130 \times 10^{-3}} \\
\theta &= 5.16 \times 10^{-6}\text{rad}
\end{aligned}
$$

(d) Using Kepler's Third Law, it is possible to determine the separation between star A and star B:

$$
\begin{aligned}
T^2/a^3 &= 1/M \\
(108/365.25 years)^2/a^3 &= 1/((18.9 + 16.2)) \\
a &= 1.45\,AU
\end{aligned}
$$

Dividing the separation between the stars by the distance to the binary system, it is possible to obtain the angular separation between the stars:

$$\alpha = a/d$$
$$\alpha = (1.45AU)/(2.29 \times 10^3 \times 206265AU)$$
$$\alpha = 3.08 \times 10^{-9}\text{rad}$$

(e) Since the angular separation between the stars is smaller than the angular resolution, the astronomer will observe both stars as a single point in his telescope. Therefore, the correct answer is NO.

2. (**20 points**) The rotation curve of a particular spiral galaxy is modeled by an exponential function of the form $V(r) = V_0(1 - e^{-r/R})$, where $V_0 = 250$ km/s, $R = 7.5$ kpc, and r is measured radially from the center of the galaxy. Throughout parts (a)-(d), you may assume the galaxy is disk-shaped. Further, we'll assume that the distribution of mass in the galaxy depends only on the radial coordinate r (and is thus radially symmetric).

   (a) Find the period of rotation (in years) of a particle 10 kpc from the center of the galaxy. Also, find the mass enclosed within the (circular) orbit in solar masses, i.e. the mass within $r = 10$ kpc from the center of the galaxy.

   (b) Find the angular velocity of the galaxy very close to the center ($r \ll R$). Hint: $e^x \approx 1 + x$ for $|x| \ll 1$.

   (c) Determine how the (gravitational) mass per unit area must vary with distance from the center of the galaxy in order to yield the given rotation curve. Find the expressions only for regions very far from the galactic center.

   (d) An astronomer measures the absolute bolometric magnitude of the galaxy to be -21.2. For comparison, the bolometric magnitude of the sun is 4.75. Assume that the luminous mass per unit area follows a profile given by $\sigma_L = \frac{k}{r}$ for $k = 2.55 \times 10^8$ M_{Sun}/kpc and that all of the luminous mass is in the form of Sun-like stars. Approximate the percentage of the galaxy's mass that is dark matter, out to the maximum distance (radius) that is still visibly defined.

Solution:

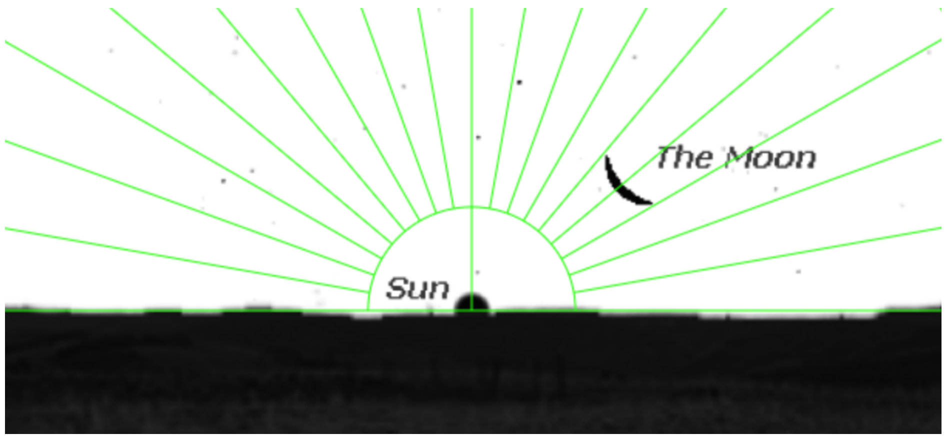
(a) The period is $T = \frac{2\pi r}{V}$, so we obtain $T = 334$ million years. Equating the centripetal force with the gravitational force (as is the condition for a circular orbit), $M = \frac{V^2 r}{G}$, so we obtain $M = 7.88 \times 10^{10}$ solar masses.

(b) $\omega = \frac{V}{r}$ is the angular velocity. Very close to the center, $e^{-r/R} \approx 1 - r/R$, so $\omega \approx V_0/R = 1.08 \times 10^{-15}$ rad/s.

(c) Recall $M = \frac{V^2 r}{G}$. Consider an annulus at radius r with small radial thickness δr. Letting σ be the desired mass per unit area, we have that $\sigma 2\pi r \delta r = \delta M$. But note that $\delta M \approx \frac{V^2 \delta r}{G}$ very far away from the center (as V is essentially constant). Thus we obtain $\sigma = \frac{V_0^2}{2\pi r G}$ very far away from the center ($r \gg R$).

(d) $M_{bol} = 4.75 - 2.5\log(L/L_{Sun})$, so $L/L_{Sun} = 2.40 \times 10^{10}$. The luminous mass enclosed by an annulus at radius r with small radial thickness δr is $2\pi r \delta r \sigma_L = 2\pi k \delta r$, which means we seek r_{max} such that $2\pi k r_{max} = 2.40 \times 10^{10}$. This gives $r_{max} = 14.97$ kpc as the maximum distance from the center of the galaxy for which the galaxy is still visibly defined. The total gravitational mass enclosed at this radius is $M = \frac{V^2 r}{G} = 1.63 \times 10^{11}$ solar masses. The fraction of dark matter is approximately $1 - \frac{2.40\times10^{10}}{1.63\times10^{11}} = 85\%$.

