2017 NAO

National Round of USAAAO - Problems with Solutions

May 2017

Instructions (Please Read Carefully)

(Start the exam after reading the instructions carefully and affirming that the contestant understood the instructions given.)

The top 5 eligible scorers on the NAO will be invited to represent USA at the next IOAA. Alongside the top 5 scorers, the next 10 eligible students in order of their scores will be invited to attend a training camp July 23-29th. In order to qualify for the national team, you must be a high school student with US citizenship or permanent residency, and you must be less than 20 years old on June 30th, 2017. The 10 next students invited to attend the training camp must less than 20 on June 30th, 2018 (such that they would be eligible to participate at IOAA 2018).

This exam consists of 3 parts: Short Questions (5 points each), Medium Questions (10 points each), and Long Questions (15 points each). The maximum score is 90 points.
The test must be completed within 3 hours (180 minutes). The proctor should mark the start and end time of the exam on the front page.

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will *not* be graded. Partial credit will be available given that correct and legible work was displayed in the solution. This exam document, solution, and all the scratch paper used should be turned in to the proctor at the end of the exam.

This is a pen and paper exam. Contestants can only use scientific calculator for this exam. Table of physical constants will be provided. Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on May 20th. Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: scientific calculator, pencil (or pen), eraser, sheets of papers, and the exam. No books, notes, laptops, mobile phones, or any other devices should be allowed while taking the exam.

After reading the instructions, please make sure to sign at the bottom of this page, affirming that:
1. All work on this exam is mine.
2. I took this exam under a proctor's supervision.
3. I did not receive any external aids besides the materials provided.
4. I am not allowed to discuss the test with others throughout the period of this examination.
5. Failure to follow these rules will lead to disqualification from the exam.

To be completed by the proctor:

```
Full name (First, Last): ____________________________________________________________
Position (e.g. Physics Teacher at High School x): ___________________________________
Email address: ______________________________________________________________________
Student began exam at (hh:mm):        _________________
Student submitted the exam at (hh:mm): _________________
Signature:________________________________________________ Date (mm/dd/yy):________
```

To be completed by the student:

```
Last name:  _________________________________
First name: _________________________________  Middle name: _________________________
Date of birth (mm/dd/yy): _________________  Email address: _________________________
Signature:________________________________________________ Date (mm/dd/yy):________
```

I Short Questions

1. [5pt] Estimate to order of magnitude the pressure, in Pa, at the center of a neutron star with a mass of 2 $M_\odot$ and a density of 5.00×10^{17} kg/m^3. Assume that the neutron star is in hydrostatic equilibrium and that its density is constant with radius. *Solution:*
Full answer (involves calculus): Integrate hydrostatic equilibrium ($dP/dr = -\rho g = -\rho \frac{GM_r}{r^2} = -\rho \frac{G}{r^2} \frac{4\pi r^3 \rho}{3} = -\frac{4\pi \rho^2 G}{3} r$) to find $P_c = \frac{2}{3}\pi \times G\rho^2 R^2$. Plug in to get $P_c = 5.38 \times 10^{33}$Pa.
Simple answer (order-of-magnitude estimation): $P_c \sim \rho GM/R \sim G\rho^2 R^2$, plug in to find $P_c \sim 2.57 \times 10^{33}$Pa. So accept any answer with $P_c \sim 10^{33}$Pa.

2. [5pt] The current angular momentum of objects in our Solar System tells us about its formation. Given that the rotation rate of the Sun is $\Omega = 2.9 \times 10^{-6} s^{-1}$, calculate its rotational angular momentum in $kg\, m^2\, s^{-1}$. Similarly, given that the mass of Jupiter is 1.90×10^{27} kg and its semi-major axis is 5.2 AU, calculate its orbital angular momentum in $kg\, m^2\, s^{-1}$. Which is larger? What does that tell us about the formation process of the Sun and our Solar System? [5pt]

Solution:
$L_\odot = \frac{2}{5} M R^2 \Omega = 1.12 \times 10^{42} \mathrm{kgm^2s^{-1}}$.
$L_{\mathrm{Jup}} = M_{\mathrm{Jup}} \sqrt{G M_\odot a} = 1.9 \times 10^{43} \mathrm{kgm^2s^{-1}}$.
The angular momentum of Jupiter's orbit is larger. This tells us that there was significant partitioning of angular momentum from the Sun into its protoplanetary disk, which is now encapsulated in planets.

