3rd Singapore Astronomy Olympiad
Suggested Answers

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Provided by 4th SAO Committee

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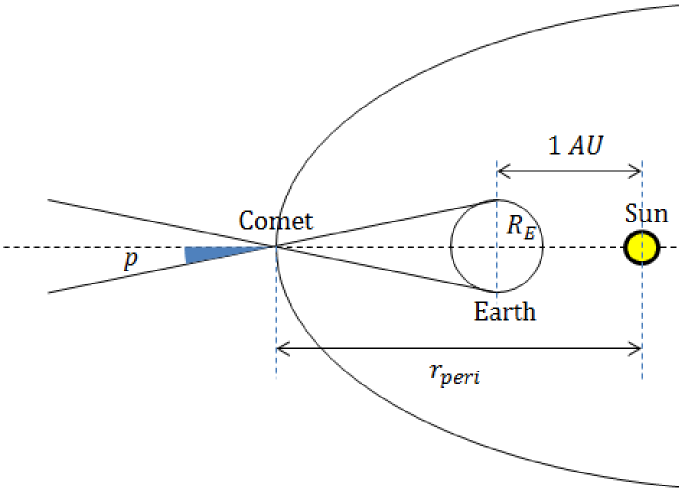
Q1 Stellar observations (9m)

a)	For a black body, its luminosity $L = 4\pi R^2 \sigma T^4$ The brightness of the star, assumed a black body, a distance d away is $b = \frac{L}{4\pi d^2}$ Given the distance d between the Earth and the star has not changed significantly, thus $b \propto L$ Hence, comparing apparent magnitudes $m_{MS} - m_0 = -2.5\lg\left(\frac{b_{MS}}{b_0}\right)$ $m_{MS} = m_0 - 2.5\lg\left(\frac{L_{MS}}{L_0}\right)$ $= m_0 - 2.5\lg\left(\frac{R_{MS}^2 T_{MS}^4}{R_0^2 T_0^4}\right)$ $= 5.0 - 5.0\lg\left(\frac{(0.7 \times 10^9\ m)(5780\ K)^2}{(0.5 \times 10^{11}\ m)(3000\ K)^2}\right)$ $= \mathbf{11.4}$	**M1** **B1**
b)	As the protostar contracts to a main-sequence star, its luminosity L and hence brightness f observed has decreased by a factor of $\frac{L_0}{L_{MS}} = \left(\frac{0.5 \times 10^5\ m}{0.7 \times 10^9\ m}\right)^2 \left(\frac{3000\ K}{5780\ K}\right)^4$ $= 370.3$ To achieve the same flux to detect the star (assuming a limiting magnitude of 5.0), the Light Gathering Power/light grasp of the telescope has to increase by 370.3 times, and since $LGP \propto D^2$ Assuming the human eye has aperture $D_{eye} = 6 \times 10^{-3}\ m$, $\frac{D_{scope}}{D_{eye}} = \sqrt{370.3}$ $\therefore D_{scope} = (6 \times 10^{-3}\ m) \times \sqrt{370.3}$ $= \mathbf{0.115\ m}$	**B1** **M1** **B1**

c)	The absolute magnitude of the main-sequence star is $M_{MS} = M_{Sun} - 2.5\lg\left(\frac{L_{MS}}{L_{Sun}}\right)$ $= 4.75 - 2.5\lg\left(\frac{4\pi(5.67\times10^{-8}\ W\ m^{-2}\ K^{-4})(0.7\times10^{9}\ m)^2(5780\ K)^4}{3.85\times10^{26}\ W}\right)$ $= 4.737$ Then, comparing the absolute and apparent magnitudes, the star is at a distance d, $M_{MS} - m_{MS} = -2.5lg\left(\frac{d^2}{d_{10pc}^2}\right)$ $\therefore \frac{d}{d_{10pc}} = 10^{\frac{m_{MS}-M_{MS}}{5.0}}$ $d = (10\ pc)\times10^{\frac{11.42-4.737}{5.0}}$ $= (10\ pc)\times10^{\frac{11.42-4.737}{5.0}}$ $\boldsymbol{= 217\ pc}$ $\boldsymbol{= 6.68\times10^{18}\ m}$	**B1** **M1** **B1**
d)	By Wien's law, $\lambda_{max}T = b$ Thus, $\lambda_{max} = \frac{b}{T}$ $= \frac{2.90\times10^{-3}\ m\ K}{5780\ K}$ $\boldsymbol{= 5.02\times10^{-7}\ m}$	**B1**

Q2 Celestial mechanics (10m)

a)



The parallax angle p is defined as the half-angle subtended by lines from antipodes on Earth tangent to the comet. Since p is small, by small-angle approximation,

$$\sin p \approx p = \frac{R_E}{d_{peri} - d_{Earth}}$$ **M1**

$$\therefore d_{peri} = d_{Earth} + \frac{R_E}{p}$$

$$= 1\ AU + \left(\frac{6370 \times 10^3\ m}{\frac{35''}{205265''}\ rad \times (1.50 \times 10^{11}\ m)} \right) AU$$

$$= \mathbf{1.25\ AU}$$ **B1**

b) Given the aphelion distance $d_{aph} = 28.5\ AU$, the semi-major axis of the comet a is

$$a = \frac{d_{peri} + d_{aph}}{2}$$

$$= \frac{28.5\ AU + 1.250\ AU}{2}$$

$$= 14.88\ AU$$

For bodies orbiting the Sun, Kepler's Third Law can be expressed in the form

$$T^2 = a^3$$

where T is measured in sidereal years and a in AU.

Assuming no external gravitational influences,

	$T = \sqrt{a^3}$ $= \sqrt{14.88^3}$ $= \boldsymbol{57.4\ years}$	**B1**
c) **i.**	By conservation of energy, for a bound orbit, $\frac{mv_p^2}{2} - \frac{GMm}{r_p} = \frac{mv_a^2}{2} - \frac{GMm}{r_a}$ $v_p^2 - v_a^2 = 2GM\left(\frac{1}{r_p} - \frac{1}{r_a}\right)$ By conservation of angular momentum, $r_a v_a = r_p v_p$ $v_a = \frac{r_p}{r_a} v_p$ Substituting into the above, $v_p^2\left(1 - \frac{r_p^2}{r_a^2}\right) = 2GM\left(\frac{1}{r_p} - \frac{1}{r_a}\right)$ $v_p = \sqrt{\frac{2GM\left(\frac{1}{r_p} - \frac{1}{r_a}\right)}{\left(1 - \frac{r_p^2}{r_a^2}\right)}}$ $= \sqrt{\frac{2G(1.99 \times 10^{30}\ kg)\left(\frac{1}{1.50 \times 10^{11}\ m\ AU^{-1}}\right)\left(\frac{1}{1.250\ AU} - \frac{1}{28.5\ AU}\right)}{\left(1 - \left(\frac{1.250}{28.5}\right)^2\right)}}$ $= \boldsymbol{3.68 \times 10^4\ m\ s^{-1}}$	**M1** **M1** **M1** **B1**
c) **ii.**	For a small time Δt, the areal velocity can be approximated to the area bounded by the radius and velocity vectors $\dot{A} \approx \frac{1}{2} r_p v_p \Delta t$ $= \frac{1}{2}(1.250\ AU)\left(\frac{36830\ m\ s^{-1}}{1.50 \times 10^{11}\ AU\ m^{-1}}\right)(1\ s)$ $= \boldsymbol{1.53 \times 10^{-7}\ AU^2\ s^{-1}}$ $= \boldsymbol{3.45 \times 10^{15}\ m^2\ s^{-1}}$	**M1** **B1**
c) **iii**	By Kepler's Second Law (law of areas), the areal velocity is constant throughout the orbit, hence $\boldsymbol{\dot{A} = 1.53 \times 10^{-7}\ AU^2\ s^{-1}}$	**A1**

