1. Ensure that all materials (answer sheets, graphing paper, practical question paper) you intend to submit for grading have been combined to one PDF file. Amendments to any answer script after it has been submitted will not be entertained.

2. Details on each side of answer sheet:
   - Your participant code.
   - Question number.
   - Page number – in increasing order.
   - **DO NOT WRITE YOUR NAME ON YOUR ANSWER SCRIPT.**

3. For **all** components of this Olympiad, order your answer scripts as follows:
   - The corresponding Summary Answer Sheet
   - Answer sheets with your workings
     i. Within each section, sort by question number. (EG. 2.1 before 2.3 before 2.4)
     ii. For the practical round, include your star chart responses here

FOR OFFICIAL USE ONLY			
Question	**Marks**	**Question**	**Marks**
1.1		3.1	
1.2		3.2	
1.3		3.3	
1.4		**Total**	
1.5		4.1	
Total		4.2	
2.1		**Total**	
2.2		P(A)	
2.3		P(B)	
2.4		P(C)	
2.5		**Total**	
2.6		**OVERALL**	
Total		Theory	
		DA	
		Practical	
	GRAND TOTAL		

FOR OFFICIAL USE ONLY	
Marker's Initials	**Signature**

Participant Code	
Total Number of Pages of SAO Answer Sheets	

Singapore Astronomy Olympiad 2022

Theory

Instructions

1. The theoretical portion of this Olympiad lasts for **105 minutes** and is worth **a total of 102 marks**.
2. Fill in these details on the Theory Summary Answer Sheet (Attached as a MS word document) and each side of an SAO answer sheet:
   - Year of competition
   - Your participant code
   - The page number – which should be continuous from 1 to N
   - The part of the paper, and the question number
3. Cross out all workings or answers you do not wish to be evaluated.
4. If you require assistance (enquire about an ambiguity or possible errata, etc.), please get the attention of the invigilators.
5. Once the Theory Round is over, you are to combine all documents into one singular PDF file to be uploaded. More details on the order can be found in the Cover Page word document.

Competition Rules and Regulation

1. Only the use of scientific calculators is permitted. No graphing or programmable calculators are allowed.
2. Disruptive behaviour, cheating, collusion to cheat or any integrity-related offences are grounds for immediate disqualification.
3. You may opt to retain the question paper, constants sheet and answer script for personal use.
4. Toilet breaks are prohibited for the duration of the paper owing to its decentralized nature. Participants are encouraged to use the washroom before the paper commences.

Contents

1 Section A [12] **4**
1.1 Exoplanet Exploration **[4]** 4
1.2 Galactic Collision **[2]** 4
1.3 Space Elevators **[2]** 5
1.4 Lunar Evaporation **[2]** 6
1.5 Artificial Illumination **[2]** 6

2 Section B [54] **7**
2.1 Eruption **[9]** 7
2.2 Kerr **[9]** 7
2.3 Magnetic Sails **[9]** 8
2.4 Standard Candle **[9]** 9
2.5 Radio Array **[9]** 10
2.6 Strömgren Spheres **[18]** 11

3 Section C: Long Question [36] **12**
3.1 Part 1: Lagrange Points **[8]** 12
3.2 Part 2: Roche Lobes **[13]** 12
3.3 Part 3: Evolution **[15]** 13

1 Section A [12]

Write all your responses in the summary answer sheet provided.

1.1 Exoplanet Exploration [4]

Consider the following hypothetical exoplanets:

	Radius (Earths)	Mass (Earths)	Mean Density (kg/m^3)	Average Temperature (°C)	Atmosphere	Parent Star Classification
SAO001	0.000012	6.27×10^{-19}	2000	125	None	B2 III (Blue Giant)
SAO002	1.57	7.3	11026	-59	None	Wolf-Rayet Star

Based on humanity's current technology, which of the following modes of transport is most practical for exploration of **each** of the exoplanet's surface: **[4]**

(A) Rover with wheels

(B) Helicopter

(C) Ion Thruster

(D) Aeroplane

1.2 Galactic Collision [2]

In five billion years, the Milky Way is projected to collide with its closest neighbour, the Andromeda Galaxy.

	Number of Stars	Galactic Disk Diameter (kly)	Central Black Hole Mass ($M_\odot$)
Milky Way	3×10^{11}	185	3.6×10^{6}
Andromeda Galaxy	10^{12}	220	1.5×10^{8}

Based on your background knowledge and the estimates in the table above, which of the following statements are **false**?