3. [5pt] A jet was emitted from quasar A towards the Earth, with velocity v (see Figure 1). After τ hours, the jet arrived at point B. Let us set time t_A as the time when the jet from A arrived at the Earth, and set time t_B as the time when the jet from B arrived at the Earth. What is the time difference Δt = t_B-t_A? [5pt]

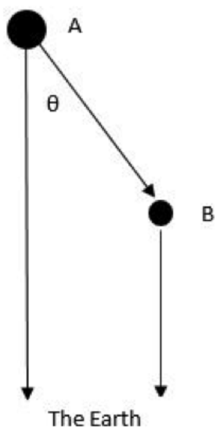

Figure 1: Image for problem 3.

Solution:

The distance between A and B is $v\tau$. Also if you draw a horizontal line from point B and the point which intersects with the line A is point C. then you will notice that the time that takes for a jet to reach the Earth from point B or C are basically the same. Therefore we only need to examine the distance before that. Now, And therefore solve for the time it takes light to reach point C from A (we label that with t_1) and subtract that from τ.

$$ct_1 = v\tau \cos\theta$$

$$t_1 = \frac{v\tau \cos\theta}{c}$$

$$\Delta\tau = \tau - \frac{v\tau\cos\theta}{c} = \frac{c - v\cos\theta}{c}\tau$$

II Medium Questions

4. [10pt] Let us think about a hypothetical universe where matter is uniformly distributed and the universe looks the same regardless of the direction one is looking at. 15 billion years ago, the expansion of the universe started. After 3 billion years from the initial point of the expansion, the expansion suddenly stopped. After 7 billion years from the sudden halt of the expansion, suddenly the size of the universe doubles and afterwards the size of the Universe remained constant until now. We observe from Earth two stellar objects: A and B. A is 3 billion light years away and B is 10 billion light years away. Determine the redshift z of each object.

Solution:

The stellar object A's age is 3 billion years, which means that it was created after the sudden increase of the size of the universe. However, after the sudden increase the size is being maintained constant and therefore the z value of this object should be 0, since there is no change in the value of energy of photon emitted from A.

The stellar object B's age is 10 billion years. This object was formed when the universe stopped expanding. 5 billion years after stellar object B's formation, the universe will suddenly expand the size of it by twice, which means that the energy of a photon emitted will be decreased to half. $E = \frac{hc}{\lambda}$ *and therefore the wavelength will be doubled. Since* $z = \frac{\lambda - \lambda_0}{\lambda_0}$, *the value of z for B should be 1.*

5. [10pt] The Eddington luminosity (or Eddington limit) is the maximum luminosity that an astrophysical object can achieve when there is a balance between the force of radiation acting outward and the gravitational force acting inward.

a) Consider a spherical cloud of ionized hydrogen with mass m and opacity κ. Taking the opacity to be $\kappa = \sigma_T/m_p$, where $\sigma_T$is the Thomson cross-section for electrons, and considering that the cloud is at a distance R from a source with luminosity L and mass M, what is the correct expression for the Eddington luminosity? Express your answer in terms of the mass M, opacity κ, and any universal constants you might need.

b) Consider (A) an AGB star (e.g. Mira), (B) the Sun, and (C) a Gamma-ray burst (GRB). The three objects have rough luminosities (in units of Eddington luminosity): i) $L \sim 3\times10^{-5} L_{Edd}$, ii) $L \sim 0.2 L_{Edd}$, iii) $L \sim 10^{12} L_{Edd}$ (Not necessarily in order). Match each luminosity (i through iii) to the corresponding astrophysical object (A through C) and justify your answer. Remember: The Eddington luminosity is the maximum luminosity that an astrophysical object can achieve when there is a balance between the force of radiation acting outward and the gravitational force acting inward.