Q3 Celestial coordinate systems I (15m)

a) Object A's declination ranges from

$$-23°02'19'' \leq \delta_A \leq +23°26'13''$$

The declination extremes are approximately equal to the *axial tilt of the Earth*, which is 23.4°. **B1**

Furthermore, object A reaches its most positive (northerly) declination in June 22, which coincides with the *Summer Solstice*. **B1**

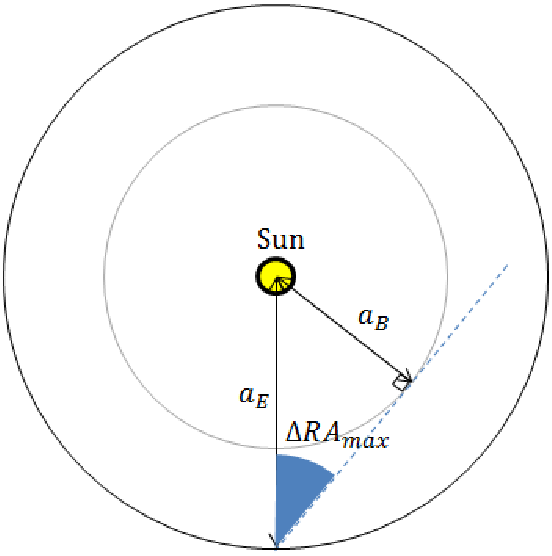
Hence, object A is likely to be the **Sun**. **A1**

b) Object B's declination varies between

$$-24°12' \leq \delta_B \leq +22°50'$$

Hence, object B is inclined $\approx 1°$ w.r.t. the plane of the Earth's orbit (the *Ecliptic*).

Notice that the difference between the RA of object B and object A (the **Sun**) is never greater than about 3 hours, thus it is likely to be an **inner planet**. **B1**



OR

The maximum elongation of object B from the Sun, reached in Mar-08, is approximately ΔRA as object B's orbital inclination to the ecliptic is small,

$$\begin{aligned}\Delta\theta_{max} &\approx \Delta RA_{max}\\ &= RA_A - RA_B\\ &= 23^h12^m40^s - 20^h12^m02^s\\ &= 3^h0^m38^s\\ &= 45.2°\end{aligned}$$

M1

This give rise to an orbital radius of object B of

$$\sin(\Delta RA_{max}) = \frac{a_B}{a_E}$$

$$\Rightarrow a_B = a_E \sin(\Delta RA_{max})$$
$$= (1.00\ AU) \sin(45.2°)$$
$$= 0.710\ AU$$

M1

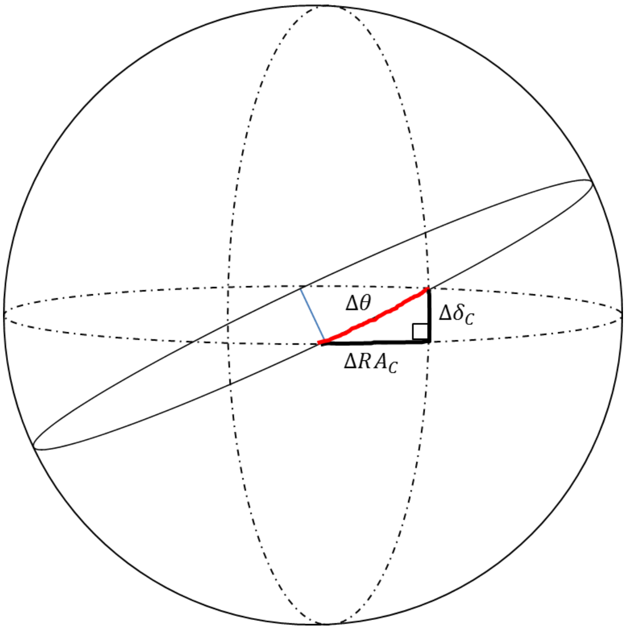
Since Mercury orbits at about 0.4 AU, $h$ence, object B is likely to be the planet **<u>Venus</u>**. **A1**

c) Notice that object C's RA is approximately 12^h away from that of the Sun on Oct-08, i.e. it is in opposition. Hence it is an **outer planet** since inner planets can never be in opposition to the Sun. **B1**

<u>OR</u> **<u>OR</u>**

Notice that object C's RA and declination changes very slowly, hence it is likely it is an **outer planet**. **B1**

Only award 1m for the above deduction.



For Object C, since the Earth is in a similar place in its orbit in both Jan-01, notice that the increase in RA over one year is only

$$\Delta RA_C = 00^h46^m40^s - 00^h32^m14^s$$
$$= 14^m26^s\ (i.e. 3.608°)$$

Then note that the increase in declination is only

$$\Delta\delta_C = 04°17'29'' - 02°43'56''$$
$$= 1°33'33''\ (i.e. 1.559°)$$

	Only award 1m for either of the above calculated.	
	Using spherical trigonometry, the angle Δθ object C completes in its orbit (out of 360°) in 1 year is	
	$\cos\Delta\theta = \cos\Delta\delta_C \cos\Delta RA_C + \sin\Delta\delta_C \sin\Delta RA_C \cos 90°$ $\therefore \Delta\theta = \cos^{-1}(\cos 1.559° \cos 3.608°)$ $= 3.930°$	M1
	Hence, the planet will take approximately the sidereal period T to complete one revolution around the Sun,	
	$T \approx \frac{360°}{\Delta\theta}$ $= \frac{360°}{2.049°}$ $= 91.6\ years$	M1
	Since object C orbits the Sun, Kepler's Third Law gives $T^2 = a^3$ $\Rightarrow a_C = \sqrt[3]{91.6^2}$ $= 20.3\ AU$	A1
	Hence, object C is likely to be **Uranus**, whose orbital semi-major axis is around 20 AU. Thus, the distance to object C ranges from **19 AU** (when in opposition) to **21 AU** (when in conjunction).	A1
d)	Since we know object B is **Venus**, we can calculate its elongation on both dates to obtain the distance to the planet. Award each of the following 1m for the deduction, regardless if the elongation was calculated. $\Delta\theta_{22Mar} \approx \Delta RA_{22Mar}$ $= 00^h03^m59^s - 21^h05^m42^s$ $= 2^h58^m17^s\ (mod\ 24^h)$ $= 44.6°$ By observation, we can see that on Mar-22, Venus was near maximum Western elongation as $RA_{Sun} - RA_{Venus} > 0$. $\Delta\theta_{22Oct} \approx \Delta RA_{22Oct}$ $= 13^h45^m11^s - 13^h43^m34^s$ $= 1^m37^s$ $= 0.404°$ By observation, we can see that on Oct-22, Venus was in conjunction with the Sun, since Venus and the Sun have the same RA then.	A1

Between Mar-22 and Oct-22, 7 months have elapsed, which is approximately n days,

$$
\begin{aligned}
n &\approx \frac{7\ months}{12\ months} \times 365.25\ days \\
&= 213\ days
\end{aligned}
$$

