SAO

2019 SAO Answers

S/N	Name	Setter
Short Questions – 2 Marks each, 12 Marks total		
1.1	Rigel	Arnav
1.2	Galilean Moons	Tommy
1.3	Achromatic Lens	Tommy
1.4	The Flying Eagle	Darren
1.5	Perseverance	Darren
1.6	Cosmological Scale Factor	Tommy
Medium Questions – 9 Marks each, 7 Choose 6, 54 Marks total		
2.1	Spacefill	Darren
2.2	Sunrise at Le Havre	Darren
2.3	Space City	Zhiyuan
2.4	Geminids	Vint Ve
2.5	Gravitational Mapping	Vint Ve
2.6	Interstellar Proton	Zhiyuan
2.7	Age of the Universe	Tommy
Long Question – 34 Marks		
3.1	Star Birth [9]	Tommy
3.2	Star Mid-Life [15]	Tommy
3.3	Star Death [10]	Tommy
Data Analysis Question – 40 Marks		
4	Seeing Double	Darren/Vint Ve
Practical MCQ – 2 Marks each, 10 Marks total		
5.1	Holiday in Sapporo	Ming Jie
5.2	Telescope Setup	Ming Jie
5.3	Visual Equipment	Ming Jie
5.4	Night Vision Light	Ming Jie
5.5	Capturing Solar Transit	Ming Jie
Star Chart – 28 Marks		
6.1	Constellation Identification [6]	Ziming
6.2	Winter Sky [9]	Ziming
6.3	Summer Sky [13]	Ziming
Practical Experience – 22 Marks		
7.1	Newtonian Telescope Details [7]	Ziming
7.2	Binoculars and Stargazing Objects [12]	Ziming
7.3	DSOs Crossing Horizon [3]	Ziming

1.1 ***Rigel [2]***

Read the peak wavelength value off the graph to be around 0.24 μm, yielding a temperature of ~12000K(S.F. rounding), upon which luminosity formula can be used:

$$L_{star}/L_{\odot} = (A_{star} * \epsilon * \sigma * T_{star}{}^4)/(3.828 * 10^{26}W)$$
$$= ((4 * \pi * (79 * 6.957 * 10^8)^2) * 1 * 5.670 * 10^{-8} * 12000^4)W/(3.828 * 10^{26}W) \approx 1.19 * 10^5$$

Final Answer: $1.19 * 10^5 L_{\odot}$

1.2 ***Galilean Moons [2]***

Synodic period of moon 1 and moon 2 of periods P_1 and P_2 each are

$$\frac{1}{P_S} = \frac{1}{P_1} - \frac{1}{P_2}$$

[0.5] for stating equation of synodic period correctly Resonant orbits happen if $P_s = mP_1$, where $m \in \mathbb{Z}^+$, i.e. a positive integer. The orbital periods of Io, Europa, Ganymede and Callisto are 1.769 days, 3.551 days, 7.155 days, 16.69 days respectively. **[0.5] for getting orbital periods for the 3 moons Ganymede, Europa and Io** Ratio is 1 to 2 to 4 **[1] for stating ratio clearly, showing some form of pairwise division between the respective periods**

Final Answer: 1:2:4 (Io, Europa, Ganymede)

1.3 ***Achromatic Lens [2]***

Write the two simultaneous equations for red and blue light. Each correct simultaneous equation is worth **[0.5] each**. Final answer is $R_A = 150$ mm and $R_B = -300$ mm **[0.5] each for correct final answer**

Final Answer: R_A = 0.09mm and R_B = -0.18mm

1.4 ***The Flying Eagle [2]***

Let the parallax angle be p.

$$M = m - 5\, lg\, lg\, \frac{d}{10\, pc} = m - 5\, lg\, lg\, \frac{1}{10 \times p} = +2.21$$

Final Answer: M_{Altair} = +2.21

1.5 ***Perseverance [2]***

$$\sin\theta = \frac{1.22\lambda}{D}$$
$$\frac{2.3\text{ m}}{17300\text{ km}} = \frac{(1.22 \times 500 \times 10^{-9}\, m)}{D}$$
$$D = 4.6\text{ m}$$

Final Answer: D = 4.6m

1.6 ***Cosmological Scale Factor [2]***

Fraction of critical density contributed by:

- radiation is sum of that of photons and neutrinos, so is 9.0×10^{-5}
- matter is sum of that of dark matter and baryons, so is 0.69

Then, we have

$$(1+z_{eq})^4\Omega_{r,0} = (1+z_{eq})^3\Omega_{m,0} \implies z_{eq} = 3443$$

Final Answer: Z_{eq} = 3443

2.1 ***Spacefill [9]***

Assume the black hole is a perfect black body:

$$\begin{aligned} L &= 4\pi\sigma R^2 T^4 \\ &= 4\pi\left(\frac{\pi^2 k_B^4}{60\hbar^3 c^2}\right)\left(\frac{4G^2M^2}{c^4}\right)\left(\frac{\hbar^4 c^{12}}{2^{12}\pi^4 G^4 M^4 k_B^4}\right) \\ &= \frac{\hbar c^6}{15360\pi G^2 M^2} \end{aligned}$$

[M1 – correct Stefan-Boltzmann formula]

[M1 – correct substitutions]

[M1 – correct expression for L]

Let $\dot{m} = \frac{2\times10^{12}}{86400\times365.25}$ kg/s be the rate at which mass will be dumped into the black hole.

[M1 – correct mass flux used]

Using the mass-energy equivalence, $L = \dot{m}c^2$. Hence

[M1 – correct use of mass-energy equivalence]

$$\begin{aligned} \dot{m} &= \frac{\hbar c^4}{15360\pi G^2 M^2} \\ M &= \frac{c^2}{G}\sqrt{\frac{\hbar}{15360\pi\dot{m}}} \\ &= 2.50 \times 10^5 \text{ kg} \end{aligned}$$

[M1 – correct expression for M]

[A1 – correct final value for M within 0.1×10^5 kg]

The radius of the black hole is

$$\begin{aligned} r &= \frac{2GM}{c^2} \\ &= \sqrt{\frac{\hbar}{3840\pi\dot{m}}} = 3.71 \times 10^{-22} \text{ m} \end{aligned}$$

[A1 – correct value for r within 0.2×10^{-22} m]

The temperature of the black hole is

$$T = \frac{\hbar c^3}{8\pi G M k_B} = 4.91 \times 10^{17} \text{ K}$$

[A1 – correct value for T within 0.2×10^{17} K]

2.2 ***Sunrise at Le Havre [9]***

Let ϕ be the latitude of the observer, H be the local hour angle, h be the altitude, and A be the azimuth.
Using spherical trigonometry,

$$\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos H$$
$$\cos H = \frac{\sin h - \sin\phi \sin\delta}{\cos\phi \cos\delta}$$

[M1 – correct cosine rule]
[M1 – correct formula for H]

The solar disc has a radius of 16’, hence the Sun’s centre will be at 34’ + 16’ = 50’ below the horizon at the point of sunrise.

[M1 – correct value of solar altitude]

$$\cos H = \frac{\sin(-50') - \sin(49°\ 29') \sin(-18°\ 05'\ 38.20'')}{\cos\phi(49°\ 29') \cos(-18°\ 05'\ 38.20'')}$$
$$H = 68.975°$$

[M1 – correct value of hour angle (sign notwithstanding) within 0.2°]

The local hour angle at sunrise is thus $H = -68.975° = -4\text{ h }35\text{ m }54\text{ s}$.
This corresponds to a local apparent solar time of $12\text{ h} - 4\text{ h }35\text{ m }54\text{ s} = 7\text{ h }24\text{ m }6\text{ s}$.

[M1 – correct local apparent solar time, within 2 min]

Finally, to convert to local mean solar time, we subtract the equation of time:

$$T = 7\text{ h }24\text{ m }6\text{ s} - (+15\text{ m }33\text{ s})$$
$$= 7\text{ h }09\text{ m}$$

Therefore, sunrise occurred at 7:09 am local mean solar time on that day.