Solution:
$F_g = \frac{GMm}{R^2}$, $Flux(R) = \frac{L}{4\pi R^2}$, $F_{rad} = \frac{Flux}{c} \times A = \frac{L\sigma_T}{c4\pi R^2}$

When the two forces balance out, we have $L_{Edd} = c4\pi GM\frac{m_p}{\sigma_T} = \frac{4\pi GMc}{\kappa}$.

The Sun is a star that is stable over long periods of time, so we expect its luminosity to be much lower than the Eddington luminosity (gravity dominates): $L \sim 3 \times 10^{-5} L_{Edd}$. *AGB stars such as Mira have luminosities thousands of times greater than the Sun but they are still sub-Eddington, with* $L \sim 0.2 L_{Edd}$ *in the case of Mira. Both Gamma-ray bursts and Supernovae are powerful explosions where the force of radiation dominates gravity by several orders of magnitude. Gamma-ray bursts are the brightest electromagnetic events known to occur in the universe, with* $L \sim 10^{12} L_{Edd}$.

6. [10pts] You are observing a star with declination $\delta = 42°21'$N and hour angle H $= 8^h16^m42^s$. If you are in a place with latitude $\phi = 60°$, compute the star's azimuth angle (A) and its height above the horizon (h) at the moment of observation.

Solution:

This solution uses the convention in Fundamental astronomy by Karttunen, Kroger, Oja, Poutanen and Donner, measuring A clockwise from South.

$H = 8^h\ 16^m\ 42^s$ = 124 deg 10.5 arcmin

Let Z denote the zenith, P the North celestial pole, and X the position of the star.

Let H = ∠ ZPX = 124 deg 10.5 arcmin

PZ = 90 deg - ϕ = 30 deg

PX = 90 deg - δ = 47 deg 39 arcmin

$\cos ZX = \cos PZ * \cos PX + \sin PZ * \sin PX * \cos H = 0.3758 \Rightarrow ZX = 67.92$ deg

ZX = 90 deg - h

$\Rightarrow$ h = 22.08 deg = 22 deg 4.6 arcmin

$\cos PX = \cos PZ * \cos ZX + \sin PZ * \sin ZX * \cos 180\text{deg} - A = \cos PZ * \cos ZX - \sin PZ * \sin ZX * \cos A$

$\cos A = (\cos PZ * \cos ZX - \cos PX / (\sin PZ * \sin ZX) = -0.75134$

$\Rightarrow A = 138.37$ deg = 138 deg 22 arcmin 12 arcsec

III Long Questions

7. [15pt] The exact source of viscosity for accretion disks is unknown, so astrophysicists often use a parameterization for the viscosity such that the viscosity (ν) is $\nu = \alpha c_s^2 \Omega^{-1}$, where α is a free parameter, c_s the disk sound speed, and Ω the disk rotation rate.

a) For protoplanetary disks around Sun-like stars, it is thought that the temperature profile follows the power-law $T \sim 100K(R/1AU)^{-1/2}$. Using this, derive an expression for how the disk viscosity depends on α, disk temperature, disk radius, and stellar mass. Note that the sound speed is $\sqrt{P/\rho}$, and one can write the ideal gas law $P = k_b/(\mu m_p)\rho T$, where k_b is Boltzmann's constant, μ is the mean molecular weight and m_p the mass of the proton. What interesting property does this solution have?

b) The evolution timescale of a disk can be approximated by the viscous diffusion timescale, $\tau_{diff} \sim R^2/\nu$. Assuming that $\alpha = 10^{-3}$ and $\mu = 2$, what is the diffusion timescale (in years) at both 1 AU and 100 AU for a disk around a Sun-like star?

c) How does your answer from part b) compare with typical protoplanetary disk lifetimes? What does this imply about protoplanetary disk dispersal?

