2019 National Astronomy Olympiad (NAO)- Solutions

1 Short Questions

1. (7 points) Assuming that the present density of baryonic matter is $\rho_{b0} = 4.17 \times 10^{-28}$ kg m^{-3}, what was the density of baryonic matter at the time of Big Bang nucelosynthesis (when T $\sim 10^{10}$ K)? Assume the present temperature, T_0 to be 2.7 K.

   Solution: Answer: $\rho_{BBN} = 21$ kg m^{-3}.
   $\rho_{b,BBN} = \rho_{b0} a^{-3}$, where a is the scale factor.
   We also know that $T_0 = aT(a)$, giving us:
   $\rho_{b,BBN} = \rho_{b0} (\frac{T_{BBN}}{T_0})^3$.
   Using $\rho_{b0} = 4.17 \times 10^{-28}$ kg m^{-3}, $T_{BBN} = 10^{10}$ K and $T_0 = 2.7$ K, we get $\rho_{BBN} = 21$ kg m^{-3}.

2. **Note: constants required are the semimajor axis of the moon's orbit (384399 km), the semimajor axis of the earth's orbit (1.4960×10^8 km), the radius of the sun (695700 km), the radius of the earth (6371 km), the radius of the moon (1737 km), the mass of the earth (5.972×10^{24} kg), the mass of the sun (1.989×10^{30} kg), and G.**

   (7 points) On the night of January 21st, 2019, there was a total lunar eclipse during a supermoon. At the time, the moon was close to perigee, at a distance of 351837 km from the earth, which was 1.4721×10^8 km from the sun. The gamma (γ) of a lunar eclipse refers to the closest distance between the center of the moon and the center of the shadow, expressed as a fraction of the earth's radius. For this eclipse, $\gamma = 0.3684$. Given this information, find the closest estimate for the duration of totality of the eclipse.

   Solution: First, calculate the radius of the shadow cast by earth. The size of the umbra at a distance of 351837 km is $r_{umbra} = 6371 - \frac{695700-6371}{1.4721\times10^8} \times 351837 = 4723$ km. However, the distance travelled by the moon through the umbra is far less than twice the umbral radius. Firstly, the moon does not pass through the center of the shadow; secondly, totality occurs when the moon is completely within the shadow, not when the center of the moon is in the shadow. So, the distance travelled by the moon during totality is $d = 2 \times \sqrt{(r_{umbra} - 1737)^2 - (0.3684 \times 6371)^2} = 3693$ km. The velocity of the moon relative to the earth can be calculated using the vis-viva equation: $v_{moon} = \sqrt{G \times 5.972 \times 10^{24} \times \left(\frac{2}{351837000} - \frac{1}{384399000}\right)} = 1.108$ km/s. However, the earth is also moving around the sun, so the umbra is also moving relative to the earth. The velocity of the earth is $v_{earth} = \sqrt{G \times 1.989 \times 10^{30} \times \left(\frac{2}{1.4721\times10^{11}} - \frac{1}{1.4960\times10^{11}}\right)} = 30.259$ km/s. Therefore, the velocity of the umbra is $v_{umbra} = \frac{v_{earth}}{1.4721\times10^8} \times 351837 = 0.072$ km/s. The earth and the moon orbit in the same direction, so the velocity of the moon relative to the umbra is $v = v_{moon} - v_{umbra} = 1.036$ km/s. Finally, the duration of totality is $t_{totality} = \frac{d}{v} = 59.426$ minutes.

3. (7 points) You are in the northern hemisphere and are observing rise of star A with declination $\delta = -8^o$, and at the same time a star B with declination $\delta = +16^o$ is setting. What will happen first: next setting of the star A or rising of the star B?

Solution: Star B will rise first. At the same time as the star A rise, a point on the opposite side of the sky with declination $+8^o$ is setting. Star B sets at the same time as that point, but having a higher declination, and being in the north hemisphere, will spend less time under the horizon.

4. (7 points) Consider a star with mass M and radius R. The star's density varies as a function of radius r according to the equation $\rho(r) = \rho_{center}(1 - \sqrt{r/R})$, where ρ_{center} is the density at the center of the star. Derive an expression for dP/dr in terms of G, M, R, and r, where P is the pressure at a given radius r

Solution: The mass inside a given radius r is $m(r) = \int_0^r 4\pi r^2 \rho(r) dr$. Thus, we have

$$m(r) = \frac{4}{21}\pi\rho_{center} r^3 (7 - 6\sqrt{\frac{r}{R}})$$

Note that $M = m(R)$, so we can solve for $\rho_{center} = \frac{21M}{4\pi R^3}$. Hydrostatic equilibrium requires that $dP/dr = -\rho(r)\frac{Gm(r)}{r^2}$. Thus, substituting, we have

$$dP/dr = -\frac{21GM^2 r}{4\pi R^6}(7 - 13\sqrt{\frac{r}{R}} + 6\frac{r}{R})$$

2 Medium Questions

5. (15 points) An alien spaceship from the planet Kepler 62f is in search of a rocky planet for a remote base. They're attracted to Earth because of a fortunate coincidence: its axis of rotation points directly at their home planet. That means they can have uninterrupted communication with home by planting fixed transmitters on The North Pole. But first, they need to find out if Earth's axis will always point in the same direction or if it undergoes precession. They can't know without years of observation, so they hope that we, its now-extinct intelligence, have left behind the answer.