[A1 – correct sunrise time within 2 min]

Our calculation corresponds very closely with that obtained from https://www.esrl.noaa.gov/gmd/grad/solcalc/
To find A at sunrise:

$$\sin\delta = \sin\phi \sin h + \cos\phi \cos h \cos(360° - A)$$
$$= \sin\phi \sin h + \cos\phi \cos h \cos A$$
$$\cos A = \frac{\sin\delta - \sin\phi \sin h}{\cos\phi \cos h}$$

[M1 – correct cosine rule]
[M1 – correct formula for A
Accept variations as long as concept is correct]

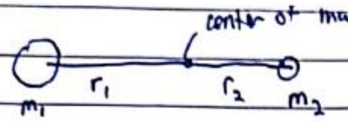
We must be careful to pick the correct quadrant for A.
Using the values obtained gives us $A = 117°$. Therefore, the azimuth of the sun at sunrise is about 117°.

[A1 – correct value of azimuth within 2°]

2.3 ***Space City [9]***

2.3(a)

Ans $F_{rad} = m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = G\frac{m_1 m_2}{(r_1+r_2)^2} = G\frac{m_1 m_2}{r^2} \quad (r = r_1 + r_2)$



$m_1 r_1 \omega^2 = \frac{G m_1 m_2}{r^2} \quad -①$

[1] for eqn ① and ②

Since $m_1 r_1 = m_2 r_2$, $r_2 = \frac{m_1}{m_2} r_1$, $r = r_1 + r_2 = \left(1 + \frac{m_1}{m_2}\right) r_1$

$\frac{1}{r_1} = \frac{m_1 + m_2}{m_2 r} \quad - ②$

From ① and ② $\omega^2 = G\frac{m_2}{r^2}\frac{1}{r_1} = G\frac{m_2}{r^2}\frac{m_1+m_2}{m_2 r} = G\frac{m_1+m_2}{r^3}$

$\left(\frac{2\pi}{T}\right)^2 = \omega^2$

$\frac{T^2}{4\pi^2} = \frac{r^3}{G(m_1+m_2)}$

$T^2 = \frac{4\pi^2 r^3}{G(m_1+m_2)}$

[1] for to get to expr of T^2

$= \frac{4\pi^2 (0.10)^3}{6.67\times10^{-11}(1.0+1.5)} = 2.3675\times10^8$

$T = 1.5386\times10^4 = 1.5\times10^4 s$ (or 4.3hr) [1]

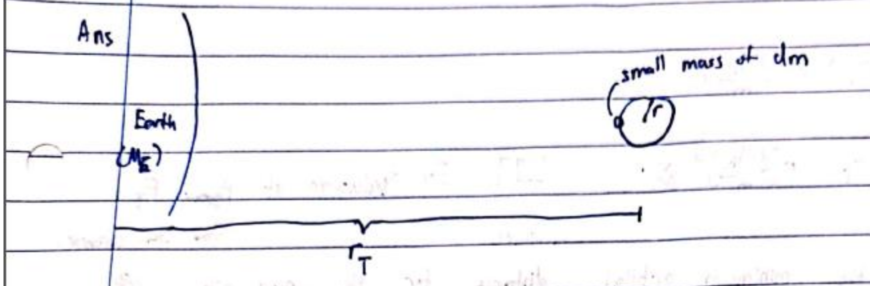
Q1(b)

Part B: Prove ~~Derive~~, for a spherical satellite of radius r and mass m, that its Roche limit is as below. [5] r_T is the distance of center of Earth to the center of the satellite.

$$r_T = r\left(2\frac{M_E}{m}\right)^{\frac{1}{3}}$$

*Hint, approximation can be used.

Ans



The satellite we assume is falling freely into the Earth. The force that holds small mass dm to the satellite is its gravitational attraction to the satellite. The force tearing it away is the tidal force. It can be seen as the difference in gravitational field strength at dm and r (multiplied by dm), or the attraction of the Earth towards dm minus the fictitious force due to the satellite accelerating to the earth.

↗ the fictitious ~~force term~~ acceleration

[2] to realise how to get to the tidal force and for looking at dm.

$$\frac{\text{Tidal force}}{dm} = g_T = G\frac{M_E}{(r_T-r)^2} - G\frac{M_E}{r_T^2}$$

$$= GM_E\frac{r_T^2-(r_T-r)^2}{r_T^2(r_T-r)^2} \quad [1]$$

$$= GM_E\frac{2r_Tr-r^2}{r_T^2(r_T-r)^2}$$

But since $r \ll r_T$

$$2r_Tr - r^2 \approx 2r_Tr$$

$$r_T^2(r_T-r)^2 \approx r_T^4$$

$$\approx GM_E\frac{2r_Tr}{r_T^4}$$

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$$\frac{\text{Tidal force}}{dm} = g_T = G\frac{M_E}{(r_T - r)^2} - G\frac{M_E}{r_T^2}$$

how to get to the tidal force and for looking at dm.

$$= GM_E \frac{r_T^2 - (r_T - r)^2}{r_T^2 (r_T - r)^2} \quad [1]$$

$$= GM_E \frac{2r_T r - r^2}{r_T^2 (r_T - r)^2}$$

But since $r \ll r_T$

$2r_T r - r^2 \approx 2r_T r$

$r_T^2 (r_T - r)^2 \approx r_T^4$

$$\approx GM_E \frac{2r_T r}{r_T^4}$$

$$= GM_E \frac{2r}{r_T^3} \quad [1] \text{ for correct estimation used}$$

Then $F_{grav} = F_T$

$$Gm\,dm \frac{1}{r^2} = GM_E\,dm \frac{2r}{r_T^3}$$

$$m\frac{1}{r^2} = M_E \frac{2r}{r_T^3}$$

$$m r_T^3 = M_E 2r^3$$

$$r_T^3 = \frac{2M_E}{m} r^3$$

$$r_T = r\left(\frac{2M_E}{m}\right)^{\frac{1}{3}}$$

[1] For equation of $F_{grav} = F_T$ for final answer.

altitude

Hence ...

Q1(c) Hence, find the minimum orbital altitude for the space city, assuming it is spherical, its mass is 1.0*10^10kg and its radius is 1km. [1]

$$r_T = 1.00 \times 10^3 \times \left(\frac{2 \times 5.97 \times 10^{24}}{1.0 \times 10^{10}}\right)^{\frac{1}{3}} = 1.06088 \times 10^8\ \text{m}$$

Orbital ~~distance~~ altitude is $r_T - R_E = 1.06088 \times 10^8 - 6371 \times 10^3 = 9.9717 \times 10^7\ \text{m}$

$= 1.0 \times 10^8\ \text{m}$

or $1.0 \times 10^5\ \text{km}$

[1]

2.4 ***Geminids [9]***

Using the formula for angular velocity per unit mass, and the vis-viva equation, we get the tangential and total velocities of the meteoroids, v_T and v, as well as the velocity of the Earth, v_E.

$$\frac{L}{\mu} = \sqrt{GMa(1-e^2)} = v_T R \;\Rightarrow\; v_T = \frac{\sqrt{GMa(1-e^2)}}{R} = 15\,319\ ms^{-1}\ [1]$$

$$v^2 = GM\left(\frac{2}{R} - \frac{1}{a}\right) \;\Rightarrow\; v = \sqrt{GMa\left(\frac{2}{R} - \frac{1}{a}\right)} = 32\,838\ ms^{-1}\ [1]$$

$$v_E = \sqrt{\frac{GM}{R}} = 29\,788\ ms^{-1}$$

We also need the rotational velocity of the Earth's surface

$$v_{rot} = \frac{2\pi R_E}{t} = 465\ ms^{-1}$$

From these, and using Pythagoras' theorem, we can get the relative tangential and radial velocities

$$v_R = \sqrt{v^2 - v_T^2} = 29\,046\ ms^{-1}$$

$$v_{T,rel} = v_T - (v_E + v_{rot}) = -14\,934\ ms^{-1}[2]$$

$$v_{rel} = \sqrt{v_R^2 + v_{T,rel}^2} = 32\,660\ ms^{-1}[1]$$

For (b), we first find the angle between the antisolar point and the radiant, θ.