I. A large number of stellar collisions will occur

II. Some stars will be ejected from the combined galaxy

III. Many supernovae will be triggered by the merger

IV. Central black holes from both galaxies are likely to merge

(A) I and III

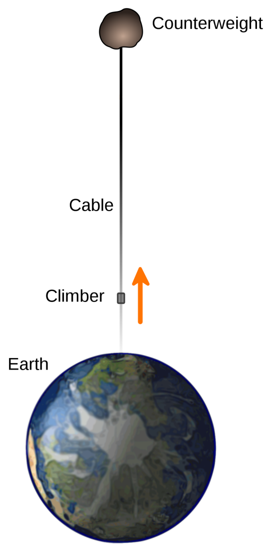
(B) I and IV

(C) I and II

(D) II and IV

1.3 Space Elevators [2]

Space elevators are a planet-to-space transportation system involving a cable structure extending perpendicularly from the planet surface. Modern space elevator concepts rely on a tensile design, with a counterweight in space using centrifugal force to negate the cable's gravitational weight.



Which of the following statements are **true**?

I. The counterweight must always be above the altitude for geostationary orbit

II. The total mass of climber and payload being lifted should be less than the mass of the counterweight

III. To minimize counterweight mass, space elevators with tensile designs should be deployed at the equator

IV. Space elevators will never work on a tidally locked planet

(A) I and III

(B) I and II

(C) II and III

(D) III and IV

1.4 Lunar Evaporation [2]

Hawking radiation refers to the thermal radiation emitted by a black hole due to relativistic quantum effects. The radiation power and temperature of a black hole are, respectively, related to mass by:

$$P \propto \frac{1}{M^2} \qquad\qquad T \propto \frac{1}{M}$$

Assume the moon has collapsed into a moon-mass black hole. From Earth, what would be observed in the final moments before its evaporation?

I. The black hole's radiant flux increases rapidly

II. Rate of DNA ionization increases rapidly

III. Spaghettification occurs to Earth due to extreme tidal forces

IV. Earth is irradiated primarily by infrared radiation

(A) III and IV

(B) I and III

(C) I and II

(D) II and IV

1.5 Artificial Illumination [2]

For two decades, the Proton-M rocket has been a cornerstone of Russia's heavy-lift rocket launches. The rocket has a burn time of 130s and a first stage carrying 458,900 kg of fuel (8.11kJ/kg).

Calculate the apparent magnitude of the rocket as seen by a observer 1500km away, given that solar flux at Earth is 1361 W/m^2 and the Sun's apparent magnitude is -26.74. Assume 0.5% of energy from combustion is converted into isotropic light.

(A) 16.1

(B) 1.83

(C) 11.7

(D) -3.49

2 Section B [54]

Section B(1): Short Questions [36]

This section contains a total of 5 short-length questions worth 9 marks each.

Answer any **FOUR (4)** questions.

2.1 Eruption [9]

In 2009, the islands of Hunga Tonga and Hunga Ha'apai (20.57° S, 175.38° W) merged due to the eruption of its submarine volcano. At 1726 Tongan civil time (UTC+13), January 15, 2022, the newly-combined island was torn back apart due to another volcanic eruption, thought to be one of the strongest in three decades. This eruption released a pressure wave that spread radially outward, resulting in several tsunami advisories in Hawaii and Japan.

Residents of Anchorage, Alaska (61.22° N, 149.90° W) reported hearing a sonic boom at 0300 AKST (UTC-9).

(a) Calculate the average velocity of this pressure wave, stating any assumptions made. **[4]**

(b) How long after the eruption is sunset? Take the effect of atmospheric distortion at the horizon to be $\theta_{\text{atm}} = 34'$. **[5]**

2.2 Kerr [9]

The optical Kerr effect is the change in refractive index of a medium induced by the electric field of light. For a beam of light, this creates a refractive index gradient analogous to a gradient index lens and results in self-focusing. It has been conjectured that the optical Kerr effect also occurs in a vacuum.

The equation for critical power is given by:

$$P_{cr} = \alpha \frac{\lambda^2}{4\pi n_0 n_2}$$

For relativistic momentum:

$$E^2 = (pc)^2 + (m_0 c^2)^2$$

Take the constants to be $\alpha = 1.862$, $n_2 = 1.555 \times 10^{-37}\text{m}^2/\text{W}$ (for a vacuum), and n_0 to be the refractive index of the medium.