From part (b), we have that $a_{Venus} \approx 0.710\ AU$, hence from Kepler's third law,

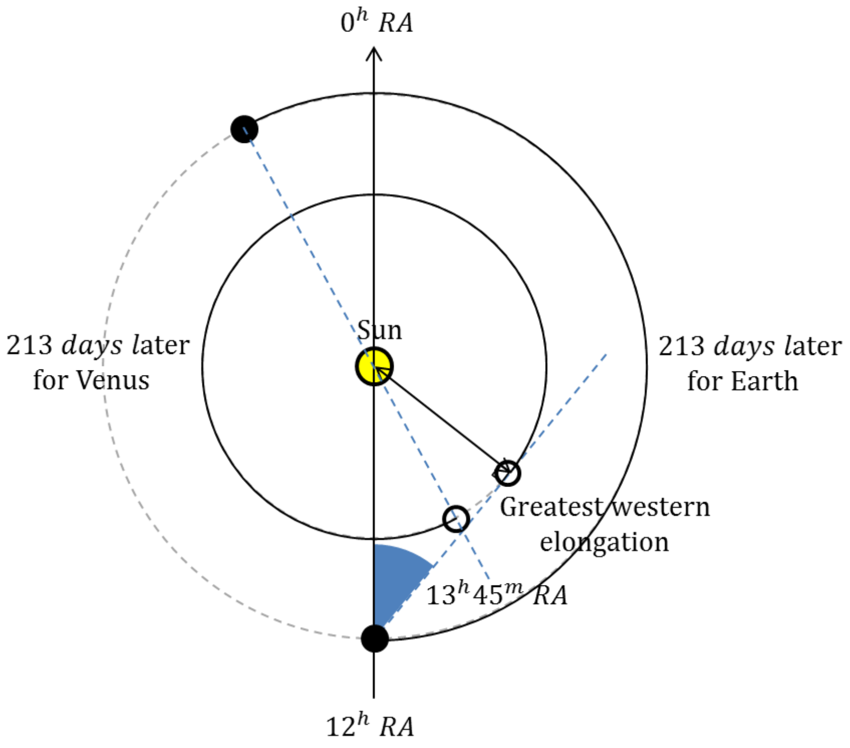
$$
\begin{aligned}
T &\approx \sqrt{a_{Venus}^3} \\
&= \sqrt{0.710^3} \\
&= 0.598\ years \\
&= 219\ days
\end{aligned}
$$

Hence, Venus has nearly completed a full revolution around the Sun since reaching greatest Western elongation in Mar-22. Likewise, in 7 months, Earth's has traveled an angle ψ_E around the Sun,

$$
\begin{aligned}
\psi_E &\approx \frac{213\ days}{365.25\ days} \times 360° \\
&= 210°
\end{aligned}
$$

Using this information and constructing a diagram, it is clear that since Venus is in **superior conjunction** with the Sun on Oct-22. Note that RA increases counter-clockwise in the diagram.

B1



Only award this 1m for deducing the type of conjunction, with explanation.

On Mar-22, the distance to Venus is

$$\tan \Delta\theta_{22Mar} = \frac{a_{Venus}}{d_{22Mar}}$$

$$\begin{aligned}\therefore d_{22Mar} &= \frac{a_{Venus}}{\tan \Delta\theta_{22Mar}}\\ &= \frac{0.710\ AU}{\tan 44.6°}\\ &= 0.720\ AU\end{aligned}$$

B1

On Oct-22 the distance to Venus is trivially

$$d_{22Oct} = 1.710AU$$

Hence, using the small angle approximation when the distance to Venus *d* is much greater than Venus' diameter *D*, the angular diameter of Venus θ_{Venus} at a given distance *d* is given by,

$$\theta_{Venus} \approx \frac{D}{d}$$

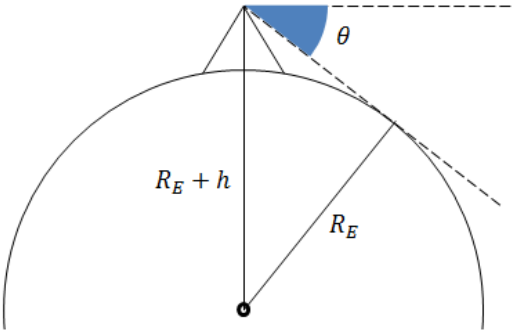
Hence as the diameter of Venus *D* remains unchanged,

$$\begin{aligned}\frac{\theta_{Venus,22Mar}}{\theta_{Venus,22Oct}} &= \frac{d_{22Oct}}{d_{22Mar}}\\ &= \frac{1.710}{0.720}\\ &= \mathbf{2.38}\end{aligned}$$

Hence the answer is $\mathbf{1 + \sqrt{2}}$, which evaluates to 2.41. **B1**

Q4 Celestial coordinate systems II (16m)

a) To find the minimum declination, we have to consider various effects in addition to latitude ϕ. We note that as the observer is at an high altitude, he/she will be able to see beyond the standard horizon. This effect is known as horizon depression, θ, see following diagram. There is also atmospheric refraction at the horizon α which is about 34'.



For this observer, the horizon depression effect is

$$\cos\theta = \frac{R_E}{R_E + h}$$

$$\therefore \theta = \cos^{-1}\left(\frac{R_E}{R_E + h}\right)$$

$$= \cos^{-1}\left(\frac{6370\ km}{6370\ km + 1.3km}\right)$$

$$= 1.157° \ (i.e.\, 1°9')$$

B1

Hence, for the observer in the Southern Hemisphere, the minimum declination is

$$\delta_{min} = -90° + \phi + \theta + \alpha$$

$$= -90° + 6°49' + 1°9' + 34'$$

$$= \mathbf{-81°28'}$$

B1

Accept answers given in decimals instead of arcseconds for δ_{min}. Award the 1m for correcting either for horizon depression or atmospheric refraction.

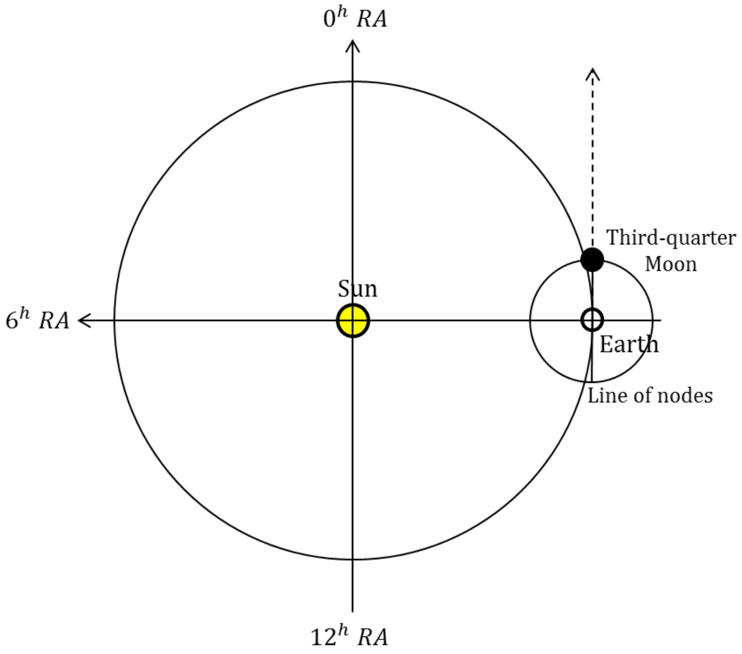
b) **This problem is similar to Q3 in the theory paper of the 2008 IOAA.**