$$\theta = tan^{-1}\left(\frac{v_{T,rel}}{v_R}\right) = 27.210°[1]$$

Dec 14 is around 7 days before winter solstice, thus the sun's RA will be about $18 - 7\left(\frac{24}{365}\right) = 17^h 32^m$[1]

Hence the radiant's RA will be $17^h 32^m + 12 + 27.210\left(\frac{24}{360}\right) - 24 = 7^h 20^m$[2]

2.5 ***Gravitational Mapping [9]***

Method 1: Lorentz transforms

We start from the reference frame of the neutron star, with the x-axis parallel to the pulse's path. Let $L(\beta)$ denote a Lorentz transform (shown below), and $R(\theta)$ denote rotation of coordinate axes. In this frame, the pulse has a frequency of f_1, momentum four-vector p, and travels between two points on the circular orbit separated by an angle φ. Note that $\varphi \neq 67.210°$, since the satellites move while the light travels. Thus, the velocity vectors of both satellites are at an angle $\frac{\varphi}{2}$ from the x-axis.

$$L(\beta) = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can get the momentum four-vectors of the pulse in Tom and Jerry's frames as follows:

$$p_T = L(\beta)R\left(\frac{\varphi}{2}\right)p$$

$$p_J = L(\beta)R\left(-\frac{\varphi}{2}\right)p$$

The frequency of light is determined by the time component of its momentum four-vector. Furthermore, from $L(\beta)$ we can see that the y-component (perpendicular to direction of Lorentz transform) does not affect the frequency of light after transformation.

Therefore, because $R\left(\frac{\varphi}{2}\right)p$ and $R\left(-\frac{\varphi}{2}\right)p$ only differ in their y-components, we can conclude that p_T and p_J have the same time components, and the pulse's frequency in both frames are **1575.25MHz**.

Method 2: Rotating Frame

We can also use a frame centered on the neutron star, rotating such that both satellites are stationary. Since this is a non-inertial frame, it will introduce an additional "gravitational" field. However, this additional field is rotationally symmetric, and since the satellites are the same distance from the center of the neutron star, the result is that Tom and Jerry are equipotential. Thus, when the pulse reaches Jerry, **its energy (and hence frequency) must be the same as when it was emitted**.

2.6 ***Interstellar Proton [9]***

Ans:

Energy $E = \gamma m c^2$ $\quad \gamma = \frac{E}{mc^2}$

$$= \frac{1.0\times10^{12} \times 1.60\times10^{-19}}{1.67\times10^{-27}\times(3.00\times10^{8})^2} = 1064.54.$$ ①

$\delta = 1-\frac{v}{c} = \frac{c-v}{c}$

$\frac{c+v}{c} + \frac{c-v}{c} = \frac{2c}{c} = 2$ ①

$\frac{c+v}{c} = 2-\delta$

$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ $\quad \frac{1}{\gamma^2} = 1-\frac{v^2}{c^2}$

$$= \frac{c^2-v^2}{c^2}$$

$$= \left(\frac{c-v}{c}\right)\left(\frac{c+v}{c}\right)$$ ①

$$= (2-\delta)\delta$$

Hence, $\frac{1}{\gamma^2} = \frac{1}{1064.54^2} = 8.8242\times10^{-7}$

$$\delta^2 - 2\delta + \frac{1}{\gamma^2} = 0$$ ② M1, A1.

$\delta = 2.00$ or 4.4121×10^{-7}
(rej.)

$\delta = 4.4\times10^{-7}$ (no units)

Ans) $F = \gamma m a \qquad a = \frac{v^2}{r}$ due to circular motion.

Hence $F = \gamma m \frac{v^2}{r} \iff r = \frac{\gamma m v^2}{F} = \frac{\gamma m}{F} v^2$. — (a)

} 2 mark

But, $\frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2}$

$1 - \frac{1}{\gamma^2} = \frac{v^2}{c^2}$

Hence $v^2 = c^2\left(1 - \frac{1}{\gamma^2}\right)$ — (b)

} 1 mark

sub (b) into (a)

Hence $r = \frac{\gamma m v^2}{F} = \frac{\gamma m}{F} c^2 \left(1 - \frac{1}{\gamma^2}\right)$

$= \frac{E}{F}\left(1 - \frac{1}{\gamma^2}\right)$

$\approx \frac{E}{F}$ since $\frac{1}{\gamma^2} \approx 10^{-7}$

} 1 mark

Thus $r = \frac{E}{F}$ #

2.7 *Age of the Universe [9]*

Using $E = mc^2$, one can consider that the energy within the expanding volume of unit comoving radius is

$$E = \frac{4\pi}{3}a^3\rho c^2$$

and the change of energy is

$$\frac{dE}{dt} = 4\pi a^2\rho c^2\frac{da}{dt} + \frac{4\pi}{3}a^3\frac{d\rho}{dt}c^2$$

while the change in volume with time is

$$\frac{dV}{dt} = 4\pi a^2\frac{da}{dt}$$

[1.5] for getting all of the above Assuming a reversible expansion, change in entropy $dS = 0$ such that by first law of thermodynamics $dE + pdV = TdS$, we obtain

$$\frac{d\rho}{dt} + 3\frac{1}{a}\frac{da}{dt}\left(\rho + \frac{p}{c^2}\right) = 0$$

where p is the pressure and not momentum. This is known as the fluid equation. **[1.5] for this part** Differentiating the Friedmann equation with respect to time yields

$$2\frac{1}{a^3}\frac{da}{dt}\left(a\frac{d^2a}{dt^2} - \left(\frac{da}{dt}\right)^2\right) = \frac{8\pi G}{3}\frac{d\rho}{dt} + 2\frac{kc^2}{a^3}\frac{da}{dt}$$

Substituting $\frac{d\rho}{dt}$ from the fluid equation, one obtains

$$\frac{1}{a}\frac{d^2a}{dt^2} - \left(\frac{1}{a}\frac{da}{dt}\right)^2 = -4\pi G\left(\rho + \frac{p}{c^2}\right) + \frac{kc^2}{a^2}$$

Using Friedmann equation again, one obtain the acceleration equation

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$

[3] for this part We consider Pressureless dust: with $p = 0$, **[1] for this part**

(Not required but for the marker's understanding:) the fluid equation becomes

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \implies \frac{1}{a^3}\frac{d}{dt}(\rho a^3) = 0 \implies \rho = \rho_0 a^{-3}$$

Substituting this into the Friedmann equation with $k = 0$, we obtain

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3a}$$

which has the solution a directly proportional to $t^{2/3}$. As we have fixed $a_0 = 1$, we get

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

Note: [0] for solving this DE

In this solution, the Universe expands forever but an ever decreasing rate:

$$H(t) \equiv \frac{\dot{a}}{a} = \frac{2}{3t}$$

[1] for finding the expression of H after differentiating a

This is one of the classic cosmological solutions known as the Einstein-de Sitter cosmology. Then, $t_0 = \frac{2}{3H_0}$ such that t_0 is the current age of the Universe and $\frac{1}{H_0}$ is known as the Hubble time, which is 14 Giga years. $t_0 = 9.6$ billion years. **[1] for correct final answer with relevant working**

3.1 **Star Birth**

3.1(a)

Ans Imagine pulling the earth apart shell by shell successively, starting with the ~~m~~ outermost shell.

Mass of shell $m_{shell} = 4\pi r^2 \rho \Delta r$

Mass of sphere inside shell $m_{int} = \frac{4}{3}\pi r^3 \rho$

Energy to remove shell to infinity is -tive of normal GPE definition

(2) for correct m_{shell}, m_{int}, and approach.

$$\Delta U = G\frac{m_{shell}\, m_{int}}{r} = \frac{G}{r}(4\pi r^2 \rho \Delta r)\left(\frac{4}{3}\pi r^3 \rho\right) = G\frac{16}{3}\pi^2 \rho^2 r^4 \Delta r$$

(1) ~~1 for correct GPE formula, and~~ ~~1 for correct sub.~~

Since ρ is constant $\rho = \frac{M}{\frac{4}{3}\pi R^3}$

$\rho^2 = \frac{M^2}{\frac{16}{9}\pi^2 R^6}$

$$\Delta U = G\frac{16}{3}\pi^2 \frac{M^2}{\frac{16}{9}\pi^2 R^6} r^4 \Delta r = G\,3\frac{M^2}{R^6} r^4 \Delta r = 3G\frac{M^2}{R^6} r^4 \Delta r$$

(3) ~~(2)~~.