   While orbiting Earth, they see a few remarkable structures, including the Hoover Dam in Nevada. Zooming in on the dam, a colorful plaza with peculiar markings on its floor catches their attention. Descending on the plaza, they realize the markings are a map of the sky when the dam was built, left to indicate the date to posterity. Figure 1 is an overhead architectural map of this plaza. The center-point depicts the north ecliptic pole, and the large circle represents the path of the Earth's axis throughout its counter-clockwise procession. As they interpret the map, they're dismayed to realize that their star has not been and will not be Earth's north star for very long.

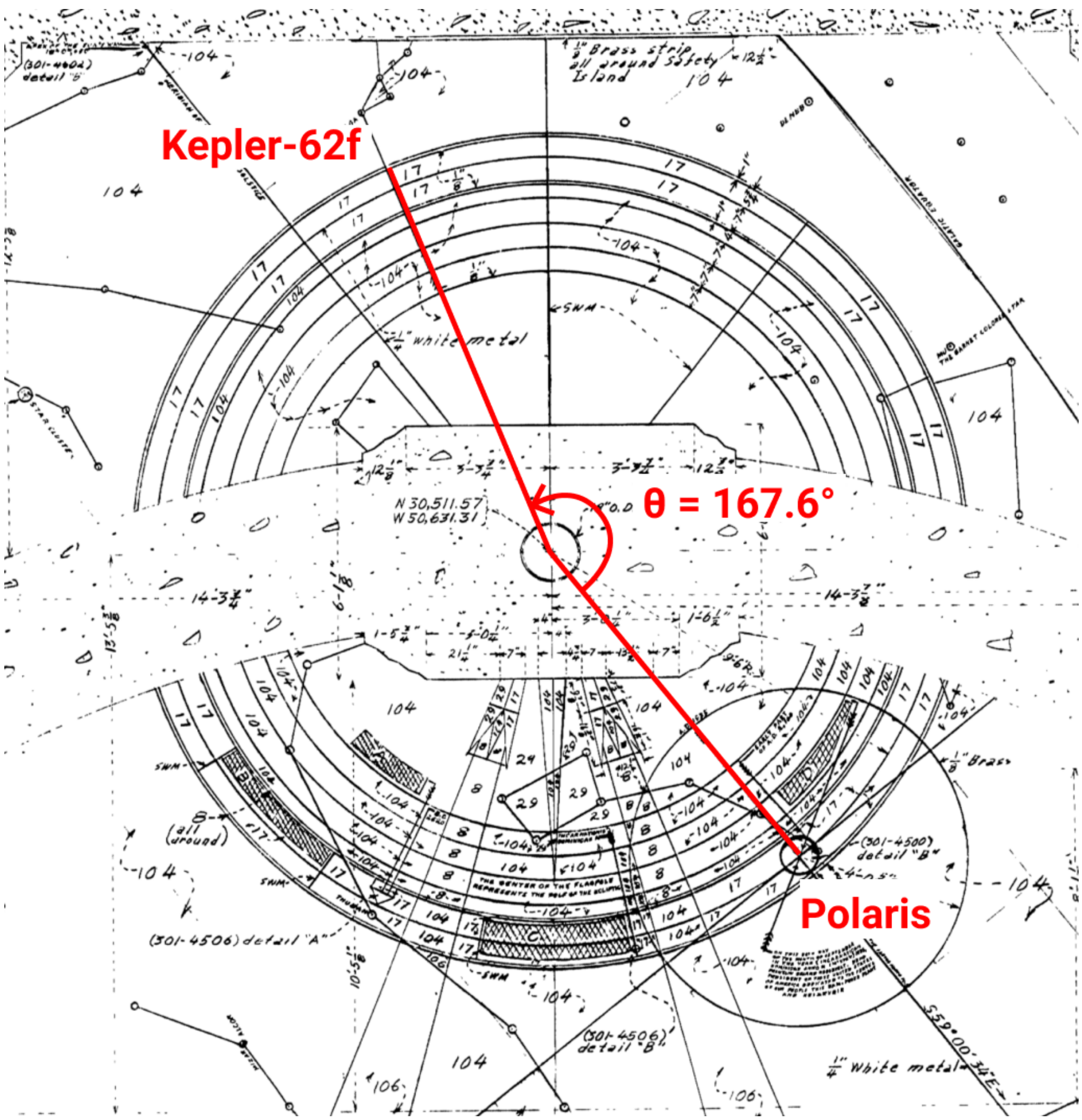

Figure 1: Overhead architectural plan of the Hoover Dam plaza depicting Polaris as north star

For the purpose of this question, assume that the Earth's axial tilt is a constant $i = 23.5^\circ$ and its axis precesses at a constant rate.

a) Using the values on the map, and knowing that the aliens used carbon-aging to determine that the dam is 12,000 years old, find all possible values for the period of Earth's axial precession.

b) Using the most optimistic answer (longest period) from part (a), calculate how many arcseconds the Earth's axis precesses each day. Use the period you calculate here in the next two sections.