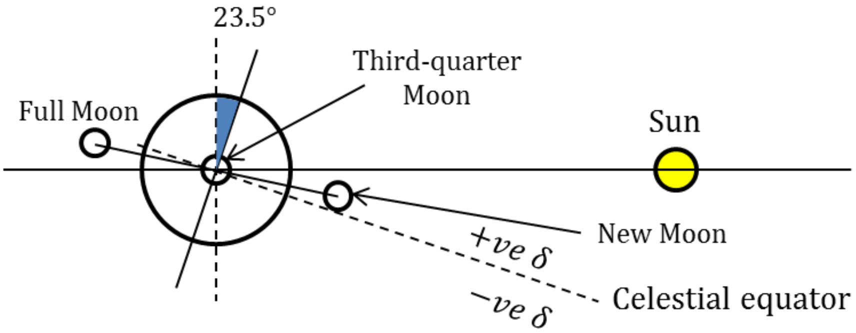
First, note that the date of observation, June 28, is very close to the Summer Solstice, hence the equatorial coordinates of the Sun are approximately

$$(\delta_{Sun}, RA_{Sun}) \approx (+23.5°, 6^h)$$

By drawing a diagram, we can see that for the Moon in the third-quarter phase, assuming it is at a lunar node (the Moon's orbit is inclined 5.1° w.r.t the ecliptic), its equatorial coordinates are

$(\delta_{Moon}, RA_{Moon}) \approx (0°, 0^h)$ **B1**





Using the Sunrise equation, the half-diurnal arc H is

$$\cos H = -\tan \delta_{Moon} \tan \phi$$
$$\Rightarrow H = \cos^{-1}(-\tan \delta_{Moon} \tan \phi)$$
$$= \cos^{-1}(-\tan 0° \ \tan(-6.82°))$$
$$= 90°$$

B1

Hence the diurnal arc $2H$, i.e. the time expressed in RA it takes the Moon to move from its rising to setting point is,

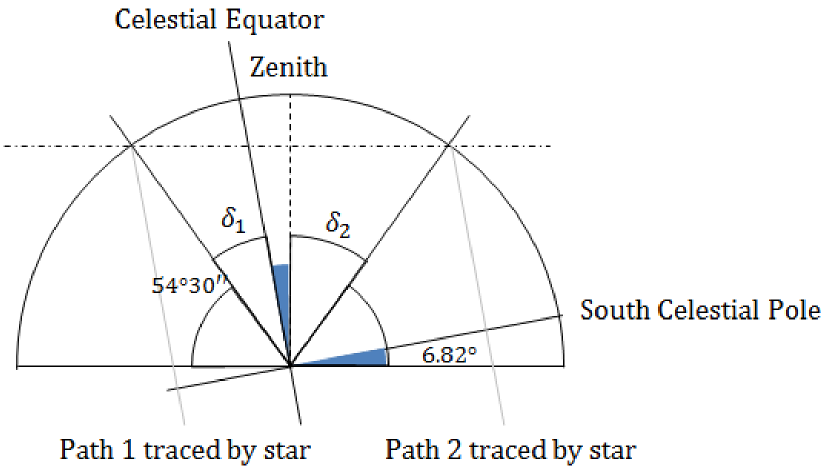
$$2H = 180°$$
$$= 12^h$$

B1

As a first approximation, the duration the Moon remains above the horizon is 12 hours.

However, we have to include a couple of corrections to this value. Firstly, the Moon is close to the Earth and hence its' orbital motion over a day cannot be neglected. The Moon's RA increases over the course of a day by

	$\Delta t_1 \approx \frac{1}{27.3} \times 24^h = 0.879^h = 52^m45^s$ every day. Hence, in 12 hours, it will move “backward” by nearly 26.5^m and will be visible for 26.5 more minutes.	**B1**
	Award this 1m for explanation of the non-negligible lunar orbital motion. Secondly, as the refraction near the horizon is nearly half a degree, the moon will be seen for an extra $\Delta t_2 \approx \frac{0.5}{15} \times 1^h \times 2 = 4^m$	**B1**
	The factor of 2 arises as the effect occurs at both rising and setting. Third, as the Moon has a relatively large angular diameter of about half a degree as well, it will be seen by an additional $\Delta t_3 \approx 2 \times 2^m$	**B1**
	The factor of 2 arises as the effect occurs at both rising and setting. Award 1m for either explaining the atmospheric refraction at the horizon or the angular diameter of the Moon. Thus, the total time the Moon is expected to remain above the horizon is $T = 2H + \sum_{i=1}^{3} \Delta t_i = 12^h + 26^m30^s + 4^m + 4^m = \mathbf{12^h35^m}$ Hence, the Moon was above the horizon (in part) for **12 hours and 35 minutes**.	**A1**
c)	Assume that the proper motion in RA, μ_α, has not been corrected by $\cos\delta$ (since the question intends us to find the declination of the star from altitude). Note that from Bosscha Observatory, there are two possible declinations of stars that can culminate at an altitude of 54°33”, see *path 1 traced by star* and *path 2 traced by star* in the following diagram. As the question does not provide further information identifying the sign of the declination of the star, we assume the star is situated **south of the CE.**	



Hence, we can calculate both possible declinations

$$\delta_1 = 90° - 54°30'' - 6.82°$$
$$= +28.7°$$
$$\delta_2 = -(90° - 54°30'') - 6.82°$$
$$= -42.3°$$

B1

Award this 1m only if δ_{star} has been calculated with the assumption of the sign of its declination clearly stated, or if the participants shows understanding of both possible scenarios.

The correction factor of cos δ accounts for the fact that the radius from the axis of the sphere to its surface varies as cos δ, becoming, for example, zero at the pole. Thus, the component of velocity parallel to the equator corresponding to a given angular change in α is smaller the further north the object's location.

Using δ_2, the corrected proper motion in RA is

$$\mu_{\alpha*} = \mu_\alpha \cos\delta$$
$$= (-0.0374''\ year^{-1})(\cos(-42.3°))$$
$$= -0.02766''year^{-1}$$

B1

Hence, the magnitude of the proper motion is

$$\mu^2 = \mu_\delta^2 + \mu_{\alpha*}^2$$
$$\Rightarrow \mu = \sqrt{\mu_\delta^2 + \mu_{\alpha*}^2}$$
$$= \sqrt{(1.21''\ year^{-1})^2 + (-0.02766''\ year^{-1})^2}$$
$$= \mathbf{1.21''year^{-1}}$$

B1

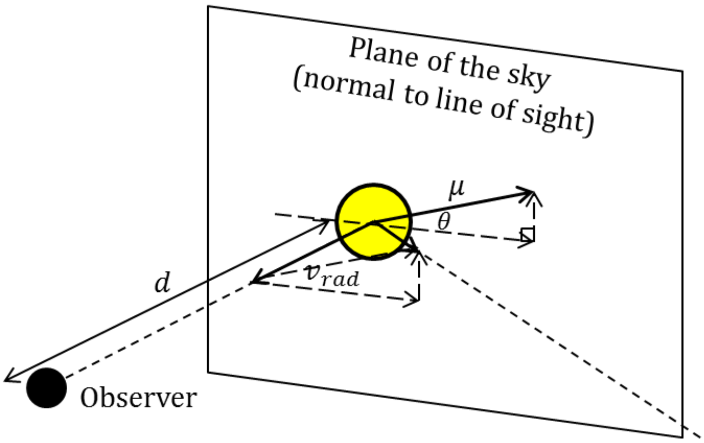
Answers that assume μ_α is corrected without stating so are capped at 1m. Answers that assume $\mu \approx \mu\_\delta$ since $\mu_{delta} \gg \mu_\alpha$ without calculating the vector addition are awarded no marks.

d) Given that the star exhibited blueshift of 7.6 km s^{-1} (*sic*), this implies its radial velocity is towards the Sun.