3. (**15 points**) An astro-photographer has taken the photo of the moon close to a new moon day shown below right before the sunset on December 21 (Winter Solstice) in a wide open area.



(a) In which hemisphere (Northern or Southern) is the photographer located?

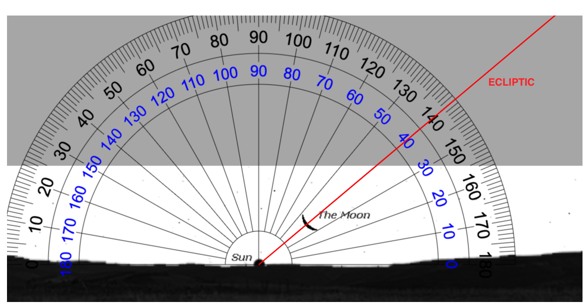
(b) Find the latitude of the photographer. Ignore the orbital inclination of the Moon and the ellipticity of the Earth's orbit. Hint: The green equiangular lines are added to the image to help you out in measuring any relevant angle.

(c) Calculate the sidereal time when the photo was taken.

Solution:

(a) By looking at the position of the moon with respect to the sun, we know that the photo has been taken from the Southern hemisphere.

(b) First off, we measure the angle between the horizon and ecliptic. It is $\sim 40°$.



Drawing the celestial spheres, let S be the celestial south pole, Z be the zenith, and K' be the South ecliptic pole. $\widehat{SZ} = 90 - \phi$; $\widehat{S(Sun)} = 90 - 23.5 = 66.5°$; $\widehat{ZK'} = 40°$; $\widehat{SK'} = 23.5°$; $\widehat{Z(Sun)} = 90°$; $\widehat{K'(Sun)} = 90°$ and $\angle K'S(Sun) = 180°$. So, Sun is the pole of a great circle crossing both K' and Z points, $\Rightarrow \angle S(Sun)Z = 40°$. Using the spherical law of cosines, $\sin\phi = \cos 23.5° \cos 40° \Rightarrow \phi = 44.6°$. Since the astrophotographer is in the Southern hemisphere, $\phi = 44.6° S$.

(c) Sidereal time=$H_\odot + RA_\odot$. Using the spherical law of cosines, $\cos H_\odot = -\tan 23.5° \tan 44.6° \Rightarrow H_\odot = 7^h 42^m$. $RA_\odot$(Winter Solstice)$= 18^h \Rightarrow ST = 7^h42^m + 18^h = 1^h42^m$.

4. (**20 points**) In general relativity, the orbit of satellites around a massive object (like a black hole) are known as geodesics and do not obey all of Kepler's laws for orbits. However, for objects that are moving at non-relativistic speeds, we can analyze the orbit using classical mechanics, with a corrective term added to Newton's Law of gravity. In this case, the potential energy of an object in orbit around a black hole is:

$$V_s(r) = -\frac{GMm}{r} - \frac{GML^2}{c^2mr^3}$$

where M and m are the masses of the black hole and the object respectively, c is the speed of light, L is the angular momentum of the object in orbit and r is the distance of that object from the black hole. Likewise, the gravitational force from a black hole has magnitude:

$$F_s(r) = \frac{GMm}{r^2} + \frac{3GML^2}{c^2mr^4}$$

You may assume that both conservation of energy and conservation of angular momentum hold in this regime.

(a) Argue which of Kepler's laws are still true.

(b) Calculate the radius of a stable circular orbit with an angular momentum L (you will get two solutions, the stable orbit generates the classical result under the proper limits)

(c) What is the radius, R_{ISCO} of the innermost stable circular orbit (the smallest stable circular obit) for a black hole of mass M? What is the numerical value of R_{ISCO} for Sagittarius A*, which has mass 3.6×10^6 solar masses?

(d) Suppose we discovered a new star orbiting Sagittarius A*, S99, that has a periapsis of $10R_{ISCO}$ and an apoapsis of 16 AU. Find the magnitude of velocity of S99 at both periapsis and apoapsis.

Solution:

(a) Only Kepler's law of areas still holds from the fact that we have a radial force.

(b) Using centripetal force, we get that

$$\frac{L^2}{mr^3} = \frac{GMm}{r^2} + \frac{3GML^2}{c^2mr^4}$$

A little algebra later, we get the quadratic equation,

$$r^2 - \frac{L^2}{GMm^2}r + \frac{3L^2}{m^2c^2} = 0$$

Using the quadratic equation, we get:

$$r = \frac{L^2}{2GMm^2}\left(1 \pm \sqrt{1 - 12\frac{G^2M^2m^2}{L^2c^2}}\right)$$

Taking the limit that $c \to \infty$ we see that the plus solution gives us the classical result. Thus the stable orbit is

$$r = \frac{L^2}{2GMm^2}\left(1 + \sqrt{1 - 12\frac{G^2M^2m^2}{L^2c^2}}\right)$$

and the unstable orbit is

$$r = \frac{L^2}{2GMm^2}\left(1 - \sqrt{1 - 12\frac{G^2M^2m^2}{L^2c^2}}\right)$$

(c) The ISCO occurs with the smallest allowed value of L which is $L^2 = \frac{12G^2M^2m^2}{c^2}$. This gives a value of $R_{ISCO} = \frac{6GM}{c^2}$ and calculating numerically we get $R_{ISCO} = 3.20 \times 10^{10} \approx 0.21$ AU.