~~1 for sub in ρ.~~

~~1 for final answer~~

1 for evaluating ρ,

1 for final correct answer.

3.1(b)

b) The condition for collapse, by Virial Theorem, **[1] for stating the inequality correctly**is

$$\frac{3MkT}{\mu m_H} < \frac{3}{5}\frac{GM^2}{R}$$

where $N = \frac{M}{\mu m_H}$ such that m_H is the mass of Hydrogen. **[1] for correct expression** Note that the radius is

$$R = \left(\frac{3M}{4\pi\rho}\right)^{1/3}$$

since the cloud has constant density. Then the Jeans Mass is

$$M := M_J = \left(\frac{5k_BT}{G\mu m_H}\right)^{1.5}\left(\frac{3}{4\pi\rho}\right)^{0.5}$$

Substituting R back into M, we get

$$R := R_J = \sqrt{\frac{15k_BT}{4\pi G\mu m_H\rho}}$$

Correct expressions and working award **[1]**

3.2 **Star Mid-Life**

1.2 Part 2: Star Mid-Life [15]

a) The inward force on the volume element $\Delta r \Delta A$ is

$$P(r)\Delta A + \frac{\Delta P}{\Delta r}\Delta r \Delta A - P(r)\Delta A = \frac{\Delta P}{\Delta r}$$

[1] for getting the pressure driven force correct Using Newton's Second Law,

$$-\rho(r)a_r \Delta A \Delta r = \rho(r)g(r)\Delta A \Delta r + \frac{\Delta P}{\Delta r}\Delta r \Delta A$$

Both the pressure-driven force and gravity are acting inwards. **[1] for getting the desired equation correct, [1] for saying hydrostatic equilibrium is achieved when the pressure-driven force is equal to that of gravity.**

b) Since the gas satisfies the ideal gas law, where ρ is directly proportional to P/T, then

$$\frac{\Delta \rho}{\rho} = \frac{\Delta P}{P} - \frac{\Delta T}{T}$$

[0.5] for getting this relation from ideal gas law Moreover, for an adiabatic process,

$$\frac{\delta \rho}{\rho} = \frac{1}{\gamma}\frac{\delta P}{P}$$

[0.5] for getting this relation from adiabatic process where we distinguish $\delta\rho$ and $\Delta\rho$ such that the former is for the displaced packet of gas and the latter is for the surroundings. We also assume that the pressure inside the packet responds rapidly to the new environment, i.e. $\delta P = \Delta P$ but there is insufficient time for heat conduction, then for convection to occur, the density of the displaced gas must be less than that of the surroundings **[1] for stating this condition for convection**

$$\delta\rho < \Delta\rho \implies \frac{1}{\gamma}\frac{\delta P}{P} < \frac{\Delta P}{P} - \frac{\Delta T}{T}$$

[0.5] for getting this inequality Since $\delta P = \Delta P$ **[0.5] for stating this and explaining why. explanation stated earlier in the solution**, we obtain

$$\frac{\Delta T}{T} < \frac{\gamma - 1}{\gamma} \frac{\Delta P}{P}$$

Rearranging terms and divide by Δr, we get our desired equation. **[0.5] for getting this equation**

On an unrelated note, the temperature and pressure gradients are both negative in this equation. Convection requires the temperature to fall off rapidly with height.

Using the hydrostatic equilibrium relation for pressure gradient, we get

$$\frac{\Delta T}{\Delta r} < -\frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{Gm(r)\rho(r)}{r^2}$$

[0.5] for getting this equation. You may award mark for getting this equation even if the previous equation was not satisfiedly shown.

c) By rearranging the given equations, we get

$$\frac{\Delta T}{\Delta r} = -\frac{3\rho(r)\kappa(r)}{4ac(T(r))^3} \frac{L(r)}{4\pi r^2}$$

This is the thermal gradient for radiation. **[1] for getting this equation correct**.

Equating this with the thermal gradient for convection,

$$\frac{3\rho\kappa}{4acT^3} \frac{L(r)}{4\pi r^2} = \frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{Gm(r)\rho}{r^2} \implies \frac{L(r)}{m(r)} = \frac{\gamma - 1}{\gamma} \frac{16\pi Gc}{\kappa} \frac{P_r}{P}$$

This is the critical value for the ratio of luminosity to mass in order for both gradients to be equal. **[1] for getting the conditional expression correct**

Note that anything below this, energy can be transported from the core by radiative diffusion without inducing convection. Conversely, anything above, convection dominates.

d) Plugging into the earlier equation, we get a value of 1.5×10^{-3}W kg^{-1}, which is slightly greater than the actual value. Hence, critical value for convection is not reached and thus radiation is dominant. **[1] for both the calculated value and show comparison with the actual value**

SΛO

e) Assuming the power generated at the core is uniform throughout the star **[1] for stating this assumption**, then the luminosity is

$$L = 1.35 \times 10^{-3} \times 0.69 \times 2 \times 10^{30} = 1.86 \times 10^{27}\text{W}$$

[1] for getting Luminosity correct Using Wien's Law, the peak wavelength is

$$\lambda = \frac{0.0029}{4715} = 615\text{nm}$$

assuming that the only wavelength the star radiates in is the peak wavelength **[0.5] for stating monochromatic assumption correctly in context to Wien's Law, and [0.5] for the right peak wavelength or peak frequency** Then the number of photons per second is

$$N = \frac{L\lambda}{hc} = 5.8 \times 10^{45}$$

[1] for correct final answer

(f) $\quad L = 4.85x10^{25}, 1.5x10^{44} photons$

3.3 **Star Death**

1.3 Part 3: Star Death [10]

a) Assumptions are **[1] each**:

- star is made up of only hydrogen and that all the hydrogen is in ionized form
- nuclear fusion has stopped at this stage

Total energy of the star is assumed to be only its gravitational energy E_G and electronic energy E_e. **[0.5]** We thus find the equilibrium radius which occurs when $dE/dR = 0$. **[0.5]**

$$\frac{dE_G}{dR} = -\frac{dE_e}{dR}$$

$$\frac{3GM^2}{5R_0^2} = KN_e^{5/3}\frac{2}{R_0^3 m_e} \implies R_0 = \frac{10KN_e^{5/3}}{GM^2}$$

Since all hydrogen is ionized, $N_p = N_e$ and since $m_e << m_p$, $N_e \approx \frac{M}{m_p}$ **[1]** and hence

$$R_0 = \frac{10K}{3Gm_e m_p^{5/3} M^{1/3}} = 2.6 \times 10^7 \text{m}$$

Correct final answer **[1]**

b) Let separation distance be x,

$$N_e \frac{4}{3}\pi x^3 \approx \frac{4}{3}\pi R_0^3$$

Then, $x = 2.6 \times 10^{-19}$m. **[2] for correct equation and final answer** Using de-Broglie relation, we have $v = \frac{h}{2m_e x} = 8.7 \times 10^7m/s= 0.29c$. **[1] for correct final answer**

c)

$$\frac{dE_G}{dR} = -\frac{dE_e}{dR}$$

$$\frac{3GM_C^2}{5R^2} = \frac{\pi^2}{4^{4/3}}\left(\frac{3}{\pi}\right)^{5/3}\frac{\hbar c}{R'^2}N_e^{4/3} \implies M_c = \frac{3(5^3\pi)^{1/2}}{16m_p^2}\left(\frac{\hbar c}{G}\right)^{1.5}$$

which is 6 solar masses. **[2] for correct working and final answer**

4 **Data Analysis**

Solutions

(a) The table above is incomplete.
P represents the number of periods elapsed since the first primary minimum at D = 44066.4870.
Copy the table onto your answer sheet, and complete the column with the appropriate values for each data point.