Given the parallax angle p, then the distance d to the star in parsecs is

$$\begin{aligned} d &= \frac{1}{p} \\ &= \frac{1}{0.376''} \\ &= 2.660\ pc \end{aligned}$$

M1



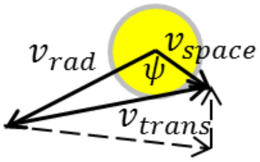
Then, for small proper motions μ, the star's transverse velocity (in the plane of the sky) is

$$\begin{aligned} v_{trans} &= \mu d \\ &= (1.21''\ yr^{-1})(2.660\ pc)(\frac{3.26\ ly\ pc^{-1}}{205265''\ rad^{-1}})(3.00 \times 10^{8}\ m\ s^{-1}\ yr\ ly^{-1}) \\ &= (1.21''\ year^{-1})(2.660\ pc)(4765\ m\ s^{-1}\ year\ pc^{-1}) \\ &= 15.34\ km\ s^{-1} \end{aligned}$$

B1

Hence its velocity in space relative to our Sun is

$$\begin{aligned} v &= \sqrt{v_{trans}^{2} + v_{rad}^{2}} \\ &= \sqrt{(15340\ m\ s^{-1})^{2} + (7.6 \times 10^{3}\ m\ s^{-1})^{2}} \\ &= \mathbf{1.71 \times 10^{4}\ m\ s^{-1}} \end{aligned}$$



From the above diagram, we can also find the angle ψ relative to our line of sight, using the Cosine Rule,

$$v_{trans}^2 = v_{rad}^2 + v_{space}^2 - 2v_{rad}v_{space}\cos\psi$$

$$\Rightarrow \psi = \cos^{-1}\left(\frac{v_{rad}^2 + v_{space}^2 - v_{trans}^2}{2v_{rad}v_{space}}\right)$$

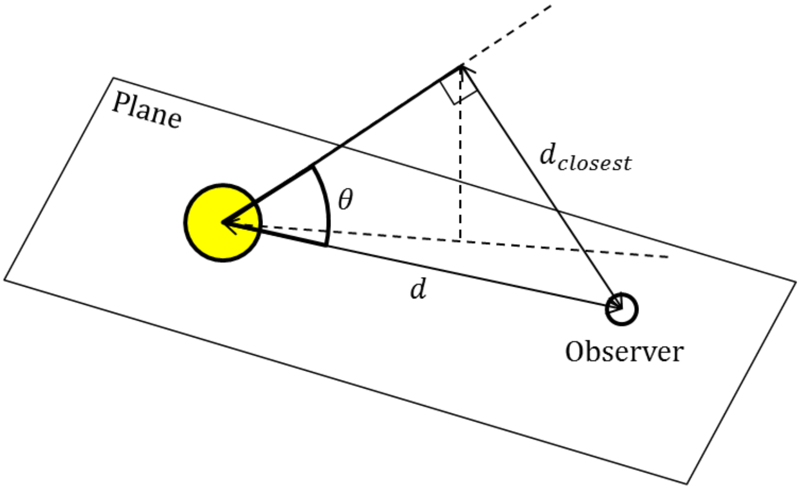
$$= \cos^{-1}\left(\frac{(7.6\times10^3\ m\,s^{-1})^2 + (1.71\times10^4\ m\,s^{-1})^2 - (1.534\times10^4\ m\,s^{-1})^2}{2(7.6\times10^3\ m\,s^{-1})(1.71\times10^4\ m\,s^{-1})}\right)$$

$$= \mathbf{63.8^\circ}$$

A1

to our line of sight.

e) Since proper motion is motion relative to the Sun, at the time of closest approach, $d_{closest}$ is a minimum.



From part (d), the angle between our line of sight and the star's path in space is

$$\psi = 63.76^\circ$$

Thus, the distance of closest approach is

$$\sin\psi = \frac{d_{closest}}{d}$$
$$\Rightarrow d_{closest} = d\sin\psi$$

M1

At the point of closest approach, the star's radial velocity disappears, as its velocity is perpendicular to our line of sight, i.e.

$$v'_{rad} = 0$$

However, the star's actual velocity in space remains unchanged. Hence

$$v'_{trans} = v_{space}$$
$$= 1.712\times10^4\ m\,s^{-1}$$

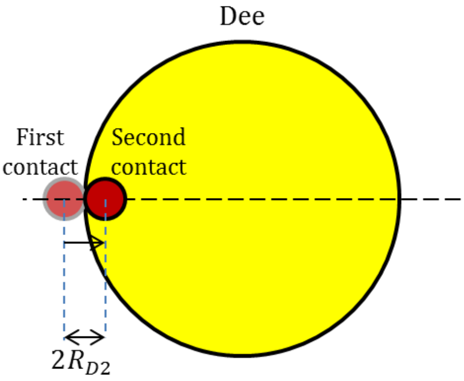
M1

Award this 1m for describing the scenario at which the star is closest to us.

	From part (d), where $k = 4765\ m\ s^{-1}\ year\ pc^{-1}$, d in parsecs, μ in “ yr^{-1}, $v_{trans} = k\mu d$ $\Rightarrow v'_{trans} = k\mu' d_{closest}$ $\therefore \mu' = \frac{v'_{trans}}{kd_{closest}}$ $= \frac{1.712 \times 10^4\ m\ s^{-1}}{(4765\ m\ s^{-1}\ year\ pc^{-1})(2.660pc)(\sin 63.76^\circ)}$ $= \mathbf{1.51''\mathit{year}^{-1}}$	**A1**

Q5 Binaries and variables (20m)

a)	Since the Doppler shift oscillation is sinusoidal and the magnitude of the semi-amplitudes are equal, the orbit of *Dee Two* around *Dee* is approximately circular. Assuming *Dee Two* orbits in *Dee*'s equatorial plane (such that the semi-amplitude of *Dee Two*'s Doppler oscillation is equal to its orbital velocity), then the lower limit for the orbital radius is $v = \frac{2\pi r}{T}$ $\therefore r_{min} = \frac{vT}{2\pi}$ $= \frac{(30 \times 10^3\ m\ s^{-1})(80 \times 24 \times 60^2\ s)}{2\pi}$ $= \mathbf{3.30 \times 10^{10}}\ \boldsymbol{m}$ Award marks even if the assumption (*Dee Two* orbiting in equatorial plane of *Dee*) which gives rise to the lower limit of *r* is not stated.	**M1** **A1**
b)	Given that $M_{D2} \gg M_D$, then the gravitational force on *Dee Two* provides the centripetal force for its orbit, i.e. $\frac{G M_D m_{D2}}{r^2} = \frac{m_{D2} v^2}{r}$ $\therefore M_D = \frac{v^2 r}{G}$ Thus the lower limit for the mass of *Dee* is $M_{Dmin} = \frac{v^2 r_{min}}{G}$ $= \frac{(30 \times 10^3\ m\ s^{-1})^2 (3.300 \times 10^{10}\ m)}{6.67 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}}$ $= \mathbf{4.45 \times 10^{29}}\ \boldsymbol{kg}$ $= \mathbf{0.224}\ \boldsymbol{M_\odot}$	**M1** **A1**
c)	Since $M_{D2} \gg M_D$, we can assume *Dee*'s orbital velocity is negligible. Considering the time Δt between first and second contact, we find that the radius of *Dee Two* is half the distance travelled by *Dee Two* between the first and second contacts. From the light curve, we see that $\Delta t = 0.1\ days$. $R_{D2} = \frac{v \Delta t}{2}$ $= \frac{(30 \times 10^3\ m\ s^{-1})(0.1 \times 24 \times 60^2)}{2}$ $= \mathbf{1.30 \times 10^8}\ \boldsymbol{m}$	**M1** **A1**



