# Singapore Astronomy Olympiad 2024

Theory

**Instructions**

1. The theory portion of this Olympiad is worth a total of **77 marks**.
2. When asked to do so, check that you have **11** printed pages.
3. Write your answers and workings clearly on the answer sheets provided.
4. Submit all used answer sheets.
5. Fill in these details on each side of your answer sheet:
   - Year of competition
   - Your participant code
   - The page number – which should be continuous from 1 to N
   - The question number
6. Cross out all workings or answers you do not wish to be evaluated.
7. If you require assistance (e.g. to visit the restroom, enquire about an ambiguity or possible errata, etc.), please get the attention of the invigilators.

**Competition Rules and Regulation**

1. Only the use of scientific calculators is permitted. No graphing or programmable calculators are allowed.
2. Disruptive behaviour, cheating, collusion to cheat or any integrity-related offences are grounds for immediate disqualification.
3. You may opt to retain the question paper and constants sheet for personal use. Return all unused answer sheets to the Organising Team.

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# 1 Supernova [35]

## 1.1 Supernova Detected [13]

In a recent communication with our extraterrestrial intelligence partners located across the galaxy, we've been alerted to the imminent occurrence of a Type II supernova 10kpc away, henceforth designated SN2024Z. This unprecedented information has prompted observatories around the Earth to direct their telescopes toward the star predicted to undergo this spectacular explosion, in anticipation of capturing valuable data. Your expertise is requested to assist in performing critical analyses and calculations that will significantly contribute to the scientific community's understanding of this event.

**(a)** You are given that during the supernova SN2024Z, a remnant of 0.07 solar masses of $^{56}$Ni was created. Given that the half life of $^{56}$Ni decay to $^{56}$Co is 6.10 days, and that of $^{56}$Co decay to $^{56}$Fe is 77.7 days, and that both decay reactions proceed by electron capture, give an expression of the number of $^{56}$Co nuclei $N_{Co}$ as a function of time. **[6]**

Give your answers and carry out your working in terms of:

- Decay constant $\lambda_1$ (for $^{56}\text{Ni} \rightarrow {}^{56}\text{Co}$)
- Decay constant $\lambda_2$ (for $^{56}\text{Co} \rightarrow {}^{56}\text{Fe}$)
- Initial number of $^{56}$Ni nuclei $N_0$.

Hint: Form a differential equation with the form $y' + P(t)y = Q(t)$, where y is a function. Multiply the entire equation by the integration factor

$$M(t) = e^{\int P(t)\,dt}$$

Then solve the equation via integration.

**(b)** Estimate the luminosity of the supernova remnant, resulting from the decay of $^{56}$Ni and $^{56}$Co, exactly 69.0 days after the supernova event. It is assumed that all energy released from the decay is instantaneously converted into the bolometric luminosity of the supernova. **[5]**

| **Isotope** | **Mass (amu)** |
|---|---|
| $^{56}$Ni | 55.942132 |
| $^{56}$Co | 55.939839 |
| $^{56}$Fe | 55.934937 |
| $e^-$ | 0.000549 |

**(c)** Determine the apparent magnitude of the object 69 days after the supernova, given bolometric interstellar extinction of 0.420. It is assumed that the radioactive decay of $^{56}$Ni and $^{56}$Co are the only source of luminosity at that time. **[2]**

### 1.2 Tick Tock [15]

Shortly after the supernova occurred, the scientists became worried that the supernova might dim too quickly for them to conduct measurements. They have asked you to help them work out how much time they have to observe the supernova. You will comply.

To commence calculations, we need to determine the angular area of the supernova remnant by using an unit spherical surface element with Earth at its center. Thereafter, we perform double integration to obtain the following expression.

$$\Omega = \int_0^{2\pi} d\phi \int_0^{\theta} \sin(\theta')\, d\theta'$$

**(a)** Solve the integral to find an expression of its solid angle $\Omega$ in terms of $\theta$, where $\theta$ is the angular radius of the object. **[2]**

**(b)** Given that the speed of the gas in the remnant moving outwards in all directions is constant at 10000 km/s, and assuming that the entire remnant is homogeneous in brightness and has negligible opacity, show that the expression of the surface brightness of the supernova remnant is given by:

$$\mathrm{SB} = 2.5 \log \left( \frac{\pi(3.2408 \times 10^{-14} t + 3.2408 \times 10^{-11})^2}{1.8186 \times 10^9 e^{-\lambda_1 t} + 3.5520 \times 10^8 e^{-\lambda_2 t}} \right) + 20.17$$