(d) One way to solve this problem is to first find the angular momentum of the orbit; to do this we use conservation of energy:

$$\frac{L^2}{2mr_1^2} - \frac{GMm}{r_1} - \frac{GML^2}{c^2mr_1^3} = \frac{L^2}{2mr_2^2} + -\frac{GMm}{r_2} - \frac{GML^2}{c^2mr_2^3}$$

Substituting $l = \frac{L}{m}$ and solving, we see that:

$$l^2 = \frac{2GM(1/r_1 - 1/r_2)}{(1/r_1^2 - 1/r_2^2) - \frac{2GM}{c^2}(1/r_1^3 - 1/r_2^3)} = \frac{R_s(1/r_1 - 1/r_2)}{(1/r_1^2 - 1/r_2^2) - R_s(1/r_1^3 - 1/r_2^3)}c^2$$

Where $R_s = 0.07$ AU is the Schwarchild radius of Sagittarius A*, $r_1 = 2.1$ AU and $r_2 = 16$ AU. Doing the calculations we get that $l = 0.36c\times$ AU. Thus the velocities are $v_1 = l/r_1 = 0.171c = 5.19 \times 10^7$ m/s and $v_2 = l/r_2 = 0.022c = 6.82 \times 10^6$ m/s.

5. (**15 points**) The following table gives the numerical values for some physical properties of four stars. The quantities that are affected by, i.e. include the effects of, interstellar extinction are marked with a star (*). You may consider that all stars are black bodies. The temperature of a star can be calculated directly from its B-V index, by using Ballesteros' formula:

$$T_{eff} = f(B - V) = 4600\left(\frac{1}{0.92(B-V) + 1.7} + \frac{1}{0.92(B-V) + 0.62}\right) K.$$

Determine the numerical values of all the other physical characteristics presented in the given table. For full credit, show your full work by writing all the mathematical expressions used in the calculation.

Hint: You might use the following empirical relation:

$$\frac{A_V}{E_{B-V}} = 3.2$$

Star	κ Velorum	β Tauri	Sirius A	Sun
Annual Parallax $p^*(10^{-3} arcsec)$	6.05	24.89	379.2	-
Distance to Sun $\Delta^*(pc)$				-
Interstellar Extinction in V Band A_V (mag)	0.20	0.08	Negligible	-
$10^{0.2\times A_V}$				-
Distance to Sun $\Delta(pc)$				4.85×10^{-6}
Annual Parallax $p(10^{-3}$ arcsec)				
Distance Modulus $\mu = m - M$				
Visual apparent magnitude $m^*(mag)$	2.86	1.68	-1.47	-26.73
Visual apparent magnitude m (mag)				
Visual absolute magnitude M_V (mag)				
Color Index $(B-V)^*$ (mag)	-0.14	-0.06	+0.01	+0.65
Extinction E_{B-V} (mag)				
Color Index $(B-V)$ (mag)				
Effective Temperature $T_{eff} = f(B-V)$ (K)				
λ_m (nm)				
Radius (Solar Radius, R_S)	9.10	4.60	1.71	1.00
Total Luminosity (Solar Luminosity,L_S				1.00
Absolute Bolometric Magnitude M_{bol} (mag)				4.64
Bolometric Correction BC for V band (mag)				-0.20

Solution:

It is known that:

$$\Delta^*_{pc} = \frac{1}{p^*_{arcsec}}$$

Also,

$$m = M + 5\log\Delta_{pc} - 5$$

without extinction, and

$$m^* = M + 5\log\Delta^*_{pc} - 5$$

with interstellar extinction. Because the luminous flux is dimmed by the interstellar extinction,

$$m^* = m + A_V$$

Thus,

$$\Delta_{pc} = \Delta^*_{pc} \times 10^{-0.2A_V}$$

Distance modulus is given by:

$$\mu = m - M = 5 \log \Delta_{pc} - 5$$

And thus, the absolute visual magnitude can be computed as:

$$M_V = m - \mu$$

For the Color Excess, we know that:

$$E_{B-V} = (B - V)^* - (B - V)$$

In order to find the temperature of a given star, we can use Ballesteros' formula, given in the question. Next, from Wien's formula:

$$\lambda \times T_{eff} = b$$

Moreover,

$$L_s = \sigma \times T^4 \times 4\pi \times R^2$$

So, we can show that:

$$M_{BOL} - M_{BOL_{Sun}} = -2.5 \log \frac{L}{L_{Sun}}$$

Lastly, the Bolometric Correction is:

$$B.C. = M_{BOL} - M_V$$

Star	κ Velorum	β Tauri	Sirius A	Sun
Annual Parallax $p^*(10^{-3}$ arcsec)	6.05	24.89	379.2	-
Distance to Sun Δ^* (pc)	165.28	40.17	2.63	-
Interstellar Extinction in V Band A_V (mag)	0.20	0.08	Negligible	-
$10^{0.2\times A_V}$	1.10	1.04	1	-
Distance to Sun $\Delta(pc)$	150.26	38.63	2.63	4.85×10^{-6}
Annual Parallax p (10^{-3} arcsec)	6.655	25.880	379.200	286.05
Distance Modulus $\mu = m - M$	5.89	2.93	-2.89	-31.57
Visual apparent magnitude m^* (mag)	2.86	1.68	-1.47	-26.73
Visual apparent magnitude m (mag)	2.26	1.60	-1.47	-27.73
Visual absolute magnitude M_V (mag)	-3.63	-1.33	1.42	4.84
Color Index $(B-V)^*$ (mag)	-0.14	-0.06	+0.01	+0.65
Extinction E_{B-V} (mag)	0.0625	0.025	negligible	-
Color Index (B-V) (mag)	-0.2025	-0.085	+0.01	+0.65
Effective Temperature $T_{eff} = f(B-V)$ (K)	13616	11270	9936	5750
λ_m (nm)	220.00	266.19	301.93	521.73
Radius (Solar Radius, R_S)	9.100	4.600	1.711	1.000
Total Luminosity (Solar Luminosity,L_S	2350	277	23	1.00
Absolute Bolometric Magnitude M_{bol} (mag)	-3.79	-1.48	+1.21	4.64
Bolometric Correction BC for V band (mag)	-0.16	-0.15	-0.20	-0.20

4 Long Questions

1. (**30 points**) M15 is a globular cluster in the constellation Pegasus. The Hertzsprung–Russell diagram (apparent visual magnitude versus color index) of the cluster is shown in fig. 1. Considering that the mass (M)– luminosity (L) relation for main sequence stars is given by $\frac{L}{M^3}$ =constant, answer the following questions. In this problem, ignore the interstellar reddening and dust extinction effects.

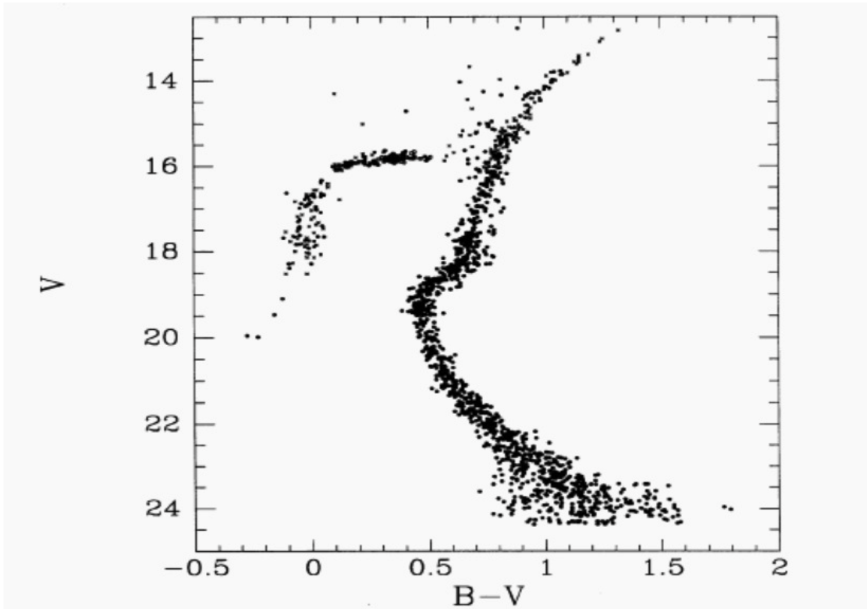


Figure 1: HR diagram for M15

(a) Given that all the stars are formed at the same time, estimate the age of the globular cluster. The color index of the sun ($(B-V)_\odot$) is 0.65 and its life time on the main sequence is 10 billion years.

(b) Estimate the distance of this globular cluster from the Earth. Give the answer in parsec. The absolute visual magnitude of the Sun is 4.83.

(c) Given that stars spend about 10% of their main-sequence life time in the post main sequence phase, find the mass of the most massive star in the post main sequence stage.

(d) The number of stars in the mass range of (M_1, M_2) can be written as:

$$N(M_1 \leqslant M \leqslant M_2) = A(M_1^{-1.35} - M_2^{-1.35}) \tag{1}$$

where A is a constant, M_1 and M_2 are in units of solar masses. Assuming that the number of stars in the post main sequence phase is 515, calculate the value of constant A in equation 1.

(e) M15 is one of the most densely packed globular clusters such that in a visual band ($\lambda \sim 5500\mathring{A}$) image of M15 taken by a telescope with diameter of 10 cm, the stars at the center of cluster cannot be resolved. Estimate the minimum number of stars in this cluster. The angular diameter of M15 is 12.3 arc minutes. Assume that the number density of stars is constant within the cluster.

(f) Use your answers from parts (d) and (e) to estimate the mass of the lowest possible mass star in this cluster. For this part, assume that the mass of the most massive star in the cluster is $20M_\odot$.

Solution:

(a) Measuring from the H-R diagram, a solar-like star with ($(B-V)_\odot$=0.65) has apparent visual magnitude $V_\odot = 21.4$. Also, this value is $V_t = 19.3$ for a star at the turning point of the main sequence. Thus, $V_\odot - V_t = 2.5\log(L_t/L_\odot)$ which gives $L_t/L_\odot \sim 7$. It allows us to estimate the life time of the star at the turning point which is the age of the cluster. Life time in the main sequence $\propto \frac{M}{L} \propto \frac{L^{1/3}}{L} \propto L^{-2/3} \Rightarrow$ Age of the cluster $= 10(L_t/L_\odot)^{-2/3}$ billion years=2.7 billion years. M15 is a very metal-poor globular cluster, so its actual age should be larger than this number (should be about the age of the Universe). The main reason for this discrepancy can be explained by a large uncertainty in B-V measurements.