Note the implicit assumption that the disk of *Dee* radiates uniformly and effects such as limb darkening are neglected (evident from the flat bottom of the light curve during the transit).

d) Given insufficient information about *Dee Two*, we assume that it does not radiate or reflect significantly. Then, the initial luminosity is L_D, and the dip from first to second contact is due to the *Dee Two*'s disk blocking part of *Dee*'s disk. We assume *Dee* radiates uniformly and neglect limb darkening effects.

Do not penalise if the above assumptions are not stated.

Comparing the visible fraction of *Dee*'s disk A,

$$\therefore \frac{L_{transit}}{L_D} = \frac{A_{transit}}{A_D} = \frac{\pi R_D^2 - \pi R_{D2}^2}{\pi R_D^2}$$

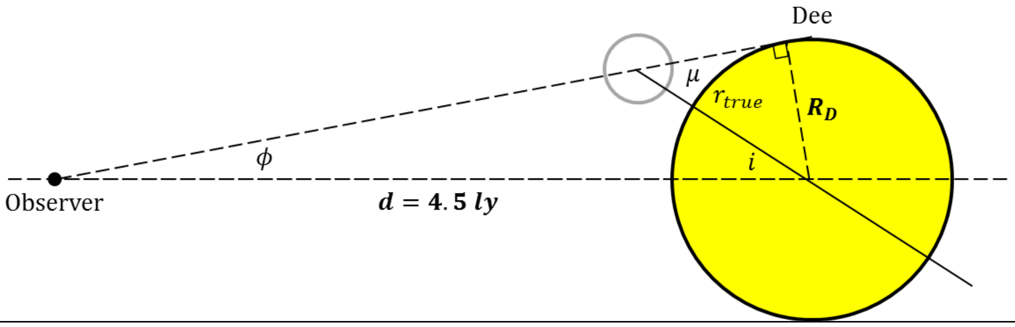
M1

$$\Rightarrow \frac{0.990}{1.000} = 1 - \frac{R_{D2}^2}{R_D^2}$$

$$\Rightarrow R_D = \frac{R_{D2}}{\sqrt{1 - \frac{0.990}{1.000}}} = \frac{1.296 \times 10^8\ m}{\sqrt{0.010}} = \mathbf{1.30 \times 10^9\ m}$$

M1

e) The inclination i of *Dee Two*'s orbital plane is defined w.r.t. our line of sight.



$$\phi = \sin^{-1}\left(\frac{R_D}{d}\right)$$
$$= \sin^{-1}\left(\frac{1.296 \times 10^9\ m}{(4.5 \times 365.25 \times 24 \times 60^2\ s)(3.00 \times 10^8\ m)}\right)$$
$$= \mathbf{1.743 \times 10^{-6}}\ \boldsymbol{degrees}$$

M1

At the maximum inclination,

$$v = v_{true} \cos i_{max}$$
$$\Rightarrow v_{true} = \frac{v}{\cos i_{max}}$$
$$\therefore r_{true} = \frac{r_{min}}{\cos i_{max}}$$

M1

Consider angle μ,

M1

$$\sin \mu = \frac{R_D}{r_{true}}$$
$$= \frac{R_D \cos i_{max}}{r}$$

Then,

$$\mu = i_{max} + \phi$$

Substitute the above equation into that for $\sin \mu$,

M1

$$\sin(i_{max} + \phi) = \frac{R_D \cos i_{max}}{r}$$

Using the provided trigonometric identity,

$$\sin i_{max} \cos \phi + \cos i_{max} \sin \phi = \frac{R_D \cos i_{max}}{r}$$

Dividing across by $\cos i_{max}$, and rearranging

$$\tan i_{max} = \frac{\frac{R_D}{r} - \sin \phi}{\cos \phi}$$
$$i_{max} = \tan^{-1}\left(\frac{\frac{1.296 \times 10^9\ m}{3.300 \times 10^{10}\ m} - \sin(1.743 \times 10^{-6}\ {}^\circ)}{\cos(1.743 \times 10^{-6}\ {}^\circ)}\right)$$
$$= \mathbf{2.249^\circ}$$

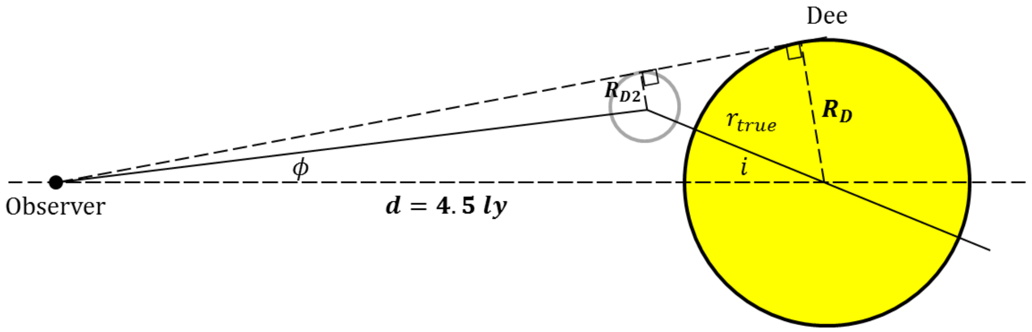
B1

In analogous working to part (ii), given that $M_{D2} \gg M_D$, simplifying from $F_g = F_c$, thus the maximum limit for the mass of *Dee* is now

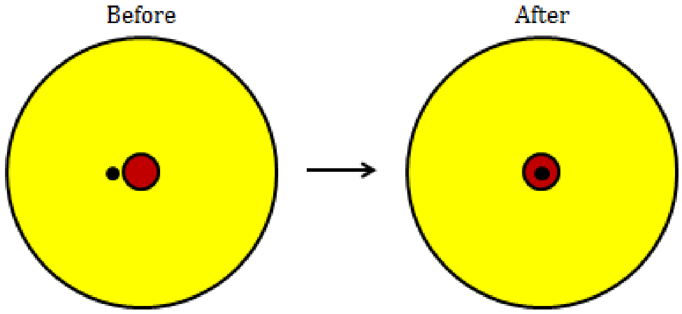
$$\therefore M_D = \frac{v_{true}^2 r_{true}}{G} = \frac{v^2 r_{min}}{G(\cos i_{max})^3} = \frac{(30 \times 10^3\ m\ s^{-1})^2 (3.300 \times 10^{10}\ m)}{(6.67 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2})(\cos 2.249°)^3} = \boldsymbol{4.46 \times 10^{29}\ kg}$$

B1

In fact, since the light curve shows a flat bottom, this implies that *Dee Two*'s inclined orbit must still bring it fully in front of *Dee*'s disk. If $i > i_{max}$, then *Dee Two*'s orbit would result in a continuously changing area of *Dee*'s disk being occulted, producing a curved dip with no flat bottom. This scenario is illustrated below.



f) During the event μ, the luminosity increases since the exomoon now transits *Dee Two* (assumed to be not significantly radiating or reflecting) instead of *Dee*. Thus, the change in luminosity observed is due to the area of the exomoon that occults *Dee*'s disk. See diagram.