The period appears to be about 4 days. Hence the values of P should be:

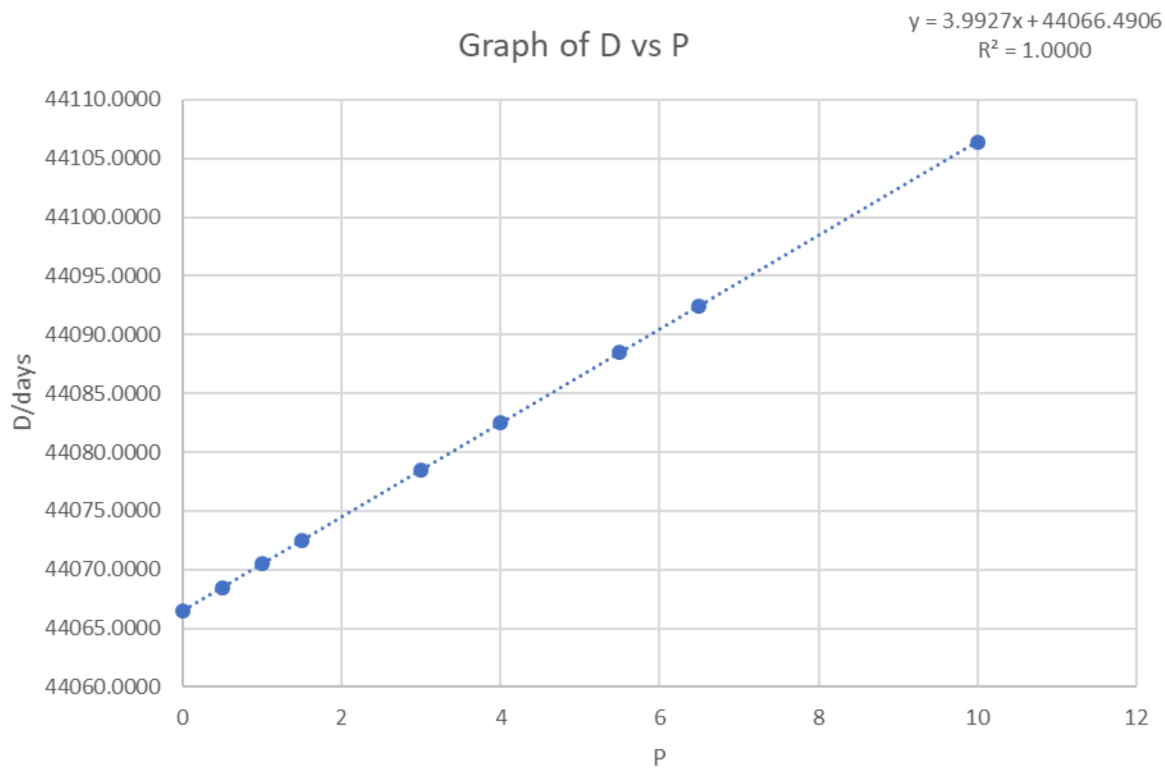
D/days	$\Delta m(V)$	P
44066.4870	0.876	0
44068.4697	0.221	0.5
44070.4973	0.890	1.0
44072.4988	0.231	1.5
44078.4522	0.937	3.0
44082.4446	0.864	4.0
44088.4629	0.200	5.5
44092.4240	0.231	6.5
44106.4225	0.921	10.0

[A0.5 for each correct value]

[Subtotal: 4]

(b) Using the values from your table, plot a suitable graph to determine the period T of the variation in Z Herculis' visual magnitude. Give your answer to 3 significant figures.

Plot a graph of D against P. The gradient of the best-fit line will give us the period P. It is acceptable to subtract e.g. 44000 from all the values of D to make the graphing slightly easier.



The best-fit line through the points has a gradient of 3.9927 days.
We will use a value of T = 3.9927 days for our subsequent calculations.
The actual period from Tümer (1984) is 3.9928012 days, which is very close to our estimate.

Marking points

Criteria	Requirements	Marks
Points	**Plotted in correct positions** **Deduct 1 for each point in wrong position**	**3**
Axes	**Axes labelled correctly with units. Deduct 0.5 for each axis not fully labelled**	**1**
Scale	**Chosen such that points should take up at least two-thirds of the graph paper**	**1**
Line	**Good visual fit of best-fit line to data points**	**1**
Gradient	**Points chosen for gradient calculation span at least half of the entire length of best-fit line** **Points and gradient triangle should be illustrated on the graph paper. Deduct 0.5 if not illustrated but shown in working**	**1**

1 – final gradient within acceptable range
Answers 4.00 ± 0.20 days are accepted.

[Subtotal: 8]

(c) Estimate the following quantities from the light curves provided:

i. τ_1, duration of primary minimum as a fraction of the period
ii. τ_2, duration of secondary minimum as a fraction of the period
iii. I_1, intensity of light at primary minimum
iv. I_2, intensity of light at secondary minimum

Estimating from the graphs gives the following values:

$$\tau_1 = (0.046) - (-0.044) = 0.090 \pm 0.004$$

[A1 – correct value]

$$\tau_2 = (0.544) - (0.460) = 0.084 \pm 0.004$$

[A1 – correct value]

$$I_1 = 0.47 \pm 0.01$$

[A1 – correct value]

$$I_2 = 0.88 \pm 0.01$$

[A1 – correct value]

Acceptable ranges are as indicated above for each value.

[Subtotal: 4]

(d) Consider an eclipsing binary with two stars, one bigger and one smaller, where $k \leq 1$ is the ratio of their radii. When the smaller star is occulted by the larger star, a fraction $\alpha_0 \leq 1$ of the smaller star is obscured. Let Δi_{oc} be the fractional decrease in observed intensity when the larger star occults the smaller star, and Δi_{tr} be the fractional decrease in observed intensity when the smaller star transits the larger star. Assuming that the stars have uniform brightness across their discs, show that

$$\alpha_0 = \Delta i_{oc} + \frac{\Delta i_{tr}}{k^2}$$

This derivation is based on Danjon (1928) p135.
Let $I_B + I_S = I$.
Since the stars have discs of uniform brightness, during occultation of the smaller star, the decrease in the amount of light is

$$\alpha_0 I_S = I\Delta i_{oc}$$

[M1 – change in fractional intensity during occultation]

During transit, the same total area must be obscured as during occultation. Hence, the proportion of the larger star covered is $\alpha_0 k^2$. Thus, the decrease in the amount of light is

$$\alpha_0 k^2 I_B = I\Delta i_{tr}$$

[M1 – change in fractional intensity during transit]

Since $I_B + I_S = I$,

$$\alpha_0 I = I\Delta i_{oc} + \frac{I\Delta i_{tr}}{k^2}$$

$$\alpha_0 = \Delta i_{oc} + \frac{\Delta i_{tr}}{k^2}$$

[Subtotal: 2]

(e) From your observational data, you also manage to draw a graph of the difference between the B-band relative intensity and V-band relative intensity of Z Herculis versus its phase. By analysing this graph, answer the following questions regarding the **primary** minimum:

i. Is this an occultation or a transit?
ii. Is the star being eclipsed the hotter or cooler star?

This derivation is based on Danjon (1928) p138.
Notice that when we observe in two wavelengths B and V, we obtain two measurements for the equation above:

$$\alpha_0 = (\Delta i_{oc})_B + \frac{(\Delta i_{tr})_B}{k^2}$$
$$\alpha_0 = (\Delta i_{oc})_V + \frac{(\Delta i_{tr})_V}{k^2}$$

Taking the difference between the two equations,

$$0 = [(\Delta i_{oc})_B - (\Delta i_{oc})_V] + \frac{[(\Delta i_{tr})_B - (\Delta i_{tr})_V]}{k^2}$$
$$k^2 = -\frac{(\Delta i_{tr})_B - (\Delta i_{tr})_V}{(\Delta i_{oc})_B - (\Delta i_{oc})_V} = -\frac{(\Delta i_B - \Delta i_V)_{tr}}{(\Delta i_B - \Delta i_V)_{oc}} = -\frac{\Delta(i_B - i_V)_{tr}}{\Delta(i_B - i_V)_{oc}}$$

This implies that we can find k^2 from the relative heights of the peak/dip in $i_B - i_V$ at transit and at occultation. The larger peak/dip must be the occultation since $k^2 \leq 1$.
Based on the graph, the primary minimum corresponds to a dip in $i_B - i_V$, and this is larger than the peak at the secondary minimum. A dip in $i_B - i_V$ means that the light lost is relatively bluer, i.e. the hotter star is being occulted.
Therefore, the primary minimum is an **occultation**.