Carry out all numerical working to five significant figures. Do not express $e^{\lambda_n}$ in numerical values. Convert all numerical quantities to SI units. You may assume that the original radius of the supernova remnant at time $t = 0$ is $10^{15}$ m. **[10]**

Hint: You may use the small-angle approximations below.

$$\tan(x) \approx x$$

$$\cos(x) \approx 1 - \frac{x^2}{2}$$

**(c)** The graph of the function derived in (b) is shown in Figure 1, with $t$ in days. Given that the surface brightness of the night sky is 21.8 mag/arcsec$^2$, find the number of days needed for the supernova to no longer be visible in the sky, based solely on the luminosity from radioactive decay of $^{56}Ni$ and $^{56}Co$. **[1]**

**(d)** In Figure 2 is the same graph, showing the first 60 days after the supernova. Explain why the gradient of the graph changes abruptly at $t = 30$ days. **[1]**

Figure 3 displays the predicted light curve of SN2024Z based on the calculations done above. Figure 4 shows the light curve of SN1987A, the first supernova to be studied by modern astronomers in detail. SN1987A is an example of a Type-II core collapse supernova where $^{56}Ni$ and $^{56}Co$ decay contributes greatly to its initial luminosity.

**(e)** Note that the initial light curve for the months immediately after the peak of SN1987A is similar to that of SN2024Z, yet the magnitude of SN2024Z increases linearly thereafter while that of SN1987A increases at a diminishing rate. Using the information already given in the question or otherwise, provide a possible reason why the magnitude of SN1987A increases more slowly. **[1]**

### 1.3 Neutrinos [7]

The scientists extend their gratitude for your efforts. Recently, our neutrino detectors registered a significant burst of neutrinos, occurring just before the supernova was observed.

**(a)** 32 neutrinos were detected by Earth's neutrino detectors during the supernova. Given that 1 in $3 \times 10^{-29}$ neutrinos that pass through Earth are detected, and assuming the neutrinos are emitted isotropically, estimate the total number of neutrinos emitted by the supernova. **[1]**

You may use the small-angle approximation:

$$cos(x) \approx 1 - \frac{x^2}{2}$$

**(b)** Assuming the supernova emitted neutrinos at a constant rate for 5s, find the ratio of the supernova neutrino flux density to the average solar neutrino flux density. **[6]**

The simplified proton-proton chain is:

$$4p^+ + 2e^- \rightarrow^4 He^{2+} + 2\nu_e$$

| **Particle** | **Mass (kg)** |
|---|---|
| $^4He^{2+}$ | $6.646 \times 10^{-27}$ |
| $e^-$ | $9.109 \times 10^{-31}$ |
| $p^+$ | $1.673 \times 10^{-27}$ |