c) If they hadn't been lucky enough to come across the star map and decided to build a radio interferometer to observe the movement of the celestial pole over the course of 30 days instead, how many kilometers would the baseline of their telescope array have to be, assuming it operated at a 20cm wavelength?

d) As a last resort, to keep Earth's axis fixed, the aliens decide to counter the forces that cause the Earth's precession by building giant nuclear thrusters on the Earth's surface. Assume Earth's pre-

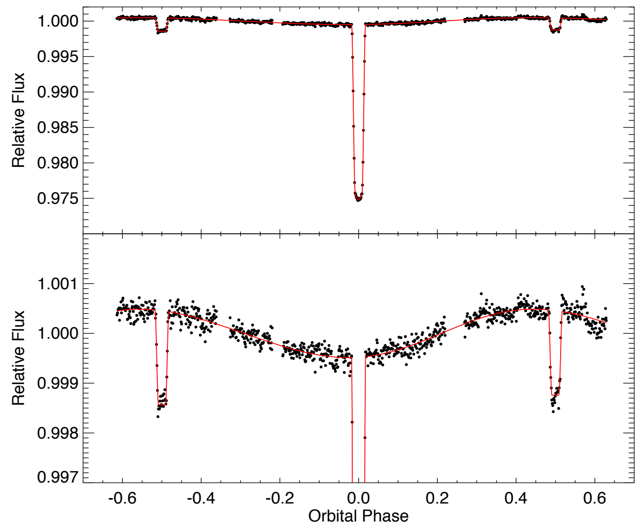

Figure 2: Full-phase light curve of HD 189733b. From Knutson et al. (2012).

cession is caused by external forces alone and calculate the average force (in kN) that a strategically positioned thruster would have to exert to counter them.

Solution:

(a) We know that $P_p = \frac{360^\circ}{\omega} = \frac{360t}{\theta_{total}}$ where $t = 12,000\, years$ and $\theta_{total} = 167.6^\circ$ or $360^\circ + 167.6^\circ$ or ... which gives: $P_p = 25775.6\, years$ or $P_p = 8188\, years$ or $P_p = 4867\, years$ etc.

(b) The axis is in the direction of the angular momentum vector, which traverses a path on a small circle of the celestial sphere, traveling a total of $2\pi \sin i$ radians in each period. So the axis moves $\frac{2\pi \sin i}{25775.6 \times 365} \approx 0.05''$ per day.

(c) $R = \frac{\lambda}{D}$ so $D = \frac{0.2 \times 3600 \times 180}{0.05 \times 30 \times \pi} \approx 25$ km

(d) $\vec{\tau} = \vec{r} \times \vec{F}$ so $\left\|\vec{F}\right\| = \frac{\|\vec{\tau}\|}{R} = \frac{\left\|\frac{d\vec{L}}{dt}\right\|}{R} = \frac{\left\|\vec{L}\right\| \frac{d\theta}{dt}}{R} = \frac{I\|\vec{\omega}\| \frac{d\theta}{dt}}{R} = \frac{8.01 \times 10^{37} \times \frac{2\pi}{24 \times 3600} \times \frac{0.055''}{3600} \times \frac{1}{24 \times 3600} \frac{\pi}{180}}{6.37 \times 10^6} \approx$ 2.82×10^{12} kN

6. (15 points) Figure 2 shows a full-phase light curve ("phase curve") of the exoplanet HD 189733b taken by the Spitzer space telescope. Use this figure to answer the following questions. The star HD 189733 has an effective temperature of 4785 K and a radius of 0.805 Solar radii.

a) Use the depth of the planet's transit to estimate the radius of HD 189733b, in Jupiter radii.

b) Use the depth of the eclipse of the planet by the host star to estimate the ratio of the flux of the planet HD 189733b to that of the host star HD 189733.

c) HD 189733b is so close-in to its host star that it is expected to be tidally locked. Use the phase curve to estimate the ratio of the dayside flux emitted by the planet to the nightside flux emitted by the planet.

d) This phase curve also noticeably has a phase curve offset, that is, the maximum in planet and star flux does not occur exactly at secondary eclipse. What process that occurs in a planetary atmosphere could cause such a phase curve offset?

Solution: a) The transit depth (reading off top plot) is ≈ 0.025. Know that transit depth $T_d \propto (R_p/R_s)^2 \rightarrow R_p = R_s\sqrt{T_d} = 0.805 * 6.95 \times 10^8$ m/Solar radius$\sqrt{0.025}/6.91 \times 10^7$ m/Jupiter radius = 1.28 Jupiter radii.
b) Eclipse depth is ≈ 0.002 in normalized flux. The bottom of the eclipse occurs at a normalized flux of 0.9985. $F_p/F_s \approx 0.002/0.9985 \approx 2 \times 10^{-3}$.
c) Max normalized flux $\approx 1.0005 \rightarrow$ max normalized planet flux $\approx 1.0005 - 0.9985 = 2 \times 10^{-3}$. Min normalized flux $\approx 0.9996 \rightarrow$ min normalized planet flux $\approx 0.9996 - 0.9985 = 1.1 \times 10^{-3}$. Ratio of max to min planet flux= $2 \times 10^{-3}/1.1 \times 10^{-3} = 1.8$.
d) Atmospheric circulation that is driven by the large dayside-to-nightside pressure gradient due to tidal locking.