[A1 – correct answer]

The **hotter star** is being eclipsed during the primary minimum.

[A1 – correct answer]
[Subtotal: 2]

(f) Assume that the orbits of the stars are circular and being viewed edge-on from Earth (inclination = 90°), resulting in total eclipses ($\alpha_0 = 1$), and also that the stars have uniform brightness across their discs. From this point onwards, star A refers to the component of Z Herculis that is eclipsed during the primary minimum. Use the information obtained so far to derive the following quantities:

i. ρ_A, ratio of the radius of star A to the distance between the two stars
ii. ρ_B, ratio of the radius of star B to the distance between the two stars

If working is correct but values used from part (d) are wrong, accept working but mark final answers as wrong.
If working is correct but minimum is identified wrongly in (e) as a transit, accept working but mark final answers as wrong.

Based on the geometry of an edge-on circular orbit, the time Δt from first contact to last contact is related to the radii of the two stars by

$$2\sin\frac{\pi\Delta t}{T} = \frac{2(R_A + R_B)}{d}$$
$$\sin\pi\tau = \rho_A + \rho_B$$

[M1 – correct expression for combined radii]

Using an average value of $\tau = 0.087$ we obtain

$$\rho_A + \rho_B = 0.27 \pm 0.01$$

[A1 – correct value]

The intensities at the minima allow us to estimate the ratio of ρ_A to ρ_B. We know from the previous part that the hotter star is being occulted at the primary minimum. Hence, star A is hotter and smaller than star B.

Assuming $\alpha_0 = 1$,

$$1 = (1 - I_1) + \frac{1 - I_2}{k^2}$$
$$k^2 = \frac{1 - I_2}{I_1}$$
$$\frac{\rho_A}{\rho_B} = k = 0.51 \pm 0.02$$

[M1 – correct application of formula from (d)]

Also accept if working uses similar principles as (d) i.e. proportion of flux obscured and relation to ratio of radii.

Solving for ρ_A and ρ_B gives

$$\rho_A = \frac{k}{k+1}(\rho_A + \rho_B) = 0.091 \pm 0.007$$

[A1 – correct value]

$$\rho_B = \frac{1}{k+1}(\rho_A + \rho_B) = 0.179 \pm 0.007$$

[A1 – correct value]

Note that the actual figures from Tümer (1984) and Popper (1988) are significantly different, likely as a result of the simplifying assumptions we have made.

[Subtotal: 5]

(g) Observe the shapes of the light curves at the primary and secondary minima. Do they support the assumption that total eclipses are being observed at the minima? Provide a brief explanation for your answer.

It is not likely that total eclipses are occurring.

[A1 – correct answer]

The shape of the light curve for a total eclipse should show a relatively flat region around the minimum (“flat-bottomed”). However, this is not seen in the light curve, suggesting that we are only seeing a partial eclipse.

[A1 – correct explanation]

[Subtotal: 2]

(h) Find the amplitude of variation K in each star's radial velocity.

$$K_A = \frac{1}{2}\left(40.5 - (-130.5)\right) = 85.5\ \text{km/s}$$

[A1 – correct value]

$$K_B = \frac{1}{2}\left(60.0 - (-150.0)\right) = 105.0\ \text{km/s}$$

[A1 – correct value]

[Subtotal: 2]

(i) Using the data, determine the following parameters:

i. d, the distance between the two stars
ii. a_A, the semi-major axis of A's orbit about the system barycentre
iii. a_B, the semi-major axis of B's orbit about the system barycentre
iv. m_A, the mass of Star A
v. m_B, the mass of Star B
vi. R_A, the radius of Star A
vii. R_B, the radius of Star B

If the orbits are circular and viewed edge-on, the amplitude of variation in the radial velocity of each star represents their orbital velocity.

$$v = r\omega = \frac{2\pi r}{T}$$

[M1 – correct concept for velocity and period]

$$a_A = \frac{K_A T}{2\pi} = 4.69 \times 10^9\ \text{m}$$

[A1 – correct value]

$$a_B = \frac{K_B T}{2\pi} = 5.76 \times 10^9\ \text{m}$$

[A1 – correct value]

$$d = r_A + r_B = \frac{(v_A + v_B)T}{2\pi} = 1.046 \times 10^{10}\ \text{m}$$

[A1 – correct value]

Using Kepler's third law,

$$\frac{4\pi^2}{G(m_A + m_B)} d^3 = T^2$$

Alternatively, by equating gravitational acceleration to centripetal acceleration,

$$\frac{Gm_A}{d^2} = \frac{4\pi^2}{T^2} r_B$$

$$\frac{Gm_B}{d^2} = \frac{4\pi^2}{T^2} r_A$$

Adding the two equations gives the expression derived from Kepler's third law.

[M1 – either method to derive total mass]

$$(m_A + m_B) = \frac{(v_A + v_B)^3 T}{2\pi G}$$
$$= 5.687 \times 10^{30} \text{ kg}$$

[M1 – correct value]

Conservation of momentum gives

$$m_A v_A = m_B v_B$$

[M1 – correct concept to determine individual masses]

Therefore

$$m_A = \frac{v_B}{v_A + v_B}(m_A + m_B) = 3.13 \times 10^{30} \text{ kg}$$

[A1 – correct value]

$$m_B = \frac{v_A}{v_A + v_B}(m_A + m_B) = 2.55 \times 10^{30} \text{ kg}$$

[A1 – correct value]

Finally, since $\rho_A = \frac{R_A}{d}$ and $\rho_B = \frac{R_B}{d}$,

$$R_A = 9.5 \times 10^8 \text{ m}$$

[A1 – correct value]

$$R_B = 1.87 \times 10^9 \text{ m}$$

[A1 – correct value]

Interestingly, star B has a larger radius but is less massive, and this result **is** corroborated with the papers.

[Subtotal: 11]

5 **MCQ Answers**

5.1 NGC 25 is circumpolar about the South Celestial Pole at 43^{o}N latitude, as it is at 57^{o}S declination.

Answer: A

5.2 Key point is to attach counterweights before Optical Tube Assembly, as doing so otherwise risks toppling the OTA and the balancing will be less accurate.

Answer: A

5.3 Use f/ratio as a representation of image brightness.

$$f - ratio = \frac{focal\ length}{aperture}$$

(a) $\frac{300*3}{80} = 11.25$

(b) $\frac{1000}{150} = 6.67$

(c) $\frac{600*2}{150} = 8.00$

(d) $Exit\ Pupil = \frac{50}{20} = 2.5mm$, this does not use up the maximum aperture of the night-adapted iris of 7mm. Will not be able to achieve the same level of brightness.

Answer: B

5.4 Light-sensitive *rod* photoreceptors are most sensitive to 498nm (green-blue) light, and least sensitive to longer visual wavelengths (red). Hence red light is used to illuminate objects without overloading our photoreceptors and cause them to contract, reducing the size of our iris and thus our eye’s light gathering power.