(a) Find the critical power for a 820nm laser in a vacuum **[1]**

(b) Currently, the most powerful laser is the ZEUS three-petawatt (peta $= 10^{15}$). Assuming the maximum power of lasers increase by an order of magnitude every eight years, estimate how long it takes before a laser will attain the power found in part (a). **[2]**

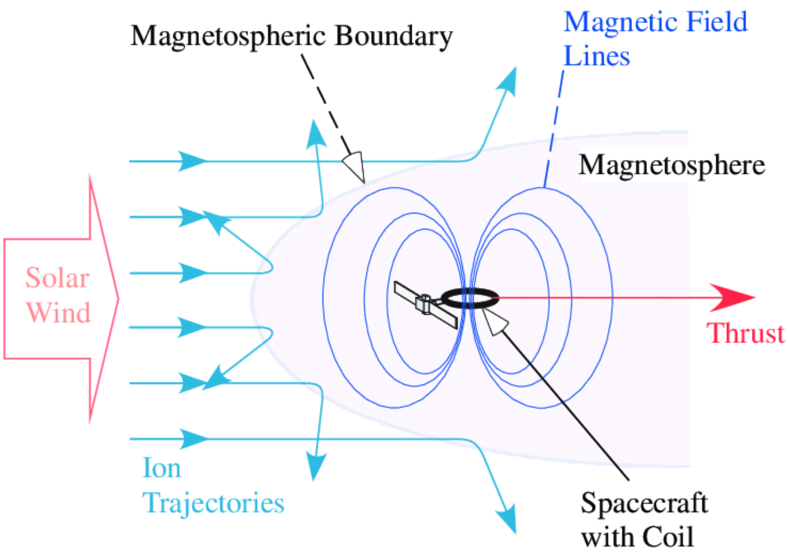
Breakthrough Starshot is a project aiming to launch miniature interstellar probes to Alpha Centauri using light sails. Assume the probes are fully reflective and propelled by aforementioned self-focusing laser on Earth with wavelength and power found in part (a) with a pulse length of ten picoseconds.

(c) For a probe with mass 5g, how long would would it take to reach Alpha Centauri, which is 4.37 light years away? The time taken for acceleration can be taken as negligible. **[6]**

2.3 Magnetic Sails [9]

Magnetic sails are a proposed method of propulsion using a static magnetic field to deflect charged particles radiated by the Sun, thus imparting momentum. Technological limitations on magnetic sails include identifying suitable high-temperature superconductors, with the highest currently known only being superconducting below 133 K.

A typical design generates the magnetic field using a large loop of superconducting wire positioned perpendicularly to the direction of charged particle flow (ignore orientation in diagram). For simplicity, take the loop to be a flat 2D disc with a radially symmetric hole.



Assume the solar wind emanates radially from the Sun with a constant velocity of 500km/s. At a distance of 1AU, the proton flow is 5×10^{14} protons/m^2s, and is inversely proportional to the square of distance from the Sun. The magnetic field resulting from the superconductor has an effective proton deflection radius of 20km, within which it deflects solar wind particles perpendicularly from their initial direction.

(a) Assuming the sail reaches thermal equilibrium with no cooling system, find the minimum distance from the Sun for the magnetic sail to function (with currently available superconductors). **[3]**

(b) Find the force by solar wind on the sail at the distance in part (a) **[4]**

(c) Find the mass of the magnetic sail such that it remains stationary (not orbiting) at the distance found in part (a) **[2]**

2.4 Standard Candle [9]

Consider a type 1A supernova in a distant galaxy with a peak luminosity of $5.8 \times 10^9 \, L_\odot$. Suppose you observe this supernova, and find it to have an apparent luminosity 1.7×10^{-8} that of Vega's. Subsequent measurements of its host galaxy finds that the 21cm line has been Doppler shifted to a wavelength of 22.5cm.

For such distant objects, the intensity of measured light is reduced due to the expansion of the universe, and the luminosity distance is no longer equal to the comoving distance. The luminosity distance is given by:

$$d_l = (1 + z) d_c$$

Where d_l and d_c are the luminosity and comoving distances respectively, and z is the redshift of the object in question.