Solution:

a) $\nu = \sqrt{R^3/GM}\frac{\alpha k_b T}{\mu m_p} = \sqrt{R^3/GM}\frac{\alpha k_b}{\mu m_p}100\text{K} * (R/1AU)^{-\frac{1}{2}} = R(GM)^{-1/2}\frac{k_b \alpha}{\mu m_p}100\text{K}\sqrt{1.5 \times 10^{11} m}.$
Linear in radius, constant : 6 points (4 if don't write constant), feature: 2 points

b) $\tau_{diff} = R\sqrt{GM}/(\sqrt{1.5 \times 10^{11} m} * 100\text{K})\frac{\mu m_p}{k_b \alpha}$. *Plug in with* $\mu = 2, m_p = 1.67 \times 10^{-27}\text{kg}, k = 1.38 \times 10^{-23}\text{JK}^{-1}, G = 6.67 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}, M = 2 \times 10^{30}\text{kg}$, *we find at* $R = 1\text{AU}$ $\tau_{diff} = 344,000\text{years}$ *and at* $R = 100\text{AU}$ $\tau_{diff} = 34\text{Myr}$. *Each 2 points*

c)Typical protoplanetary disk lifetimes are $\sim 1 - 10\text{Myr}$. *As a result, the inner regions of protoplanetary disks can be efficiently accreted due to viscosity, but the outer regions cannot. Some other process (e.g. disk winds, photoevaporation) must disperse the outer part of protoplanetary disks. 3 points, 1 for each of above sentences*

8. [15pt] Many of the stars in our Universe are in binary systems. A particular type of binary star consists of a regular star with mass m_0 and radius R and a neutron star, with a much higher mass M (but far more compact, therefore the scale difference in Figure 2). The two stars will move around their common center of mass.

Assume that the neutron star is much more massive than the regular star (M >> m_0) such that the regular star is rotating around the neutron star on a circular orbit with radius r_0 and speed v. The regular star starts to lose gas (with mass Δm) towards the neutron star with a relative velocity v_0 with respect to the regular star (see Figure 1).

a) Approximating that the neutron star is dominating the gravitational potential of the binary system and neglecting any changes in the orbit of the regular star, determine the minimum distance r_{min} that the gas will reach with respect to the neutron star.

b) Find the maximum distance r_{max} of the gas with respect to the neutron star.

Solution:

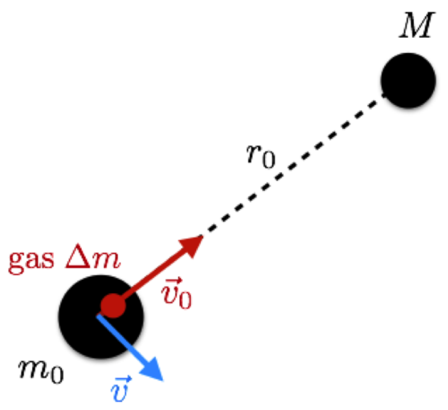

Figure 2: Image for problem 8. The blob of gas with mass Δm is shown in red. The velocity of the less massive (but larger radius) star with mass m_0 is shown in blue (v). The neutron star of mass M is a distance r_0 away.

The gas has a relative velocity v_0 with respect to the regular star. Let u be the velocity of the gas with respect to the lab frame. Then, $u^2 = v^2 + v_0^2 = \frac{GM}{r_0} + v_0^2$. Let r_{min} and v_{max} be the position and velocity of the gas at the pericenter of its orbit around the star, while r_{max} and v_{min} are the position/velocity at the apocenter.