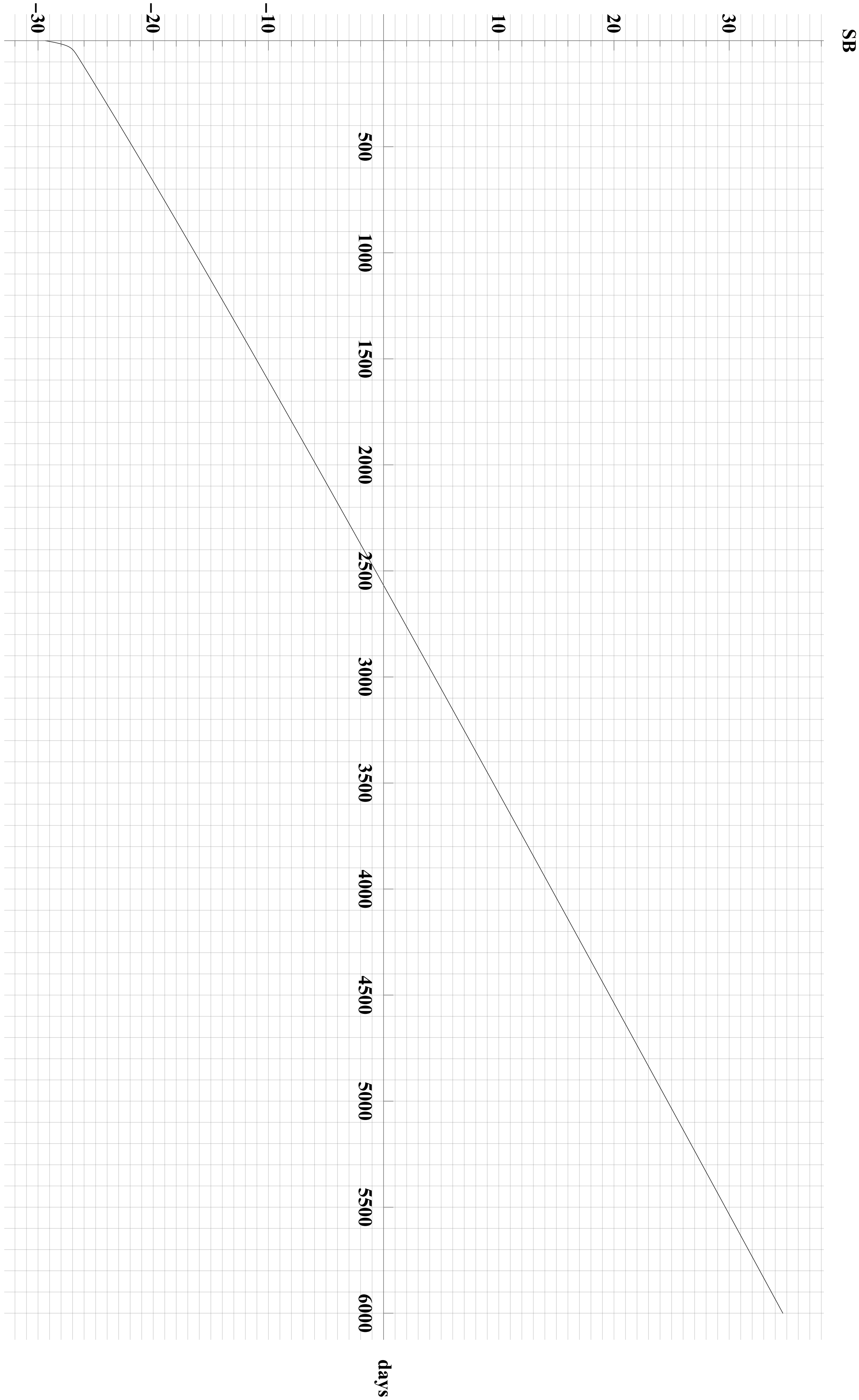


Figure 1: Surface Brightness of SN2024Z

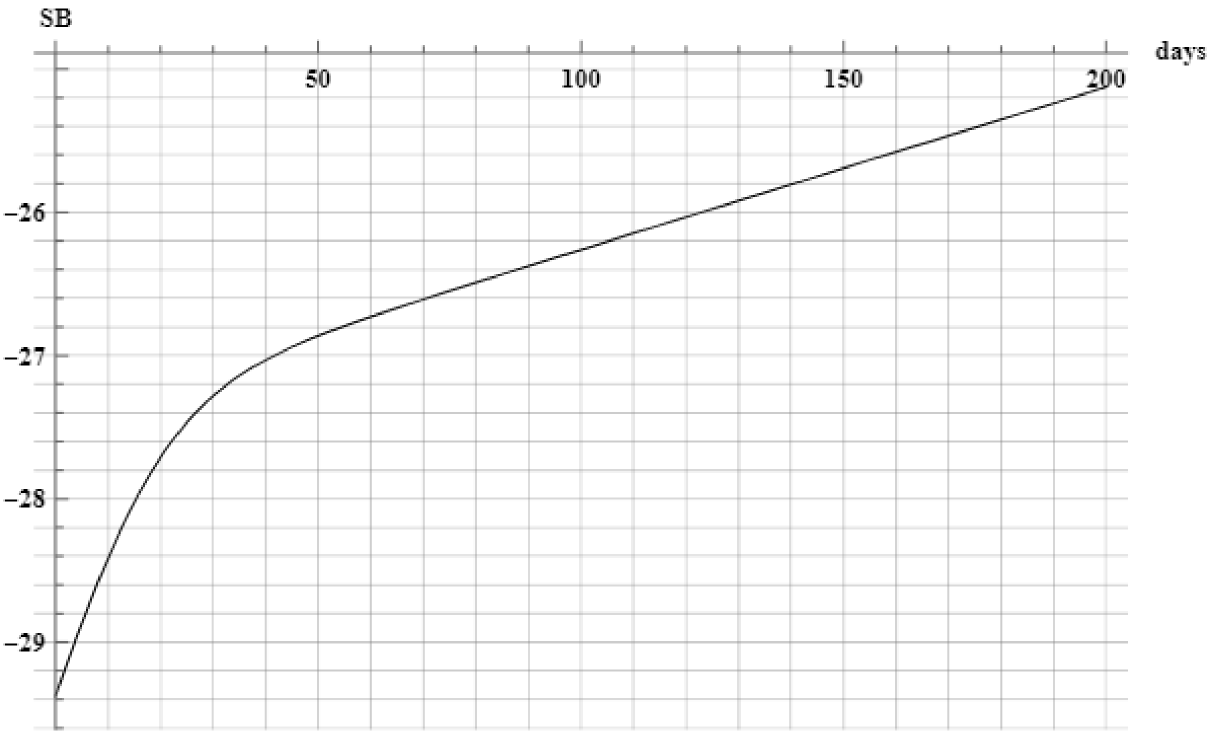


Figure 2: Surface Brightness of SN2024Z

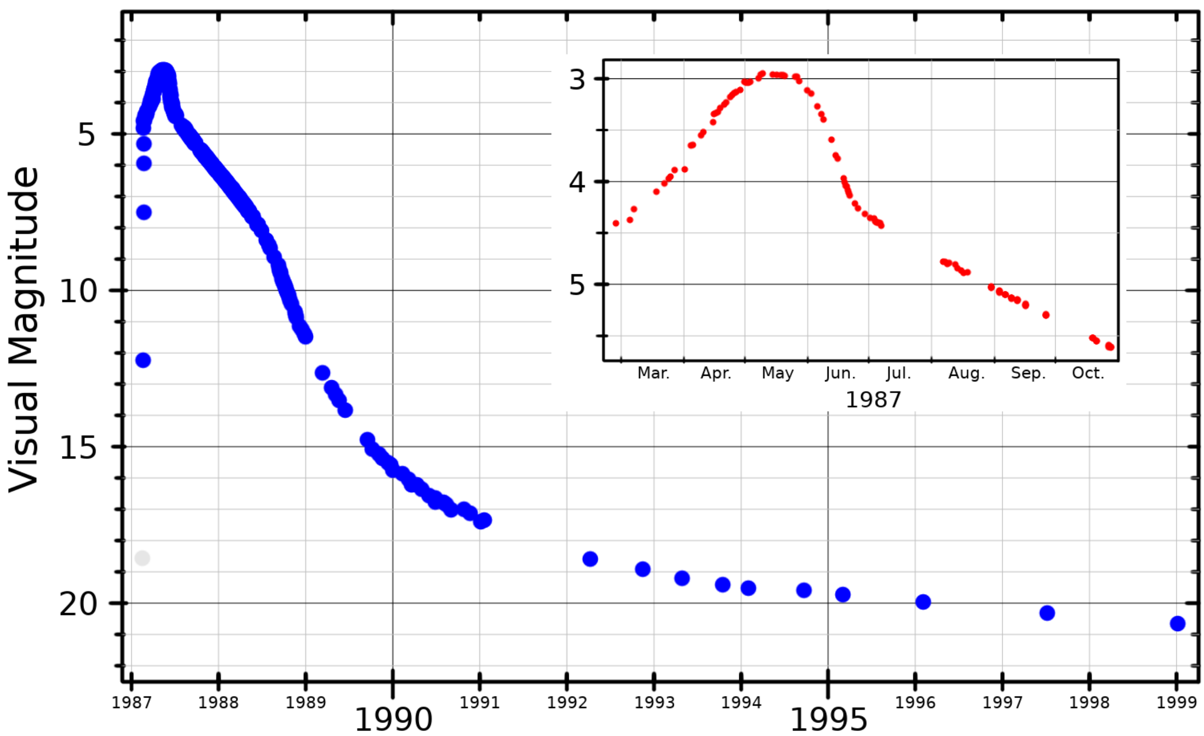


Figure 3: Light Curve of SN1987A

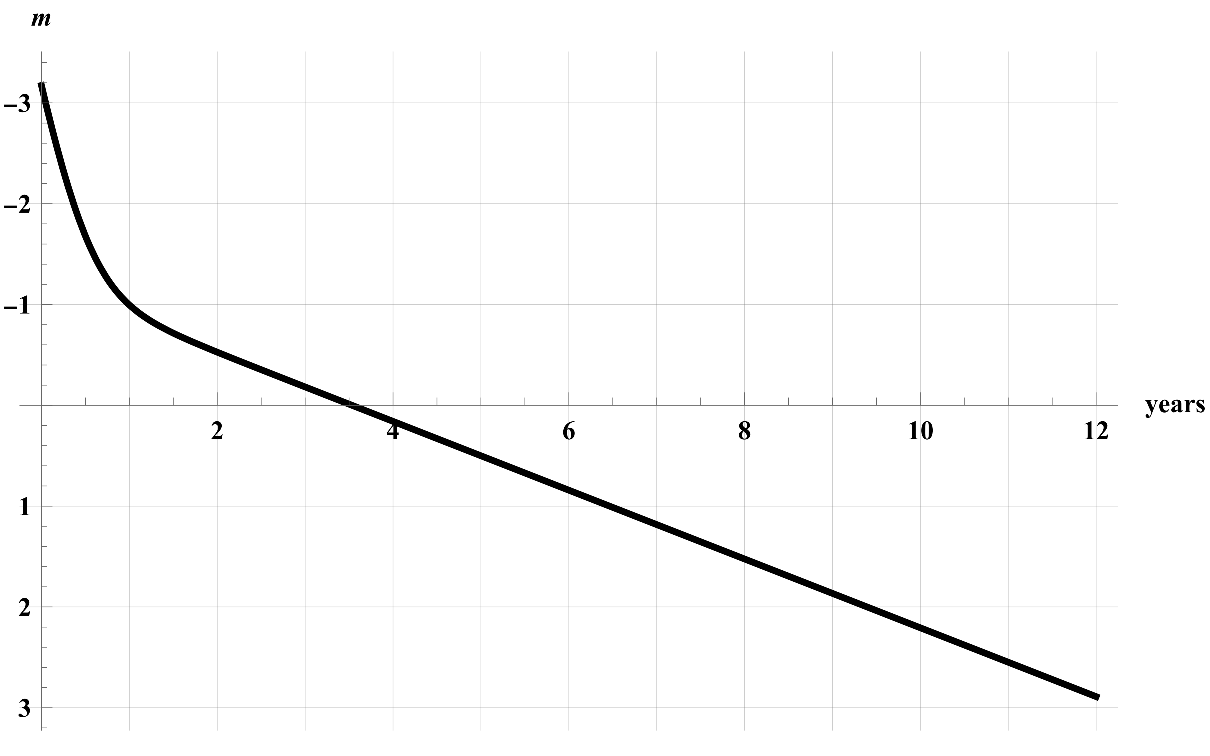


Figure 4: Light Curve of SN2024Z

## 2 The Precession of Mercury [21]

It was determined in 1859 that the perihelion of Mercury's orbit was precessing at a rate of 43" per century. This disagreed with Newton's predictions, and so scientists believed there was another planet ("Vulcan") between Mercury and the Sun that caused this precession. Yet no observations could prove the existence of Vulcan. When Einstein formulated general relativity , one of the first wins for his new theory was in being able to explain this precession. Here we will go through the calculation.

In Newtonian gravity, the effective potential energy for a body of mass $m$ orbiting another body of mass $M$ is given by:

$$V_N(r) = \frac{GMm}{r} + \frac{L^2}{2\mu r^2}$$

where $\mu = \frac{mM}{m+M}$. The frequency of oscillation of a particle of mass $m$ in a potential is given by $\omega^2 = \frac{1}{m}\frac{\partial^2 V}{\partial r^2}$ evaluated at the minimum of the potential.