(a) Calculate the redshift to the host galaxy. **[2]**

(b) Calculate the comoving distance to the type 1A supernova. **[4]**

(c) Hence, calculate the Hubble constant, and subsequently the Hubble time. **[3]**

2.5 Radio Array [9]

The SKA (Square Kilometre Array) is a large international radio telescope being built in South Africa and Australia. The SKA1-mid telescope in South Africa (30° S) consists of 133 dish antennas of diameter 12 m observing in the frequency range of 350 MHz to 15.3 GHz. Using interferometry, these dishes will function as a single telescope with a baseline length of 150 km.

An astronomer points one of the dish's antennas towards a radio source with a known incident flux of 1.3×10^{-20} W/m^2.

(a) Assume that the collected photons are evenly distributed across the frequencies 0.95 GHz to 1.76 GHz. What is the average number of photons reaching the detector of each radio dish every second? **[2]**

(b) If instead observing using a frequency of 7.825 GHz, what is the angular resolution of the entire interferometer array in arcseconds? **[2]**

To detect a faraway point source using the telescope, the source's signal must be sufficiently strong compared to the noise level of the telescope. The noise level, σ is defined as follows:

$$\sigma = \frac{2k_B T_{sys}}{A\sqrt{t_i \Delta v}}$$

In terms of the system temperature T_{sys}, aperture area of the telescope A, integration time t_i and bandwidth Δv.

A radio galaxy has a known spectral flux density of 2.5×10^{-3} Jy at an observing frequency of 0.4 GHz. At this frequency, the SKA1-mid array has $T_{sys} = 150$ K and bandwidth 12.5 MHz. The Jansky is a unit frequently used to describe the spectral flux density of radio sources with conversion 1 Jy = 10^{-26} W/m^{2}Hz. The entire radio array is used to observe the galaxy.

(c) Calculate the minimum integration time $t_{i,\text{min}}$ to achieve a 40σ detection (the signal is $40\times$ the noise level σ). **[2]**

(d) What is the minimum spectral flux density of a source located on the celestial equator, such that it may still be observed in the SKA1-mid array for a 30σ detection? Consider the maximum integration time due to the rotation of the Earth. **[3]**

Section B(2): Medium Questions [18]

Write all your responses in the summary answer sheet provided.

2.6 Strömgren Spheres [18]

Strömgren spheres are spheres of ionised hydrogen formed around hot stars of spectral types O and B, where neutral hydrogen (HI) is ionised by the Lyman-continuum photons from the star ($h\upsilon > 13.6$ eV) to form a region of ionised hydrogen (HII-region).

Consider a luminous blue star located within a uniform spherical hydrogen cloud with a density of 10^8 molecules per cubic metre and some temperature T_{HII}. The star emits 10^{49} photons per second, with **all** such photons assumed to be ionizing.

Suppose the hydrogen within the Strömgren sphere is fully ionized. Within the Strömgren sphere, the rate of recombination and ionisation is balanced. The rate of recombination is proportional to the number density of protons (n_p) and of electrons (n_e), as well as a temperature-dependent constant of recombination α. At T_{HII}, $\alpha(T_{\mathrm{HII}}) = 10^{-19}$ $\mathrm{m^3 s^{-1}}$. For simplicity, ignore the effect of any photons created via recombination.

(a) Derive an algebraic expression for the radius of a Strömgren Sphere and calculate its value using the given parameters. Express your answer in light years. **[5]**

(b) Estimate the timescale (in years) it takes for such a sphere to form, if recombination does not take place. **[2]**

In reality, 40% of recombination events actually emit line photons that will further ionize the hydrogen cloud.

(c) Hence, derive a new expression for the radius of a Strömgren Sphere taking this percentage into account. **[2]**

Hint: $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ for $|r| < 1$

The Bubble Nebula is a famous example of a Strömgren sphere located approximately 3400 pc from Earth. It is approximately spherical with observed angular diameter $3'$ and surface brightness of $14.94\,\mathrm{mag/arcmin^2}$. Ignore the contribution of the luminous blue star in the brightness of the nebula and assume the sphere is of uniform density.

(d) Calculate the apparent magnitude of the Bubble Nebula as observed from Earth. **[2]**

(e) Find the difference in apparent magnitude of the Bubble Nebula as observed from the centre of the nebula as compared to your answer in (d). **[7]**

3 Section C: Long Question [36]

Write all responses on the answer sheets provided. The marks are stated in the brackets [ ] at the end of each sub-part.