Conservation of angular momentum:

$\Delta m\, r_{min} v_{max} = \Delta m\, r_{max} v_{min} = \Delta m\, r_0 v$ (1)

Conservation of energy:

$-\frac{GM\Delta m}{r_{min}} + \frac{\Delta m v_{max}^2}{2} = -\frac{GM\Delta m}{r_{max}} + \frac{\Delta m v_{min}^2}{2} = -\frac{GM\Delta m}{r_{min}+r_{max}}$ (2)

Since m_0 is on a circular orbit around $M \Rightarrow v = \sqrt{\frac{GM}{r_0}}$.

(1) $\Rightarrow v_{max} = v_{min}\frac{r_{max}}{r_{min}}$ (2) $\Rightarrow v_{min} = \sqrt{\frac{2GM}{r_{min}+r_{max}}\frac{r_{min}}{r_{max}}}$

$-\frac{GM}{r_0} + \frac{u^2}{2} = -\frac{GM}{r_{min}+r_{max}}$

The semi-major axis of the ellipse on which the gas is moving around the regular star is:

$a = \frac{r_{min}+r_{max}}{2} = \frac{GMr_0}{GM-v_0^2 r_0}$

$r_{max}v_{min} = \sqrt{r_{max}r_{min}}\sqrt{\frac{2GM}{(r_{max}+r_{min})}} = r_0 v = \sqrt{GMr_0}$

$\Rightarrow 2r_{max}r_{min} = r_0(r_{max}+r_{min}) = 2a\, r_0$

$\Rightarrow r_{max} = a\, r_0/r_{min}$ and $r_{max} + r_{min} = \frac{ar_0}{r_{min}} + r_{min} = 2a$

We arrive at the quadratic equation:

$r^2 - 2ar + ar_0 = 0$

with the two roots:

$r = \frac{2a \pm \sqrt{4a^2 - 4ar_0}}{2} = a \pm \sqrt{a^2 - ar_0}$

The smaller root corresponds to the minimum distance and it is:

$r_{min} = r_0\frac{(v_0\sqrt{GMr_0} - GM)}{v_0^2 r_0 - GM}$ or $r_{min} = \frac{r_0}{1+v_0\sqrt{\frac{r_0}{GM}}}$.

Note that $r_0 > r_{min} > 0$.

The larger root corresponds to the maximum distance and it is:

$r_{max} = r_0\frac{(v_0\sqrt{GMr_0} + GM}{GM - v_0^2 r_0}$ or $r_{max} = \frac{r_0}{1-v_0\sqrt{\frac{r_0}{GM}}}$.

Note that $r_{max} > r_0$.

9. [15pt] **Attention: Problem continues on next page.** The Milky Way is virtually optically thin to 21cm radiation, which makes this wavelength band a good tool to study the Galactic structure. Spectroscopy is used to measure the velocity distribution of clouds along a line of sight in the galaxy. In this problem, you can assume all motions are circular about the center of the galaxy. Assume the galactic longitude, l, takes values between 0 and 90 degrees.

i) In a couple of sentences (2-3), explain what is the physical mechanism that causes the emission of the 21.11 cm photons.

ii) Explain how spectroscopy can be used to determine the radial velocity along the line of sight (1-2 sentences).

iii) Match the clouds A,B,C from Figure 3, to the X,Y,Z peaks in Figure 4.

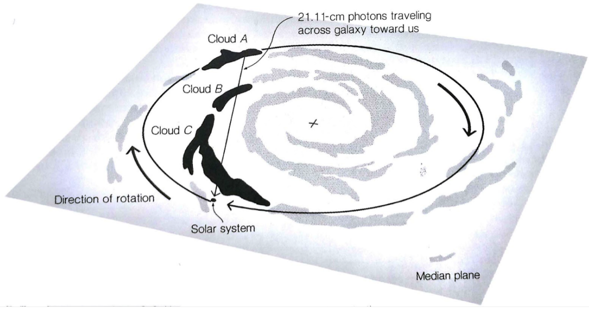

Figure 3: Figure for problem 9. Image credit: 'The Evolving Universe' - Donald Goldsmith, pg 210.