7. (15 points)

a) **Mass-Radius Relation** Stellar physics often involves guessing the equation of state for stars, which is typically a relation between the pressure P and the density ρ. A family of such guesses are known as polytopes and go as follows-

$$P = K\ \rho^\gamma \tag{1}$$

where K is a constant and the exponent γ is fixed to match a certain pressure and core temperature of a star. Given this, show that one can obtain a crude power-law scaling between the mass M of a polytopic star and its radius R of the form $M \propto R^\alpha$. Find the exponent α for polytopic stars (justify all steps in your argument). Also, indicate the exponent γ for which the mass is independent of the radius R. Bonus: Why is this case interesting?

b) **Black Holes as Blackbodies** The mass radius relation for ideal non-rotating, uncharged black holes is known from relativity to be

$$R = \frac{2GM}{c^2} \tag{2}$$

Moreover, Stephen Hawking showed that a black hole behaves like a blackbody, where its temperature (known as the Hawking temperature) is given by

$$T = \frac{\hbar c^3}{8\pi k_B GM} \tag{3}$$

Given this information, show that the lifetime of a black hole (justify this phrase!) t^* scales with its mass M as

$$t^* \propto M^\beta \tag{4}$$

where you should find the exponent β

c) **Minimal Black Holes** Using the information of the previous part, and Wien's displacement law, estimate the smallest possible mass of a black hole. State any possible flaws with this estimate.

Solution:

(a) To maintain hydrostatic equilibrium in a star, we need the pressure to balance the gravitational pull. This translates to the radial force balance equation

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \tag{5}$$

(Consider the forces on a thin shell of gas in the stellar interior to obtain this). Now to obtain a crude estimate, we do the following

$$\frac{dP}{dr} \approx \frac{\Delta P}{\Delta R} = \frac{-P}{R} \approx -\frac{GM\rho}{R^2} \tag{6}$$

where P is the pressure in the core of the star and R is the radius of the star. Thus, we obtain the scaling

$$P \sim \frac{M\rho}{R} \sim \rho^2 R^2 \tag{7}$$

Using (1), we obtain then

$$\rho \sim R^{\frac{2}{\gamma-2}} \tag{8}$$

Thus, we finally arrive at

$$M \sim R^{\frac{3\gamma-4}{\gamma-2}} \tag{9}$$

$$\implies \alpha = \frac{3\gamma-4}{\gamma-2} \tag{10}$$

Thus, if $\gamma = 4/3$, the mass of the polytopic star is independent of the radius R. This is an interesting case, since this has connections to the equation of state for a relativistic degenerate fermion gas, which is used to model white dwarves. This independence of the mass and radius can be thought of as a crude way to understand the Chandrashekar limit.

(b) We have that

$$R = \frac{2GM}{c^2} \tag{11}$$

$$T = \frac{\hbar c^3}{8\pi k_B GM} \tag{12}$$

Using the assumption that a black hole is a blackbody, we apply Stefan's law to find the power of radiation emitted by a black hole

$$\frac{dE}{dt} = 4\pi R^2 \sigma T^4 = \frac{\hbar^4 c^8 \sigma}{256\pi^3 G^2 k_B^4} \frac{1}{M^2} \tag{13}$$

Since this energy has to arise from somewhere, we take this to emerge from the mass-energy of a black hole, giving us

$$\frac{dE}{dt} = -c^2 \frac{dM}{dt} \propto \frac{1}{M^2} \tag{14}$$

Thus, we get

$$M^2 dM = -K dt \tag{15}$$

where K is a constant of proportionality involving the fundamental constants and σ. Integrating the above equation, we see that the blackhole loses mass and its lifetime scales with its mass as follows

$$t^* \propto M^3 \tag{16}$$

$$\implies \beta = 3 \tag{17}$$

(c) Under the assumption that a black hole is a blackbody, we are justified in thinking that its spectrum has the characteristic Planckian spectrum with the Wien's law peak for the emitted photon being given by:

$$\lambda = \frac{b}{T}, \qquad b \ = \ 2.9 \times 10^{-3}\text{m K} \tag{18}$$

If we estimate that one photon (or any $\mathcal{O}(1)$ photons, as required to preserve momentum) having this wavelength carries away the entire mass energy of the black hole, we estimate that

$$Mc^2 \approx \frac{2\pi\hbar c}{\lambda} \tag{19}$$

$$\implies M^2 \approx \left(\frac{\hbar c}{2\sqrt{Gk_B b}}\right)^2 \tag{20}$$

$$M \approx \left(\frac{\hbar c}{2\sqrt{Gk_B b}}\right) \approx 9.68 \times 10^{-9} \text{ kg} \tag{21}$$

$$M \approx 4.84 \times 10^{-39} M_\odot \tag{22}$$

There are several problems with this method of estimation. Firstly radiation is a fully quantum process, so our assumption of radiating a single photon is not really correct. Moreover, we use Wien's law in its wavelength form, to obtain energy but we really need it in the frequency form to actually obtain the peak of the energy of the photon spectrum. This is an order of magnitude estimation, so any reasonably justified argument should be given full credit.