3.1 Part 1: Lagrange Points [8]

The Lagrange points of a system are points of gravitational equilibrium in a two-body system. The L4 and L5 points are naturally stable and are known to harbour objects such as the Greek and Trojan asteroids in the Sun-Jupiter system.

On 25 Dec 2021, the JWST was launched towards the L2 point which is a favourable location as it offers a view into deep space unobstructed by neither the Sun nor Earth. Although the point is not intrinsically stable, station keeping manoeuvres will allow the JWST to remain in orbit for the duration of its mission.

(a) For a two-body system with masses $M_1 \gg M_2$, find the distance between L2 and the secondary body with mass M_2 in terms of the orbital separation a.

Hence or otherwise, calculate the distance between L2 and Earth for the Sun-Earth system. **[8]**

3.2 Part 2: Roche Lobes [13]

When studying the gravitational interactions of a binary system, it is also useful to look at the Roche lobe of each body. This is the region around each component where matter is gravitationally bound to and will orbit it.

The shape of the donor's Roche lobe is non-spherical with shape dependent on the mass ratio of the binary, defined as $q \equiv M_1/M_2$. In a theoretical treatment, it is convenient to approximate the Roche lobe by a spherical volume with radius given by Eggleton (1983):

$$R_L \approx a\,\frac{0.44q^{0.33}}{(1+q)^{0.2}} \qquad \text{for } 0.1 < q < 10$$

(b) Briefly describe how L1 is related to the boundary of a Roche lobe **[2]**

(c) Show that the ratio of Roche lobe radii for $0.1 < q < 10$ is given by **[4]**

$$\frac{R_{L,1}}{R_{L,2}} \approx q^{0.46}$$

In a binary system, Roche lobe overflow occurs when the stellar envelope of a component expands past its Roche lobe, and mass is siphoned onto the accreting object through L2. With mass-transferring binary systems, strong tidal forces tend to circularize the orbit relatively fast.

Conventionally, the mass ratio for such a system is defined $q \equiv M_d/M_a$, where M_d and M_a are the masses of the donor and accretor respectively. In a semi-detached binary, only one component overflows its Roche lobe. Effectively, the donor's radius is equivalent to the radius of its Roche lobe.

(d) Express the orbital period T of a semi-detached binary system in terms of the mean density of the donor star $\bar{\rho}$ and other relevant quantities. **[7]**

3.3 Part 3: Evolution [15]

The Roche lobe index ζ_L relates the response of the donor star's Roche lobe to the donor star's mass:

$$R_L \propto M_d^{\zeta_L}$$

In studying the dynamics and evolution of these systems, it is convenient to express the change in an observed quantity over time as a fraction of the original quantity. For example, if a quantity Q can be expressed as

$$Q^n = k\frac{X^a Y^b}{Z^c}$$

For some constants a, b, c, k and n, and quantities X, Y and Z, then the rate at which Q changes is given by

$$n\frac{\dot{Q}}{Q} = a\frac{\dot{X}}{X} + b\frac{\dot{Y}}{Y} - c\frac{\dot{Z}}{Z}$$

The notation $\dot{Q}$ denotes the change in quantity Q over time. For instance, the change in masses of the stellar components are related by

$$\dot{M}_a = -\eta \dot{M}_d$$

This models mass transfer with $0 \leq \eta \leq 1$ being the fraction of mass outflow from the donor star that is captured by the accretor. In reality, usually $\eta < 1$ since processes such as stellar wind carry mass out of the system.

The angular momentum of a binary system is given by

$$J^2 = G\frac{M_1^2 M_2^2}{M_1 + M_2}a(1 - e^2)$$

(e) For a circularized semi-detached binary undergoing conservative mass transfer:

- **(i)** Determine the value of $\dot{J}/J$ and $\dot{e}/e$ **[2]**
- **(ii)** Hence or otherwise, express the change in orbital separation $\dot{a}/a$ in terms of other relevant quantities of the system **[6]**

(f) Derive $\zeta_L(q)$ for the case of conservative mass transfer. **[4]**

Hint: *The expression has the form $\zeta_L(q) = a + bq$ for constants a and b*

(g) A study of a semi-detached binary system determines the masses of the donor and accretor to be $1.49M_\odot$ and $3.22M_\odot$ respectively, with a period of 0.901 days. Calculate the density of the donor. Hence or otherwise, suggest (with explanation), which of the following the donor is likely to be: **[3]**

- Main Sequence star
- Giant star
- White dwarf
- Neutron star

End of Theory Paper