(b) The distance module for the cluster is $V_{\odot} - M_v(\odot)$ where $M_v(\odot)$ is the absolute visual magnitude of the sun which is given, so distance module= 21.4-4.83=16.57. distance module= $5\log(d(pc)/10) \Rightarrow$ d$\sim$ 20 kpc. The actual distance of this globular cluster is $\sim$ 10 kpc. Our overestimation in the distance originates from the dust extinction effect that we ignored in this problem.

(c) The most massive star in the post main sequence phase spent $\frac{2.7}{110\%} = 2.45$ billion years on the main sequence. Life time in the main sequence $\propto \frac{M}{L} \propto \frac{M}{M^3} \propto M^{-2} \Rightarrow M_{max}$(post main sequence)= $(\frac{2.45}{10})^{-0.5} M_{\odot} = 2.02\ M_{\odot}$

(d) The mass of the star at the turning point is $(L_t/L_{\odot})^{(1/3)} = 7^{(1/3)} = 1.91 M_{\odot}$. Thus, $A = \frac{515}{1.91^{-1.35} - 2.02^{-1.35}} \sim 16945$. This number can be changes based on your normalization for masses (M_1 and M_2). Here we consider that they are given in solar masses.

(e) Consider a cylinder with a height of 2R (diameter of the cluster) and diameter of the telescope resolution at the center of the cluster. Within this cylinder, we should have at least two stars to observe the stars at the center of the cluster image unresolved. Telescope resolution: $D = 1.22 \frac{\lambda = 550nm}{D = 0.1m} = 1.38\ arcsec \Rightarrow \frac{2}{2R \times \pi \frac{D^2}{4}} = \frac{N_{min}}{\frac{4}{3}\pi R^3} \Rightarrow N_{min} = \frac{16}{3}(\frac{R}{D})^2 = \frac{16}{3}(\frac{12.3/2\ arcmin}{1.38\ arcsec})^2 \sim 3.8 \times 10^5$.

(f) Using the results from part d and e: $M_{min} = (\frac{N_{min}}{A} + M_{max}^{-1.35})^{\frac{1}{-1.35}} = (\frac{3.8 \times 10^5}{16987} + 20^{-1.35})^{\frac{1}{-1.35}} \sim 0.1\ M_{\odot}$.