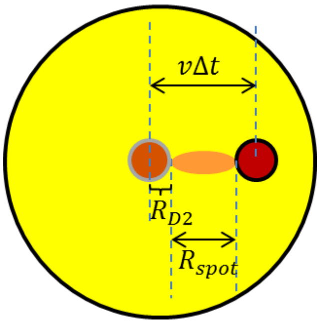
Hence, by observation, the area of *Dee Two*'s disk A_{D2} and that of its exomoon A_{moon} are related by

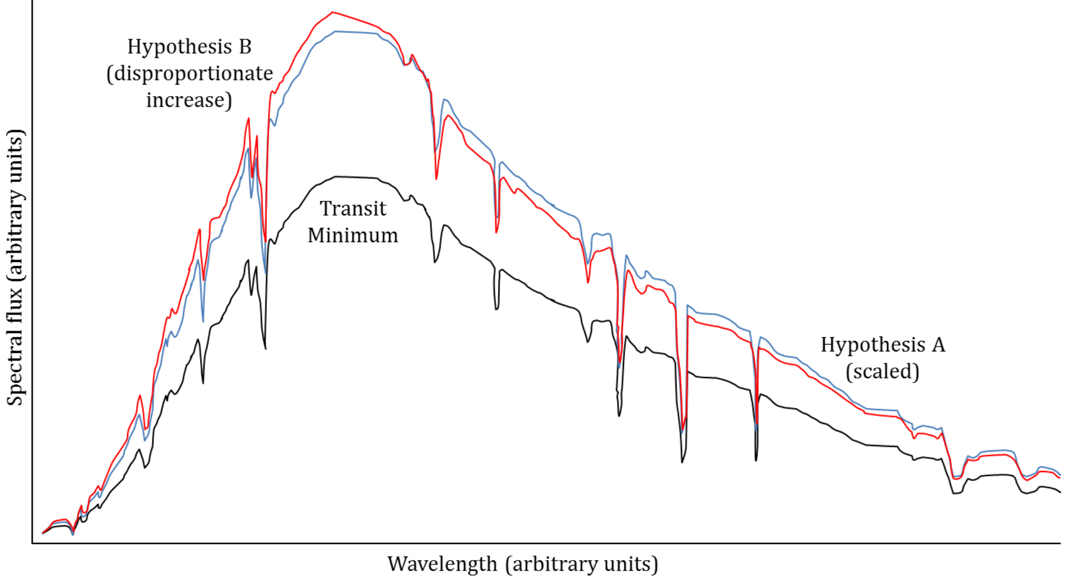
$$\frac{\Delta L_\mu}{L_D} = \frac{area\ occulted\ during\ \mu}{area\ of\ Dee's\ disk}$$

$$\Rightarrow \frac{0.992 - 0.990}{1} = \frac{A_{moon}}{A_D}$$

$$\Rightarrow \frac{R_{moon}^2}{R_D^2} = 0.002$$

M1

	$\therefore R_{moon} = R_D\sqrt{0.002}$ $= (1.296 \times 10^9\ m)\sqrt{0.002}$ $= \mathbf{5.80 \times 10^7\ m}$ Accept the alternative interpretation where there is no change in the transit condition, giving a $R_{moon} = 6.48 \times 10^7\ m$.	**A1**
g)	From the diagram, the width of the starspot is equal to the distance travelled by the transiting planet during event μ, less twice the radius of *Dee Two*.  $R_{spot} = v\Delta t - 2R_{D2}$ $= (30 \times 10^3\ m\ s^{-1})(0.3 \times 24 \times 60^2) - 2(1.296 \times 10^8\ m)$ $= \mathbf{5.18 \times 10^8\ m}$	**M1** **A1**
h)	Astronomers can distinguish between the two by observing the spectrum of *Dee* during the event μ with a spectrograph, specifically by measuring $\frac{F_\nu(shorter\ wavelengths)}{F_\nu(longer\ wavelengths)}$ A proportional increase in spectral flux (power per unit wavelength) across all wavelengths is indicative of a exomoon transit, since the disk of the exomoon is no longer occulting part of the disk of *Dee*. This part of the disk is similar to the surrounding unocculted disk of *Dee*, and thus has similar spectral characteristics and radiates such that the above ratio remains unchanged. However, when a starspot is occulted, the above ratio increases. This is because starspots are cooler than the surrounding photosphere (giving them their characteristically dim black appearance relative to rest of the disk), and hence radiate more in the longer wavelengths. Thus, when they are occulted, the spectral flux of longer wavelengths increases proportionally less compared to shorter wavelengths, since part of the initial spectral flux in longer wavelengths is due to the starspot. See following illustration.	**B1** **B1**



Accept other logical responses, e.g. starspot phenomena are transient and hence starspot transits are purely coincidence, rotation of star moving starspots creating an observation unlikely of an exomoon. Award 1m for description of the observational result that supports each hypothesis.

Reject all answers that suggest a Doppler effect within a Doppler effect.

Q6 Physics of stars and planets (20m)

a) i.	Assuming mass is conserved during the collapse, since $V \propto R^3$, then $\frac{R'}{R} = \sqrt[3]{\frac{V'}{V}}$ $\Rightarrow R' = R\sqrt[3]{\frac{V'}{V}}$ $= (10^6 \times 10^3\ m)\sqrt[3]{10^{-15}}$ $= \mathbf{1.00 \times 10^4}\ \boldsymbol{km}$ Responses that do not state the assumption that there is no mass loss are capped at 1m.	**M1** **A1**
a) ii.	Angular momentum is conserved during the collapse. Hence, as $L = mr^2\omega$ and $v = r\omega = 2\pi r T^{-1}$ $\therefore L = \frac{2\pi r^2}{T}$ Thus, $T \propto r^2$, $\therefore \frac{T'}{T} = \frac{R'^2}{R^2}$ $\Rightarrow T' = T\left(\frac{10^4\ m}{10^9\ m}\right)^2$ $= (20\ days)\left(\frac{10^4\ m}{10^9\ m}\right)^2$ $= \mathbf{2 \times 10^{-9}}\ \boldsymbol{days}$ Responses that do not state that angular momentum is conserved are capped at 1m.	**M1** **B1**
a) iii	The magnetic flux through the surface of Pock is $\Phi_B = BA$ $= 4\pi r^2 B$ As magnetic flux is conserved during the collapse, this implies Φ_B remains unchanged thus $\Rightarrow B \propto \frac{1}{r^2}$ $\therefore \frac{B'}{B} = \frac{r^2}{r'^2}$	**M1**