8. (15 points) In a rather weird universe, the gravitational constant G varies as a function of the scale factor $a(t)$.

$$G = G_0 f(a) \tag{23}$$

Consider the model $f(a) = e^{b(a-1)}$ where $b = 2.09$.

a) Assuming that the universe is flat, dark energy is absent, and the only constituent is matter, estimate the present age of this weird universe according to this model. Assume that the Friedmann equation:

$$H(a)^2 = H_0^2 \ (\Omega_m + \Omega_r + \Omega_k + \Omega_\Lambda) \tag{24}$$

still holds in this setting.

b) What is the behaviour of the age of the universe t as the scale factor $a(t) \to \infty$?

Note that all parameters with subscript $_0$ indicate their present value. Take the value of Hubble's constant as $H_0 = 67.8 \text{ kms}^{-1}\text{Mpc}^{-1}$

Hint: You might need the following integrals

$$\int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4} \qquad\qquad \int_0^1 x^2 e^{-x^2} dx \approx 0.189471 \tag{25}$$

Solution:

(a) We have from the Friedmann equations that in such a matter-only universe,

$$H(a)^2 = H_0^2 \ \Omega_m \tag{26}$$

where

$$\Omega_m = \frac{\rho_m}{\rho_c} \tag{27}$$

$$\rho_c = \frac{3H_0^2}{8\pi G} \tag{28}$$

Using eqs. (26) to (28) and the relation $\rho_m = \rho_{m_0} a^{-3}$ we infer that

$$\Omega_m = \Omega_{m_0} f(a) a^{-3} = f(a) a^{-3} \tag{29}$$

where for a matter-only universe $\Omega_{m_0} = 1$. Using now the relation $H(t) = \frac{\dot{a}}{a}$, we obtain that

$$t = \int_0^t dt' = \int_0^{a(t)} \frac{dt'}{da'} da' = \int_0^{a(t)} \frac{a'}{\dot{a}' a'} da' = \int_0^{a(t)} \frac{da'}{H(a')a'} \tag{30}$$

$$t = \frac{1}{H_0} \int_0^{a(t)} \frac{da'}{a' \sqrt{f(a') a'^{(-3)}}} \tag{31}$$

Using $f(a) = e^{b(a-1)}$, and the substitution $x = \sqrt{a' b/2}$ simplifies this to

$$t = \frac{4\sqrt{2} e^{b/2}}{b^{3/2} H_0} \int_0^{\sqrt{a(t)b/2}} dx \; x^2 e^{-x^2} \tag{32}$$

Using the given value of H_0 and the integrals in the hint, we arrive at the present age ($a(t) = 1$) of such an obscure universe to be

$$t \approx 15.0 \text{ billion years} \tag{33}$$

which is surprisingly close to the current estimate of our universe's age.

(b) What is even more surprising in this model, is that if one were to set $a(t) = \infty$ one would find that the integral above is finite, as shown in the hint.

$$t = \frac{\sqrt{2\pi} e^{b/2} b^{-3/2}}{H_0} \tag{34}$$

The value $b = 2.09$ gives us $t \approx 34.1$ billion years. Thus, the scale factor blows up in a finite amount of time, which is a weird feature of this model given this simplistic treatment.

(The above model appeared in the paper *Varying-G Cosmology with Type Ia Supernovae* by Rutger Dungan and Harrison B. Prosper (arXiv:0909.5416v2) where the authors showed that the type Ia supernovae data alone wasn't sufficient to conclude the existence of dark energy. The value $b = 2.09$ was obtained by fitting the experimental luminosity distance to the theoretical luminosity distance obtained in the model.)

9. (15 points)

   a) Find the shortest distance from Boston ($42.3601^\circ\ N,\ 71.0589^\circ\ W$) to Beijing ($39.9042^\circ\ N,\ 116.4074^\circ\ E$)traveling along the Earth's surface. Assume that the Earth is a uniform sphere of radius 6371 km.

   b) What fraction of the path lies within the Arctic circle (north of $66.5608^\circ\ N$)?

Solution:

(a) We choose a coordinate system with Boston lying on the x-axis and the axis of the Earth's rotation corresponding to the z-axis. The two cities are separated by $360 - (116.4074 + 71.0589) = 172.5337^\circ$ of longitude. Now, in spherical coordinates, we get that (θ, ϕ) is given by $(90 - 42.3601, 0) = (47.6399^\circ, 0^\circ)$ for Boston and by $(90 - 39.9042, 172.5337) = (50.0958^\circ, 172.5337^\circ)$ for Beijing. Converting to cartesian coordinates on the unit sphere (we multiply by the radius later to get the actual distance), we get that $(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ are given by $\vec{r_1} = (0.738925, 0, 0.673788)$ and $\vec{r_2} = (-0.760614, 0.0996817, 0.641506)$. The cosine of the angle θ between them is given by $\cos(\theta) = \vec{r_1} \cdot \vec{r_2} = -0.129798 \implies \theta = 97.458^\circ = 1.701$. Noting that the distance is given by $D = R_{Earth}\theta$ when θ is measured in radians, we get $D = 10840$ km.