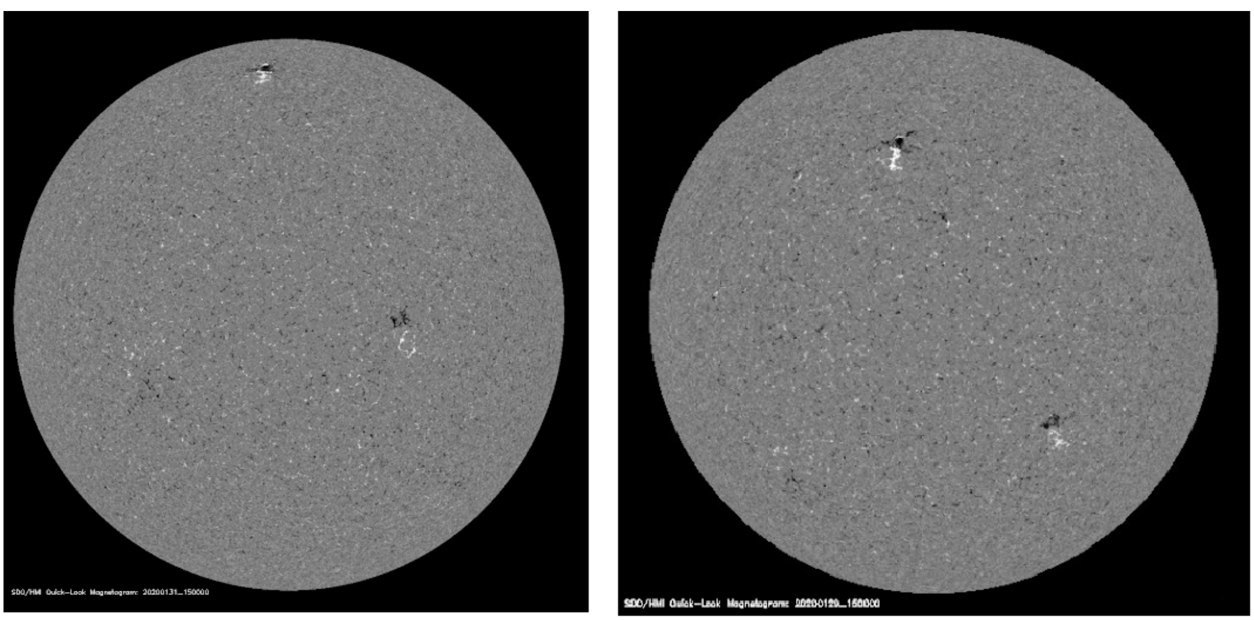
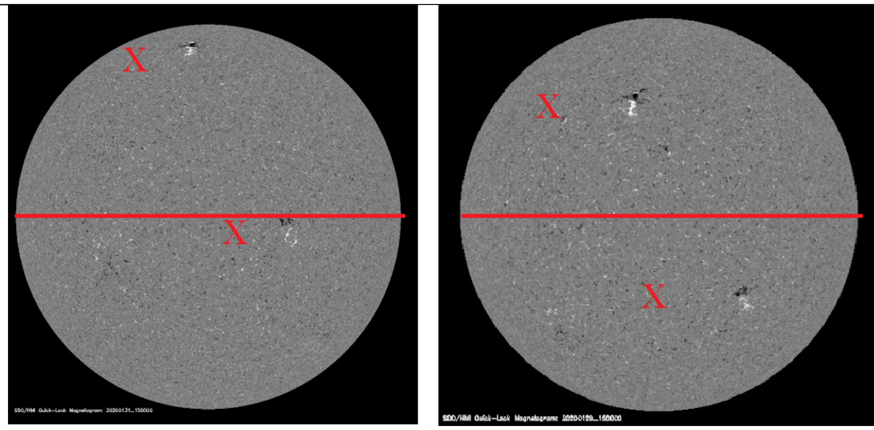
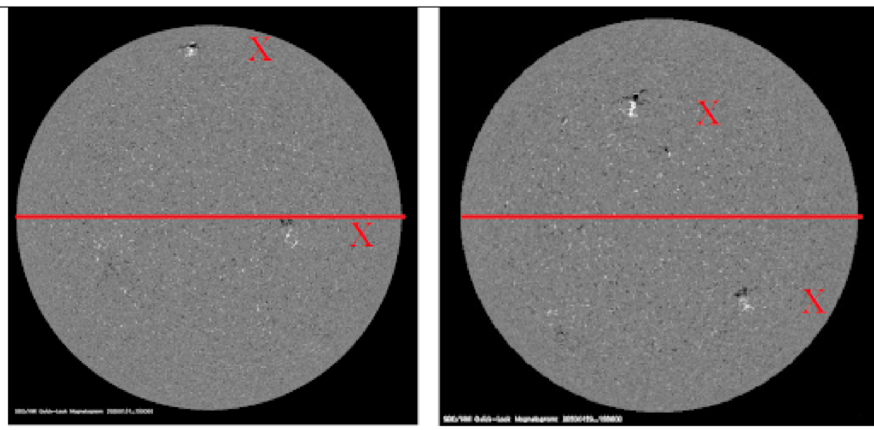
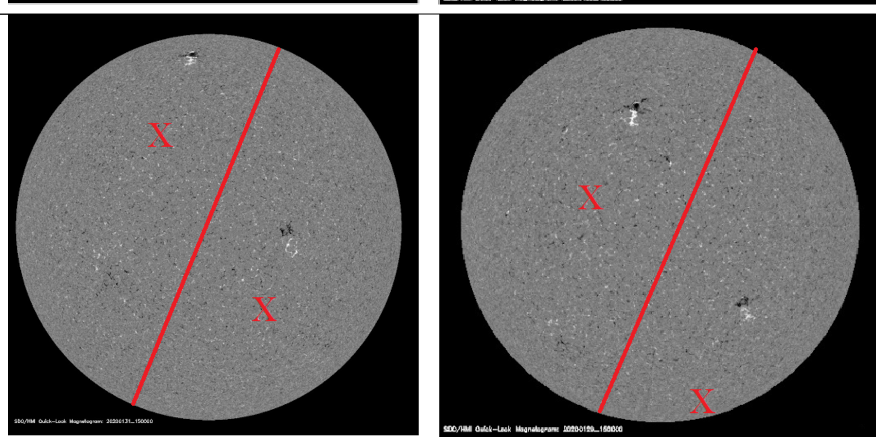
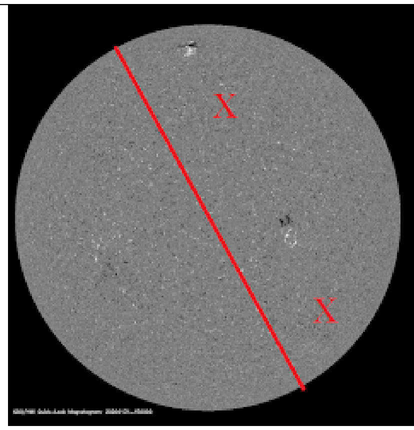
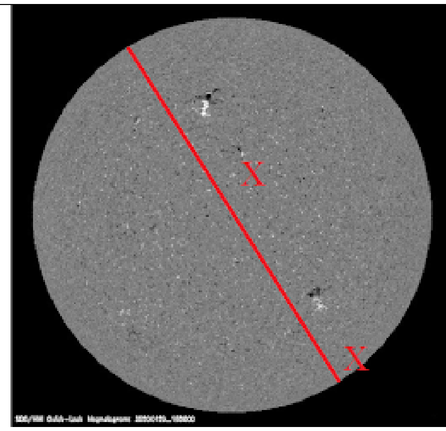
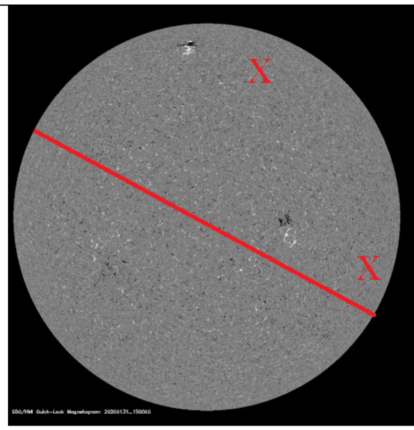
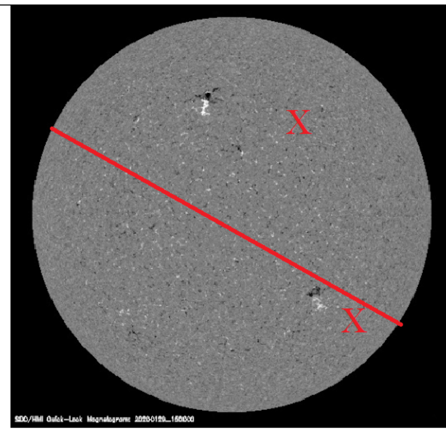