Answer: B

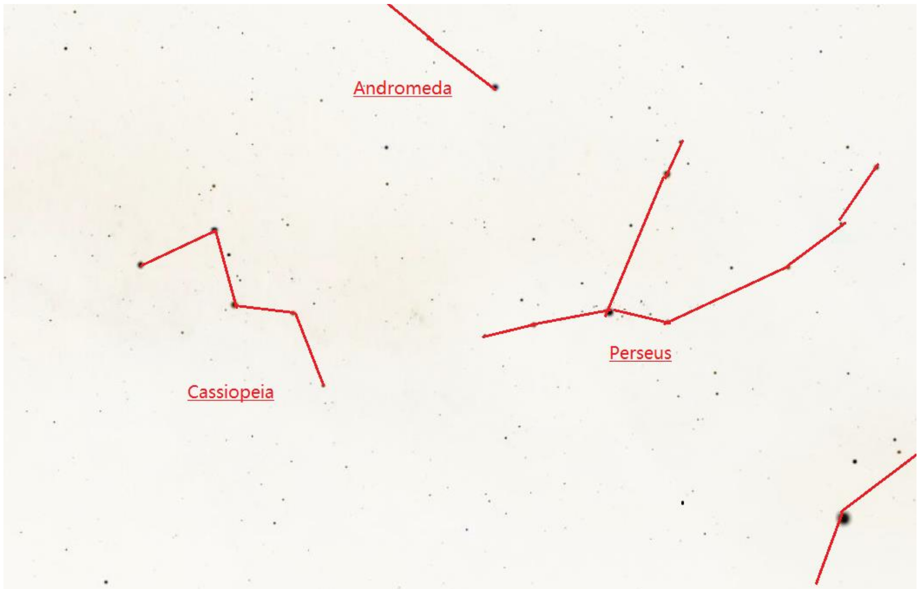
5.5 Solar transits have an extremely high brightness, thus we do not need to make the image brighter. In LEO, the ISS moves across the sky very quickly relative to background stars (and even the Sun). Thus a high shutter speed is required to “freeze” the ISS, usually between 1/2000 and 1/8000 of a second.

Answer: A

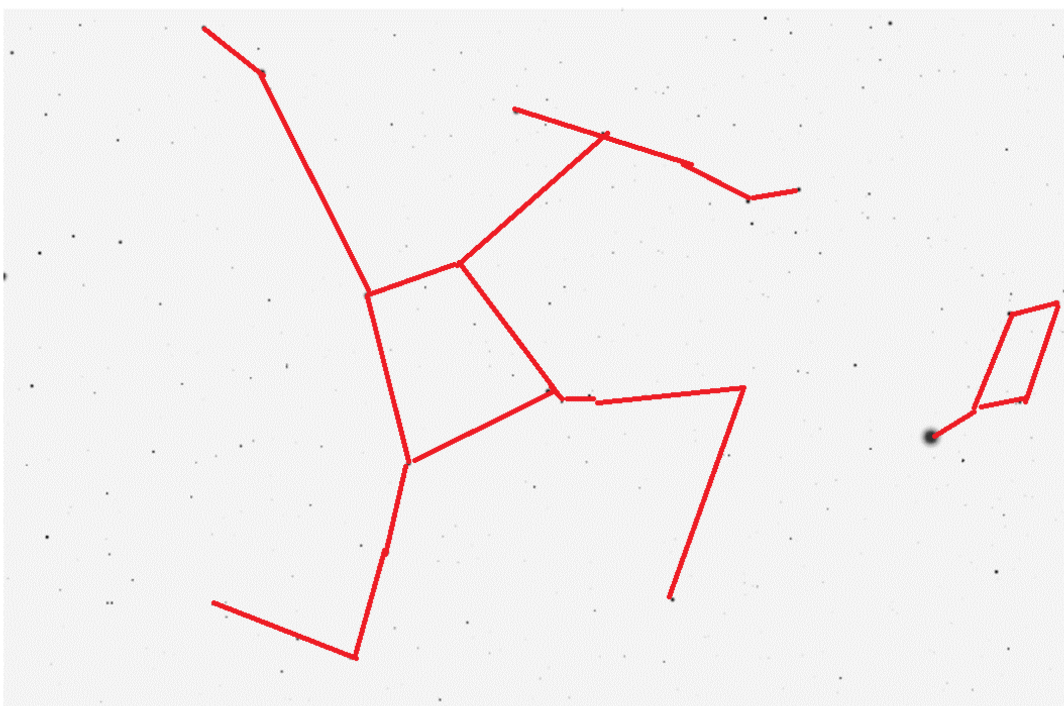
Part P(B)

Part 1: Constellation Identification **[6 marks]**

Name and link any <u>one</u> constellation in the image. Also, Name and circle <u>one</u> deep sky objects, <u>one</u> bright star in the image. [DSO, Star, constellation all one mark each]

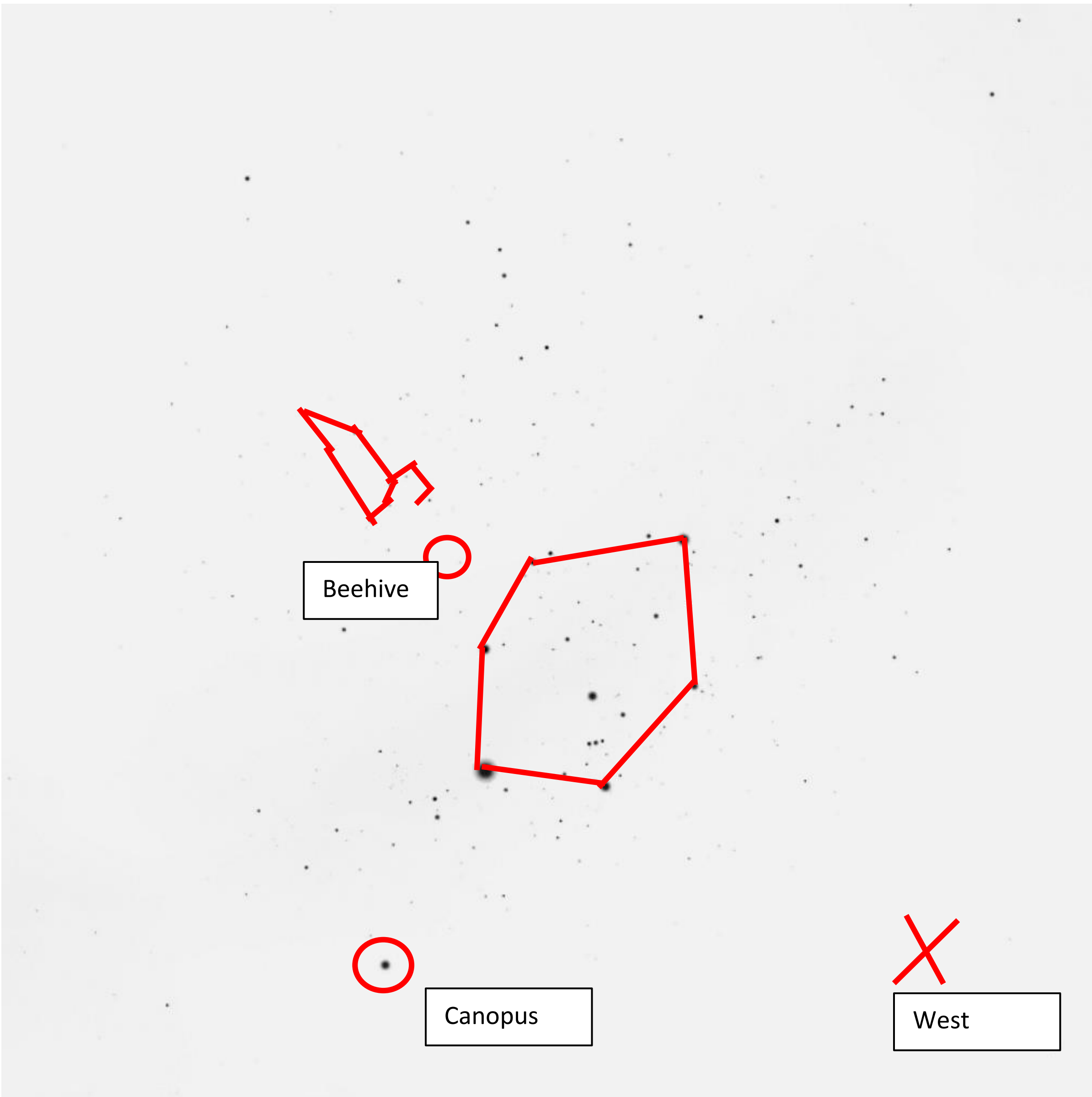


Star: Capella; DSO: Dragonfly cluster, Sailboat Cluster, Double Cluster, Heart and Soul Nebula, Alpha Persi Cluster



Star: Vega, DSO: Ring Nebula, M13 (Great Cluster in Hercules)

Part 2: Winter Sky **[9 marks]**



(a) Circle and identify the star *Canopus* in the image. [1]