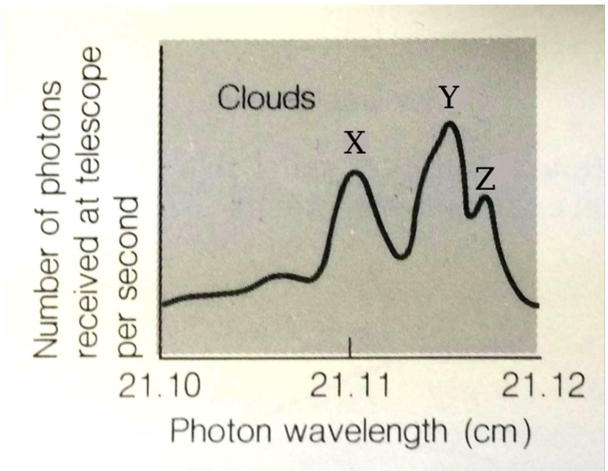

Figure 4: Figure for problem 9.

iv) Describe the procedure and show the calculations by which you can determine the galactic rotation curve. By "galactic rotation curve" we mean the rotational speed of the galaxy V(R) as a function of the distance from its center, R. You can consider your known variables to be $V_{r_{max}}$ - the maximum radial velocity along the line of sight, R_0 - distance from Sun to center of Galaxy, V_0 - the orbital velocity of the Sun around the Galactic center, and l, the galactic longitude.

v) From here on, you can consider that we know the galactic rotation curve $V(R)$ from previous studies of 21cm radiation. A new study now looks at a dense molecular cloud, and studies its ^{13}CO emission. Stars are expected to form in molecular clouds of spiral galaxies, and as a result, molecular clouds can be used to trace the spiral structure of the Milky Way.
Measuring the rotation curve was key to understanding the gas distribution in the Galaxy, since it was then possible to relate the spectroscopically observed radial velocity of a cloud to its galactocentric radius and distance. For a given cloud with galactic longitude l and radial velocity V_r, there is a unique solution for its galactocentric radius R.
Find R as a function of R_0- the galactocentric radius of the Sun, V_0 - the orbital velocity of the Sun around the Galactic center, V(R) - the rotation curve, and V_r - the radial velocity of the cloud. You can

consider V(R) is known in this case even though R is not known, coming from the studies of the galactic rotation curve that have determined the speed of the cloud at other points. Use Figure 5 for reference.

vi) Find the solution of the distance to the cloud d as a function of the galactocentric radius of the cloud R, galactic longitude l, and galactocentric radius of Sun R_0.

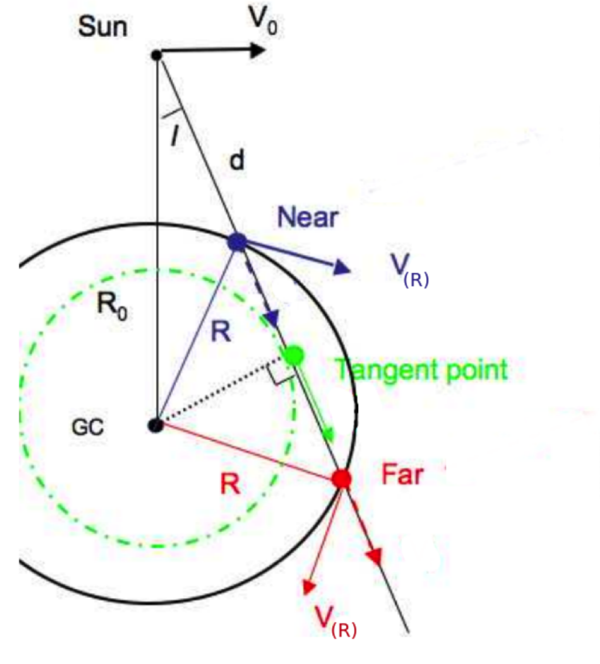

Figure 5: Figure for problem 9. Image credit: 'Kinematic Distances to Molecular Clouds identified in the Galactic Ring Survey' - Julia Roman-Duval, 2009.