(b) The equation of the plane passing through the origin and the two cities is given by $(\vec{r_1} \times \vec{r_2}) \cdot \vec{x} = 0$. Now, solving the equations $-0.0671643x - 0.986517y + 0.0736573z = 0$. Setting $z = \sin(66.5608^\circ) = 0.917483$, we get $0.0671643x + 0.986517y = 0.0675793$ and $x^2 + y^2 = 0.158225$. Writing y in terms of x, we get $x^2 + \frac{(0.0675793 - 0.0671643x)^2}{0.986517^2} - 0.158225 = 0$. Thus, the product of x values that satisfy this equation is -0.152824. Doing the same with the variables switched, we get $y^2 + \frac{(0.0675793 - 0.986517y)^2}{0.0671643^2} - 0.158225 = 0$. The product of y-values is thus given by 0.00394098. Thus, the dot product of the points of contact with the Arctic circles is given by 0.692892. This gives an angle of 46.141°. Therefore, the final ratio is $\frac{46.14}{97.458} = 0.47$.

3 Long Questions

10. (25 points) In this problem, we will try to understand the relationship between magnetic moments and angular momenta, first for charged particles and how this can be extended to planetary objects.

(a) (5 points) Consider a charge e and mass m moving in circular orbit of radius r with constant speed v. Write down the angular momentum L of the charge and magnetic moment μ of the effective current loop. Recall that the magnetic moment of a current loop with current I and radius r is given as $\mu = IA$ where A is the area of the loop.

(b) (3 points) Use the above results to find a relationship between the magnetic moment μ and angular momentum L in terms of intrinsic properties of the particle (charge,mass).

(c) (2 points) The relationship from part (b) can be expressed as $\mu = \gamma L$. γ is usually referred to as the *classical* gyromagnetic ratio of a particle. Evaluate the classical gyromagnetic ratio for an electron and for a neutron in SI units.

(d) (7 points) For extended objects such as planets, the magnetic dipole moment is not directly accessible whereas the surface magnetic field can be measured. Assuming a magnetic dipole of magnetic moment μ located at the center of a sphere of radius r, write down the expression for the surface magnetic field B_{surf} and the surface magnetic moment defined as $\mathcal{M}_{surf} = B_{surf}r^3$. You may use the value of the angular dependence at the magnetic equator for the following parts.

(e) (3 points) Assuming a gyromagnetic relationship exists between magnetic moment μ and angular momentum L of an extended object, write down the relationship between the surface magnetic moment $\mathcal{M}_{surf}$ and angular momentum L as $\mathcal{M}_{surf} = \kappa L$. You will observe that κ depends only on fundamental constants and intrinsic properties of the extended object.

(f) (3 points) The surface magnetic moments for Mercury and Sun are 5×10^{12} T m^3 and 3×10^{23} T m^3 respectively. Assuming the bodies are perfect spheres, evaluate the constant κ for Mercury and the Sun. Comment on values obtained and if they fit into the model developed in parts (c) and (d).

(g) (5 points) The surface magnetic moments $\mathcal{M}_{surf}$ and angular momenta L of various solar system bodies are plotted in the figure 3. Justify that the data implies $\mathcal{M}_{surf} \sim L^{\alpha}$ and calculate the constant α. What is the expected value of α from the model developed in parts (c) and (d)?

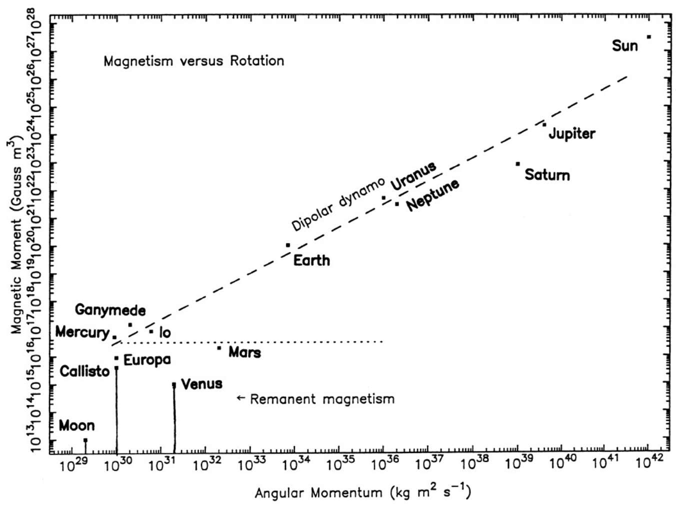

Figure 3: Surface magnetic moment vs angular momentum for solar system objects. Figure taken from Vallée, Fundamentals of Cosmic Physics, Vol. 19, pp 319-422, 1998.

(h) (2 points) Certain bodies such as Venus, Mars and the Moon are remarkably separated from the trend observed for other bodies. What can you say about magnetism in these bodies when compared to the others?

Solution:

(a) $L = mvr$ and $\mu = IA = \frac{ev}{2\pi r}(\pi r^2) = \frac{evr}{2}$

(b) $\mu = \frac{e}{2m}L$

(c) $\gamma_e = 8.794 \times 10^{10}$ C/kg and $\gamma = 0$ for a neutron, since the charge is zero.