(b) Connect the stars that make up the *Winter Hexagon*. [1]

(c) Locate the position of *Beehive cluster* in the photo by putting a "**X**". [2]

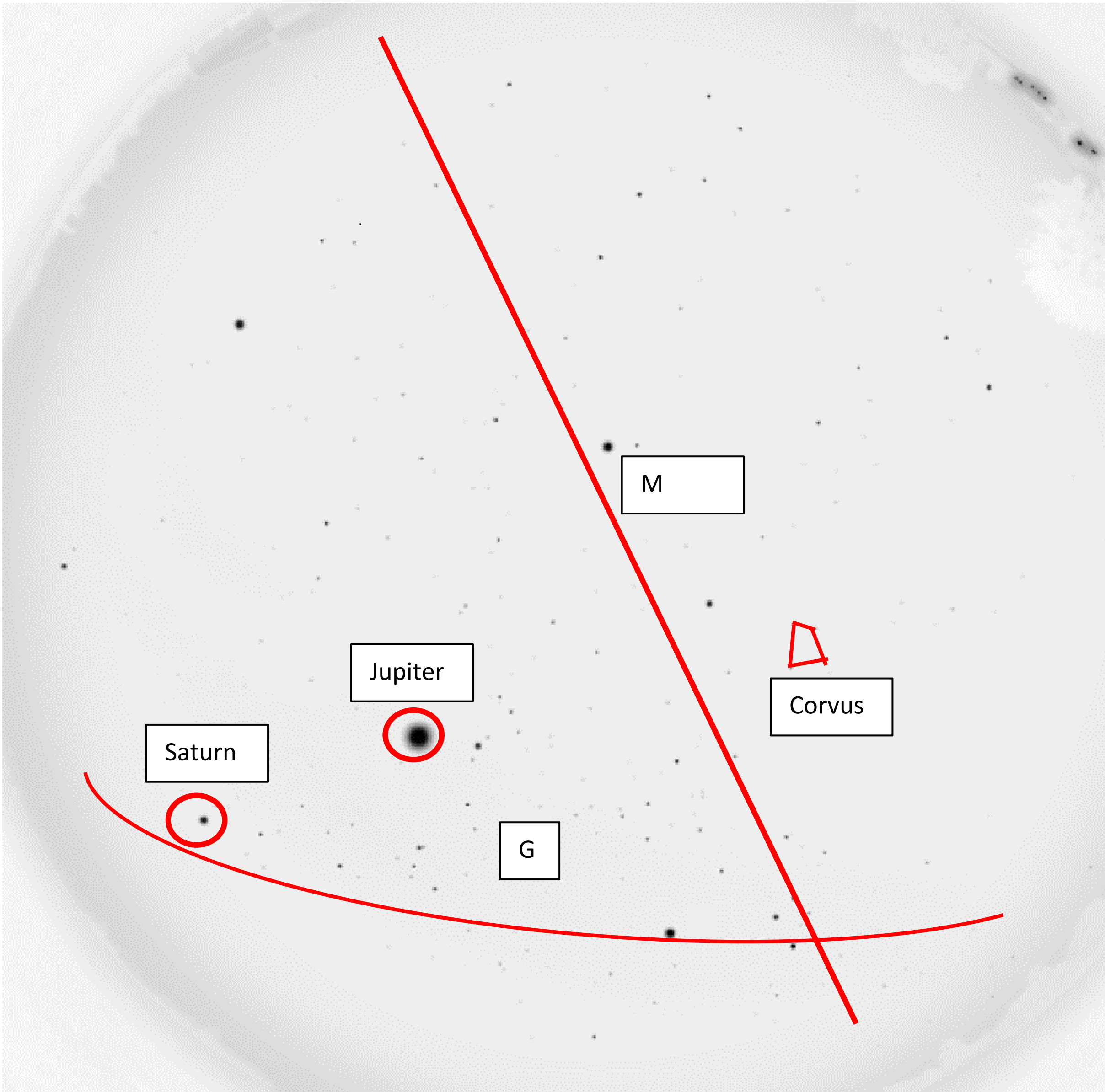
(d) Link and identify the constellation that represents the Nemean Lion which was killed by Hercules. [2]

(e) Along the horizon (circumference of the star chart), mark out the approximate location of cardinal West, with a "**X**". [1]

(f) Estimate the latitude of the observer (To the nearest 10°). [2]

Answer	Latitude = 30 N

Part 3: Summer Sky [13 marks]



(g) Label and identify the two missing stars on the star chart. [4]

Answer	Name/ designation of star: beita Cen; Mizar

(h) Label and identify any planet(s) on the star chart. Saturn, Jupiter [4]
(i) Draw and label the plane of the Milky Way with "**G**" [1]
(j) Link and Label the constellation "**Corvus**". [2]
(k) Trace out the local meridian with a solid arc, and label it "**M**". [2]

Part P(C)

Part 1: **[7 Marks]**

Shown below is the Orion Atlas 10 EQ-G GoTo Newtonian Telescope of 1200mm focal length

(a) On the image, indicate the finder scope with a circle and label it with "**F**". [1]

(b) "The Atlas 10 EQ-G reflector features has a *focal ratio of f/4.7* for breathtaking view". Explain what it means by *focal ratio of f/4.7*. [1]

Focal length/ diameter of the mirror of the reflector [1 Mark]

(c) Briefly describe the structure of a Newtonian Telescope. [2]

Newtonian Telescope is a reflector using a concave primary mirror at the back of the telescope [1] and a flat diagonal secondary mirror near the front [1].

(d) Indicate the direction of motion in declination by drawing a circular arrow around the appropriate axes. [1]

(e) Explain why it is preferred to use Astro-modified DSLR to image *Eta Carina Nebula* using this telescope as compared to unmodified camera. [2]

Astro mod removes inbuilt IR filter [1] ; Eta Car is Emission nebula [1]

Part 2: **[15 Marks]**

Shown below is the Celestron SkyMaster Giant 15x70 Binoculars.

(a) Explain what is meant by *exit pupil* and calculate the exit pupil size of this binocular. [2]

Exit pupil is the virtual aperture in an optical system, and only rays passing through the circle can exit the system [1]. The exit pupil size is simply 70/15=4mm [1]

(b) Calculate the magnification for this binocular. [1]

Answer	Magnification: 15

(c) The binocular is waterproof, and *nitrogen purged* for use in all weather conditions. Explain why the binocular is *nitrogen purged*. [1]

Prevent fogging: moisture from condensing onto the front element [1]

(d) Suppose on the day of SAO (March 23th), you are tasked to organize a stargazing event in Singapore (8pm – 11pm) with telescopes, binoculars and other commonly found objects in astronomy clubs. The committee passed you a list of objects to observe.

Butterfly Cluster; Hyades; M41; Jewel Box Cluster;

Moon; Mars; Orion Nebula; Omega Centauri; Saturn

Choose any 4, briefly explain where you can roughly find the object, object's appearance. Due credit will be given if you explain why objects cannot be seen or are too difficult to be seen in Singapore. [8]

Just see whether participants' description makes any sense: anyone who actually goes stargazing regularly should be able to score

[2 Marks capped for each objects described correctly; Under one object, 1 mark for 1 legitimate point]

Part 3:

You are given a list of the following planetary nebulae for an upcoming observatory session near the Equator. (Suppose all can be seen and the weather is clear)

Object	RA	Dec	Apparent Mag
M27 (Dumbbell Nebula)	19h 59m	22°43’	7.5
M57 (Ring Nebula)	18h 53m	33°01’	8.8
M97 (Owl Nebula)	11h 14m	55°01’	9.9
M42 (Orin Nebula)	5h 35m	-5°23’	4.0
M11 (Wild Duck Cluster)	18h 51m	6°16’	5.8

(a) Suppose that during one night, you notice that the Owl Nebula is setting. Other than the Owl Nebula, what objects are above the horizon right now? [2]

M11, M27, M57 [1 for 1 object correct, 2 marks if all three objects are correct]

(b) On one night, you notice object X in your list is near the zenith. You also notice that Orion Nebula is setting in the west, what is Object X ? [1]

M97: Owl Nebula [1]