Solution:

Note: For a full understanding of the problem, read 24.3 Kinematics of the Milky Way from Carroll and Ostlie, and the article "Kinematic Distances to Molecular Clouds identified in the Galactic Ring Survey" - Julia Roman-Duval, 2009 `https://arxiv.org/pdf/0905.0723.pdf`

i) Read `https://www.wikiwand.com/en/Hydrogen_line`

ii) The radiation from one cloud is Doppler shifted because of the effects of differential Galactic rotation, as the object who emits the 21cm radiation is moving with a certain velocity. Spectroscopy measures the shift in the wavelength, which can then be converted into velocity using the Doppler shift formula.

iii) Cloud A is at the same orbital distance as the Sun. It's velocity along the line of sight is the same as the one of the Sun's, thus this results in 0 relative velocity along the line of sight and no wavelength shift in the emitted radiation. Clouds C and B are on orbits closer to the center of the galaxy, their perceived relative velocity along the line of sight is positive, leading to a Doppler Shift of the wavelength. Cloud C is much larger than B, which makes the intensity of the radiation line higher. Thus
$X = A$
$Y = C$
$Z = B$

iv) See Figure 6 for reference. The relative radial and transverse velocities are

$$V_r = V(R)\cos\alpha - V_0 \sin l$$
$$V_t = V(R)\sin\alpha - V_0 \cos l$$
$$R\cos\alpha = R_0 \sin l$$
$$R\sin\alpha = R_0 \cos l - d$$

d is not known, but if we go to the point where the radial velocity is maximum, that point will be the *tangent point on an orbit, and thus* $d = R_0 \cos l$*, and radius of the orbit becomes* $R(V_{r_{max}}) = R_0 \sin l$*. Since* $\alpha = 0$ *for max* v_r*, we can determine that* $V(R) = V_{r_{max}} + V_0 \sin l$*, at distance* $R = R_0 \sin l$*,which gives*

$$V(R) = V_{r_{max}} + V_0 \frac{R}{R_0}$$

v) From the same equations as part iv), we get

$$V_r = V(R) R_0 \sin l / R - V_0 \sin l$$

$$R = R_0 \sin l \frac{V(R)}{V_r + V_0 \sin l}$$

vi) From the same equations as part iv),

$$R \cos \alpha = R_0 \sin l$$

$$R \sin \alpha = R_0 \cos l - d$$

we get

$$d = R_0 \cos l \pm \sqrt{R^2 - R_0^2 \sin l^2}$$

There are two distance solutions.
"In the inner Galaxy, a single galactocentric radius (determined by the radial velocity of the cloud) corresponds to two distances along the line of sight, a near (in blue) and a far (in red) kinematic distance. The near and far kinematic distances correspond to the same radial velocity Vr, which is the projection of the orbital velocity V0 of a cloud around the Galactic center onto the line of sight. At the tangent point, the orbital velocity of a cloud is parallel to the line of sight. In this case, the radial velocity is maximal and the near and far kinematic distances are identical." If you are curious to see a procedure to distinguish between the two distance solutions, see article "Kinematic Distances to Molecular Clouds identified in the Galactic Ring Survey" - Julia Roman-Duval, 2009 https://arxiv.org/pdf/0905.0723.pdf

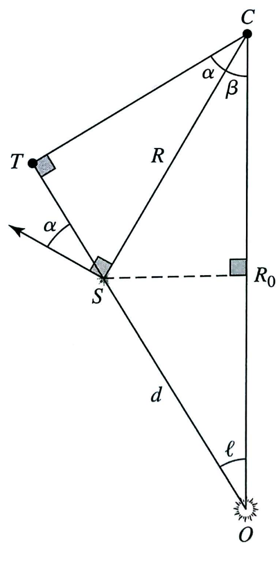

Figure 6: Figure for problem 9. Image credit: Carroll and Ostlie pg 909 .