(d) Use the magnetic field of a dipole μ as $B_{surf} = \frac{\mu_0}{4\pi}\frac{\mu}{r^3}(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta})$. The r^3 dependence is crucial here which follows for any dipole type distribution; partial credit if that is shown. $\mathcal{M}_{surf} = \frac{\mu_0}{4\pi}\mu(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta})$

(e) Neglecting the angular dependence, $\mathcal{M}_{surf} = \left(\frac{\mu_0}{4\pi}\gamma\right)L = \kappa L$

(f) For Mercury, angular momentum $L = 10^{30}$ kg m^2/s, so $\kappa = 5 \times 10^{-18}$ T s^2/(kg m). For Sun,angular momentum $L = 10^{42}$ kg m^2/s, so $\kappa = 3 \times 10^{-19}$ T s^2/(kg m). These two values are approximately close to within an order of magnitude. We therefore expect that the mechanisms for magnetic dipole generation are similar in Mercury and the Sun.

(g) Observe that the plot is on log-log scale. A linear relationship on this plot implies a power-law form. From the slope of linear trend-line on the graph, $\alpha \approx 0.8$. Full credit for realizing $\alpha < 1.0$; if $\alpha = 1.0$ is written, partial credit.

(h) Venus, Mars and Moon do not have a active dynamo and hence the magnetism present with them is of a different nature from the other bodies. These bodies had a magnetic dynamo in the past which has since died down. The magnetism is only remnant from this past dynamo.

11. (25 points) Cygnus X-1/HDE 226868 is a binary system consisting of a black hole Cygnus X-1 and blue supergiant HDE 226868. The mass of HDE 226868 is $30M_\odot$ and the period of the binary system is 5.6 days. Radial velocity data reveals that the orbital velocity of HDE 226868 is 116.68 km/s at apoapse and 123.03 km/s at periapse.

    (a) (5 points) Determine the eccentricity of the orbit of HDE 226868.

    (b) (5 points) Determine the length of the semimajor axis of the orbit of HDE 226868.

    (c) (5 points) Determine the mass of Cygnus X-1, to at least 3 significant figures.

    The peak blackbody temperature of an accretion disk occurs at a distance of r_{peak} and a temperature of T_{peak}. One can determine the peak blackbody temperature by assuming that it corresponds to the peak in the x-ray spectrum. Due to relativistic effects, the actual peak blackbody temperature T_{peak} is related to the peak color temperature T_{color} derived from observed spectral data by $T_{color} = f_{GR} f_{col} T_{peak}$, where $f_{GR} \approx 0.510$ and $f_{col} \approx 1.7$. Three x-ray spectra of Cygnus X-1 are shown in Figure 4.

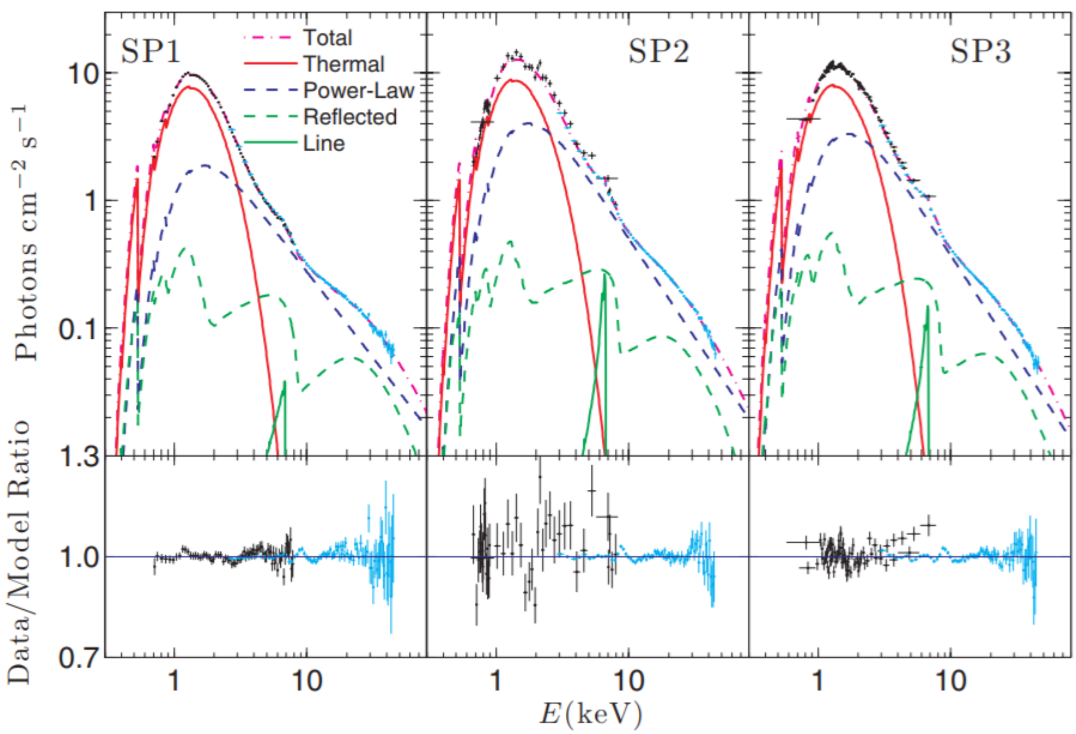

Figure 4: Three x-ray spectra from Cygnus X-1. From Gou et al. (2011).