**(a)** Find the angular velocity of the orbit in Newtonian gravity, $\omega_N$. **[4]**

In the general relativistic correction, to first order,

$$V_G(r) \approx \frac{GMm}{r} + \frac{L^2}{2\mu r^2} - \frac{G(M+m)L^2}{\mu c^2 r^3}$$

**(b)** Find the radius of the orbit in general relativity, $r_G$. **[5]**

**(c)** Expand the results of (b) to first order in $1/c^2$, and show that it agrees with the expected Newtonian result. **[3]**

**(d)** Find the angular velocity of the orbit under general relativity, $\omega_G$, and thus find the expression for the precession in radians per orbit, again to first order in $1/c^2$. **[6]**

Hint: The precession is the number of extra radians general relativity predicts Mercury will obtain in one orbit, which is just $T(\omega_G - \omega_N)$, and to first order we can approximate the period as $T \approx 2\pi/\omega_N$. Your final expression should be simplified.

**(e)** Calculate the value of the precession per century. To do this, you will need to express angular momentum in terms of the semimajor axis and eccentricity of Mercury's orbit. **[3]**

# 3 Diamond Fuji [21]

"Diamond Fuji" is a phenomenon where the rising or setting sun aligns with the peak of Mount Fuji in Japan, causing it to brightly shine like a diamond. Depending on the time of year, Diamond Fuji can be observed from a variety of locations in Japan. However, in this question, we will investigate the phenomenon as seen from the peak of Mount Takao, a popular sightseeing destination near the outskirts of Tokyo.

Figure 5: Diamond Fuji as observed from Mt. Takao

For the purposes of this question, we will define "Diamond Fuji" to have occurred when the centre of the Sun's disk intersects with the crater line, as shown in Figure 6. You can also assume Mount Fuji to be a perfect truncated cone, with a crater diameter of 780 metres.

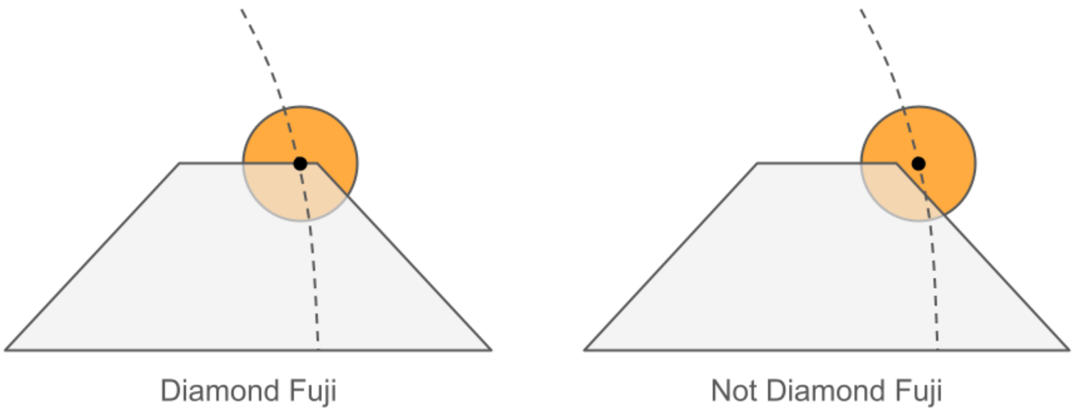


Figure 6: Definition of Diamond Fuji

The following information may be useful:

| | **Latitude** | **Longitude** | **Height above sea level (m)** |
|---|---|---|---|
| Mt. Fuji | 35° 21' 39" N | 138° 43' 39" E | 3777 |
| Mt. Takao | 35° 37' 31" N | 139° 14' 37" E | 599 |

The spherical cosine rule is given by

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

where $a$, $b$, and $c$ are the sides of the spherical triangle, and $C$ is the angle opposite side $c$.

**(a)** Find the distance between Mount Fuji and Mount Takao. **[3]**

**(b)** Find the bearing angle of Mount Fuji observed from Mount Takao. **[2]**

**(c)** Will Diamond Fuji occur at sunrise or sunset? **[1]**

**(d)** Using the information in **(a)**, calculate the angular sizes of Mount Fuji's height and crater width as seen from Mount Takao. **[2]**

**(e)** Taking the height of both mountains into account, find the range of dates in the year on which Diamond Fuji can be observed. Neglect atmospheric refraction. **[13]**