Figure 2: Solar magnetograms taken at the end of Jan 2020

2. (**25 points**) Fig. 2 shows two magnetograms of the Sun taken with the Heliseismic and Magnetic Imager (HMI) at the Solar Dynamics Observatory (SDO) towards the end of January 2020. The picture on the left was taken three days after the image on the right.

(a) Select the pair of images (numbered from I to V) in which the lines are drawn at the Sun's Equator and the Xs correspond to the position of each sunspot 4 days before the pictures were taken.

I	
II	
III	

IV		
V		

(b) Estimate the absolute value of the latitude of both Sunspots in Figure 2.

(c) Fig. 3 is a magnetogram of the Sun in normal activity. It is possible to notice that the sunspots have different orientations in different hemispheres. In one of the hemispheres, each spot has the white part on the left and the black one on the right, and vice-versa. However, this is not the case for the images presented in fig. 2. Suggest an explanation for the anomaly on the images in fig. 2.

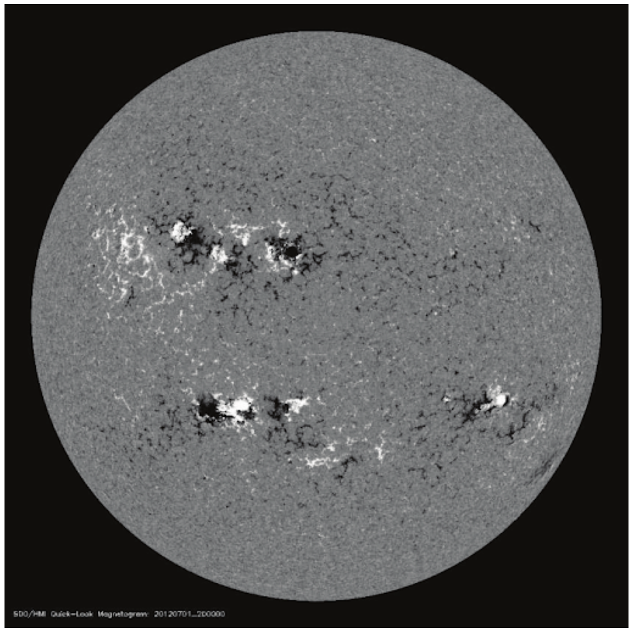


Figure 3: Sun in normal activity

(d) Assume for the sake of simplicity that a specific sunspot has a shape very similar to that of a spherical triangle. The sides of the triangle are equal to 0.176°, 0.0981°, and 0.201°. Calculate the value of the three internal angles in degrees.

For the following parts, assume that this sunspot is centered at 7.89° South and 51.74° East of the center of the Solar disk for an observer on Earth.

(e) For an observer on Earth, what is the ratio between the area of the Solar disk and the observed area of the sunspot? Note that the required ratio is between the areas observed by someone on Earth, not the ratio between the actual areas. The area of a spherical triangle is equal to $\pi R^2 E/180^\circ$, in which the spherical excess (E) in (deg) is equal to the sum of the internal angles minus 180° and R is the radius of the sphere on which the spherical triangle lies.

(f) If an observer on Earth uses a huge f/5 telescope with a focal length of 13 m to look at this sunspot, will it be possible to resolve it? Visible light is centered at 550 nm.

(g) The Sun generates its luminosity by converting Hydrogen into Helium in the proton-proton chain. In the most energetic branch of the chain, 4 protons fuse into a helium nucleus. Considering that only 10% of the solar mass can be converted into energy, calculate the time that the Sun spends in the Main Sequence.

Solution:

(a) The correct answer is III.

(b) The acceptable range will be 26^o to 29^o for full credit, and 25^o or 30^o to half of the punctuation.

It is useful to know that sunspots always come in pairs and have the same absolute value of latitude (one is positive and the other negative).

(c) Every 11 years (the period of the Schwabe Cycle), the Solar magnetosphere flips its orientation. Therefore, the orientation of the sunspots on each hemisphere is reversed. The pictures on item A were likely taken while the Sun was going through this transition, which explains their anomaly.

(d) A, B, and C are the internal angles of the spherical triangle.