    (d) (4 points) Using spectrum SP2, determine the peak blackbody temperature T_{peak} of the accretion disk around Cygnus X-1.

    The total luminosity of the blackbody component of the accretion disk can be estimated by $L_{disk} \approx 4\pi\sigma r_{peak}^2 T_{peak}^4$ (Makishima et al. 1986). The radius r_{last} of the innermost edge of the accretion disk is related to the radius r_{peak} of the peak blackbody temperature by $r_{peak} = \eta r_{last}$, where $\eta \approx 0.63$. In 1996, the blackbody luminosity of the accretion disk around Cygnus X-1 was estimated to be 2.2×10^{37} ergs/s.

    (e) (4 points) Determine the radius r_{last} of the innermost edge of the accretion disk around Cygnus X-1.

Assume that the innermost edge of the accretion disk is located at the innermost stable circular orbit (ISCO), whose radius r_{isco} is a function of the spin of the black hole. The relationship between r_{isco} and a_*, the spin parameter of the black hole, can be estimated by:

$$r_{isco} = \frac{GM}{c^2}\left(\sqrt{8.354 \cdot [(2-a_*)^2 - 1]} + 1\right)$$

(f) (2 points) Determine the spin parameter a_* of Cygnus X-1.

Solution:

(a) In the absence of external forces, the total angular momentum of the binary system is conserved. The angular momentum $l = mr^2\omega = mrv$ of each component must also be conserved. Therefore, $m_1 r_{1,a} v_{1,a} = m_1 r_{1,p} v_{1,p} \implies \frac{v_{1,p}}{v_{1,a}} = \frac{r_{1,a}}{r_{1,p}} = \frac{a(1+\epsilon)}{a(1-\epsilon)} \implies \epsilon = \frac{v_{1,p}-v_{1,a}}{v_{1,p}+v_{1,a}} = \frac{123.03-116.68}{123.03+116.68} = 0.0265.$

(b) The area swept by the vector $\mathbf{r_1}$ (from the center of mass to the primary) over time interval Δt is approximately a right triangle of base r_1 and height $v_1\Delta t$, and therefore area $\frac{1}{2}r_1 v_1 \Delta t$. Over a single orbit, the area swept is $A = \frac{1}{2}r_1 v_1 T = \pi a_1^2\sqrt{1-\epsilon^2}$, for (r_1, v_1) at any point in the orbit (by Kepler's second law, the rate of area swept is constant). So, $\pi a_1^2 \sqrt{1-\epsilon^2} = \frac{1}{2}a_1(1+\epsilon)v_{1,a}T \implies a_1 = \sqrt{\frac{1+\epsilon}{1-\epsilon}} \cdot \frac{v_{1,a}T}{2\pi} = \sqrt{\frac{1+0.0265}{1-0.0265}} \cdot \frac{116680 \cdot 5.6 \cdot 24 \cdot 3600}{2\pi} = 9.23 \times 10^9$ m.

(c) Kepler's third law states that $(a_1 + a_2)^3 = \frac{G(m_1+m_2)}{4\pi^2}T^2$, so $a_2 = \left[\frac{G(m_1+m_2)}{4\pi^2}T^2\right]^{1/3} - a_1$. Additionally, the definition of center of mass is that $a_1 m_1 = a_2 m_2$, so $a_2 = \frac{a_1 m_1}{m_2}$. Putting these two equations together yields $\left[\frac{G(m_1+m_2)}{4\pi^2}T^2\right]^{1/3} - a_1 = \frac{a_1 m_1}{m_2}$, or $\left[\frac{G(m_1+m_2)}{4\pi^2}T^2\right]^{1/3} - a_1 - \frac{a_1 m_1}{m_2} = 0$. Solving by iteration gives $m_2 = 2.40 \times 10^{31}$ kg, or $m_2 = 12.1 M_\odot$.

(d) The spectrum peaks at about 1.5 keV, or 2.4×10^{-16} J. The energy of a photon is related to its wavelength by $E = \frac{hc}{\lambda}$, which is related to the peak blackbody temperature by $T_{color} = \frac{b}{\lambda}$, where b is Wein's constant. Therefore, $E = \frac{hcT_{color}}{b}$, or $T_{color} = \frac{Eb}{hc} = \frac{2.4\times10^{-16} \cdot 2.898\times10^{-3}}{6.626\times10^{-34} \cdot 3.00\times10^{8}} = 3.50 \times 10^6$ K. Finally, $T_{peak} = \frac{T_{color}}{f_{GR} f_{col}} = 4.04 \times 10^6$ K.

(e) $L_{disk} = 2.2 \times 10^{37}$ ergs/s, or 2.2×10^{30} W. $L_{disk} = 4\pi\sigma r_{peak}^2 T_{peak}^4$, so $r_{peak} = \frac{1}{T_{peak}^2}\sqrt{\frac{L_{disk}}{4\pi\sigma}} = \frac{1}{(4.04\times10^6)^2}\sqrt{\frac{2.2\times10^{30}}{4\pi \cdot 5.67\times10^{-8}}} = 107.66$ km. $r_{last} = \eta r_{peak} = 67.83$ km.