Using the Law of the Cosines for spherical triangles: $\cos(0.176^o) = \cos(0.0981^o) \times \cos(0.201^o) + \sin(0.0981^o) \times \sin(0.201^o) \times \cos(A)$

$$\cos(A) = \frac{\cos(0.176^o) - \cos(0.0981^o) \times \cos(0.201^o)}{\sin(0.0981^o) \times \sin(0.201^o)}$$

$A = 61.11699^o$

Using the Law of the Sines for spherical triangles:

$\sin(61.11699^o)/\sin(0.176^o) = \sin(B)/\sin(0.201^o)$

$B = 89.67051^o$

Using the Law of the Sines for spherical triangles to calculate the last internal angle:

$\sin(61.11699^o)/\sin(0.176^o) = \sin(C)/\sin(0.0981^o)$

$C = 29.21265^o$

(e) First, it is necessary to calculate the spherical excess:

$E = 61.1^o + 89.7^o + 29.2^o - 180^o$

$E = (1.51 \times 10^{-4})^o$

Then, it is possible to calculate the area of the spherical triangle:

$A = \pi \times R_{Sun}^2 \times 1.51 \times 10^{-4}/180$

$A = 8.41 \times 10^{-7} \times \pi \times R_{Sun}^2$

Since the question asks about the observed area it is necessary to correct it with the inclination of the tangential plane.

Inclination of the tangential plane (Law of the Cosines with a right spherical triangle containing both coordinates):

$\cos(i) = \cos(7.89^o) \times \cos(51.74^o)$

$i = \arccos(0.613)$

$i = 52.2^o$

Now, it is possible to correct the observed area with the inclination:

$A_c = 8.41 \times 10^{-7} \times \pi \times R_{Sun}^2 \times \cos(52.2^o)$

$A_c = 5.16 \times 10^{-7} \times \pi \times R_{Sun}^2$

Considering that the area of the Solar disk is $\pi \times R_{Sun}^2$, it is possible to calculate the ratio between the areas observed:

$r = 5.16 \times 10^{-7} \times \pi \times R_{Sun}^2/(\pi \times R_{Sun}^2)$

$r = 5.16 \times 10^{-7}$

It is important to consider that small rounding errors will lead to huge differences in the spherical excess, so students won't be penalized if their answer is different than expected due to rounding errors. In fact, if students notice that the spherical excess is so small that the triangle is basically flat, they will also get full credit for this item. In this case, the calculations for the value of r would be the following (it is possible to use any pair of sides in the triangle and the angle in between them):

$A = 1/2 \times (2 \times \pi \times R_{Sun} \times 0.0981^o/360^o) \times (2 \times \pi \times R_{Sun} \times 0.201^o/360^o) \times \sin(61.1^o)$

$A = 8.37 \times 10^{-7} \times \pi \times R^2_{Sun}$

$A_c = 8.37 \times 10^{-7} \times \pi \times R^2_{Sun} \times \cos(52.2^o)$

$A_c = 5.13 \times 10^{-7} \times \pi \times R^2_{Sun}$

$r = 5.13 \times 10^{-7} \times \pi \times R^2_{Sun}/(\pi \times R^2_{Sun})$

$r = 5.13 \times 10^{-7}$

(f) First, it is necessary to calculate the diameter of the telescope:

$f/D = 5$

$13/D = 5$

$D = 2.6m$

Then, it is possible to calculate the angular resolution of the telescope:

$\theta = 1.22\lambda/D$

$\theta = 1.22 \times 5.5 \times 10^{-7}/2.6$

$\theta = 2.58 \times 10^{-7} rad$

It is possible to use the smallest side of the triangle to see if it will be resolved by the telescope. Length of the smallest size of the triangle:

$l = 2 \times \pi \times R_{Sun} \times 0.0981^o/360^o$

$l = 1.19 \times 10^6 m$

Correcting it for the inclination:

$l_c = 1.19 \times 10^6 m \times cos(52.2^o)$

$l_c = 7.31 \times 10^5 m$

Angle that corresponds to this length:

$\alpha = 7.31 \times 10^5/(1.496 \times 10^{11})$

$\alpha = 4.89 \times 10^{-6}\text{rad}$

Since $\alpha > \theta$, it is possible to resolve the sunspot.

(g) Variation in mass per reaction:

$$\Delta m = 4m_H - m_{He}$$
$$\Delta m = 4 \times 1.6725 \times 10^{-27} kg - 6.644 \times 10^{-27} kg$$
$$\Delta m = 4.600 \times 10^{-29} kg$$

Fraction of mass converted:

$$f = m/(4m_H)$$
$$f = 4.600 \times 10^{-29} kg/(4 \times 1.6725 \times 10^{-27} kg)$$
$$f = 6.876 \times 10^{-3}$$

Mass converted into energy while the Sun is in the Main Sequence:

$$\Delta M = M \times f \times 0.10$$
$$\Delta M = 1.989 \times 10^{30} \times 6.876 * 10^{-3} \times 0.10$$
$$\Delta M = 1.368 \times 10^{27} kg$$

Total energy converted:

$$E = \Delta M c^2$$
$$E = 1.368 \times 10^{27} kg \times (2.998 \times 10^{8} m/s)^2$$
$$E = 1.229 \times 10^{44} J$$

Time in the main sequence:

$$\Delta t = E/L$$
$$\Delta t = 1.229 * 10^{44} J/(3.826 \times 10^{26} W)$$
$$\Delta t = 3.213 * 10^{17} s$$
$$\Delta t \approx \text{ 10 billion years}$$