	$\Rightarrow B' = (10^{-2}\,T)\left(\frac{10^{9}\,m}{10^{4}\,m}\right)^{2}$ $= \mathbf{1.00 \times 10^{8}}\ \boldsymbol{m}$ Responses that do not state that magnetic flux is conserved are capped at 1m.	**A1**
a) **iv**	The minimum period for Pock corresponds to the period where the centripetal force for an object located at the equator is exactly equal to the force of gravity on that object. Any faster rotation will lead to mass loss. $\therefore \frac{mv^{2}}{R'} = \frac{GMm}{R'^{2}}$ $\Rightarrow GM = R'^{3}\omega^{2}$ Then, $T = \sqrt{\frac{4\pi^{2}R'^{3}}{GM}}$ $= \sqrt{\frac{4\pi^{2}(1.00 \times 10^{4}\,m)^{3}}{(6.67 \times 10^{-11}\,m^{3}\,kg^{-1}\,s^{-2})(1.5 \times 1.99 \times 10^{30}\,kg)}}$ $= \mathbf{4.45 \times 10^{-4}}\ \boldsymbol{s}$	**M1** **B1**
b)	From the formula sheet, the binding energy $BE \propto \frac{M^{2}}{R}$ Assuming the mass of Pick does not exceed the limit wherein neutron degeneracy pressure is overcome by gravity, let the mass of Pock and Pick be *m* and *2m* respectively. Let the total binding energy before and after be *BE’* and *BE*. $\frac{BE'}{BE} = \frac{BE_{Pick}}{BE_{Pock} + BE_{Puck}}$ Since *Pock* and *Puck* are identical, then $\frac{BE'}{BE} = \frac{BE_{Pick}}{2BE_{Pock}}$ Given neutron stars follow the mass-radius relationship $RM^{\frac{1}{3}} = k$, $\Rightarrow R \propto M^{-\frac{1}{3}}$ Thus,	**M1** **M1**

$$\frac{BE'}{BE} = \frac{1}{2}\left(\frac{M'}{M}\right)^{2}\left(\frac{R}{R'}\right)$$

M1

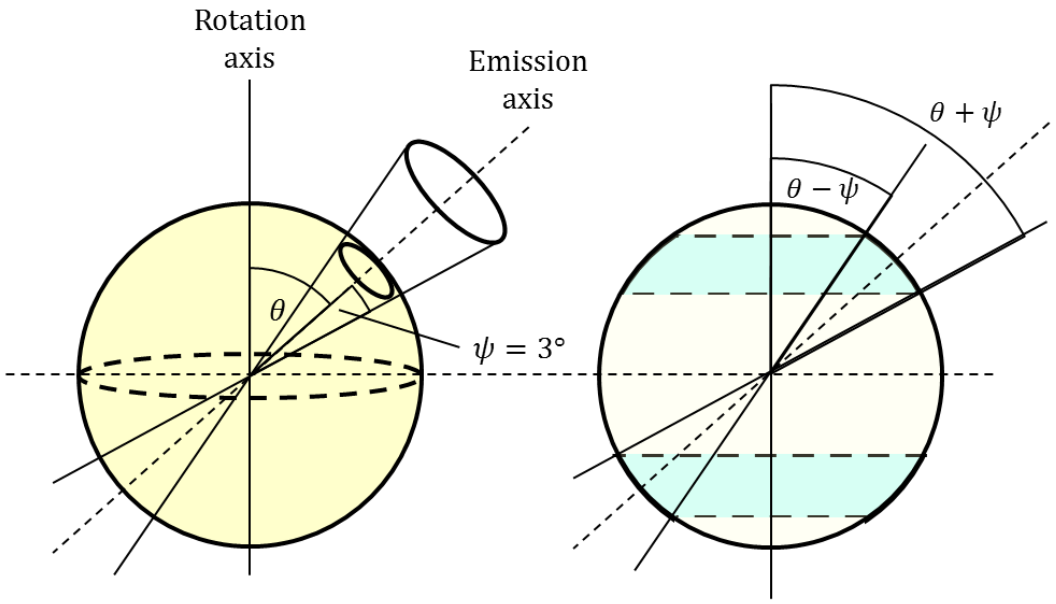
$$= \frac{1}{2}\left(\frac{M'}{M}\right)^{\frac{7}{3}}$$

$$= \frac{1}{2}\left(\frac{2m}{m}\right)^{\frac{7}{3}}$$

$$= \mathbf{2.52}$$

A1

c) **This problem is similar to Q8 in the theory paper of the 2012 IOAA.**



We assume that $\theta \geq \psi$ such that, as the pulsar rotates, that the path traced out (a solid angle) has no overlap, i.e. there is no part of the beam that can be seen continuously by an observer along any line of sight. (Note further that the bounds for non-overlap are $0 \leq \psi \leq 90°$ and $\psi \leq \theta \leq 90° - \psi$.)

The solid area Ω of a cone with apex angle 2θ is

$$\Omega = 2\pi(1 - cos\theta)$$

A1

Note that when $\theta = 90°$, the solid angle evaluates to 2π, i.e. such a cone is a hemisphere. Hence, the solid angle of a sphere is $2 \times 2\pi = 4\pi$.

Then, we can calculate the solid angle traced out by one beam as the pulsar rotates (the area in light cyan), as equal to the difference between the solid angles of two cones of different half-apex angles $\theta - \psi$ and $\theta + \psi$.

$$\begin{aligned}\Omega_{beam} &= \Omega_{\theta+\psi} - \Omega_{\theta-\psi} \\ &= 2\pi[1 - \cos(\theta + \psi)] - 2\pi[1 - \cos(\theta - \psi)] \\ &= 2\pi[\cos(\theta - \psi) - \cos(\theta + \psi)]\end{aligned}$$

M1

	Using the following trigonometric formula, $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ we can rewrite this as $\Omega_{beam} = 2\pi(\cos\theta\cos\psi + \sin\theta\sin\psi - \cos\theta\cos\psi + \sin\theta\sin\psi)$ $= 2\pi(2\sin\theta\sin\psi)$ $= 4\pi(\sin\theta\sin\psi)$	**M1**
	Since there are two beams (two of the shaded cyan areas), then the total solid angle traced out by the beams as they rotate are $\Omega_{beams} = 2 \times 4\pi(\sin\theta\sin\psi)$ $= 8\pi(\sin\theta\sin\psi)$ The probability p of detecting the pulses, assuming a random orientation of the pulsar beams w.r.t. Earth, is then equal to the ratio of the solid angle subtended by the beams as the pulsar rotates, over the total solid angle of a sphere. $p = \frac{\Omega_{beams}}{\Omega_{sphere}}$	**M1**
	$= \frac{8\pi(\sin\theta\sin\psi)}{4\pi}$ $= 2\sin\theta\sin 3°$ $= \mathbf{0.1047\sin\theta}$	**A1**
d)	Comparing the absolute bolometric magnitude of the Sun and Pick, $M_{Pick} - M_{\odot} = -2.5\lg\left(\frac{L_{Pick}}{L_{\odot}}\right)$	**M1**
	$\frac{L_{Pick}}{L_{\odot}} = 10^{\frac{M_{\odot}-M_{Pick}}{2.5}}$ $\Rightarrow L_{Pick} = 10^{\frac{M_{\odot}-M_{Pick}}{2.5}} L_{\odot}$	**M1**
	$= 10^{\frac{4.75-(-3.5)}{2.5}}(3.85 \times 10^{26}\ m)$ $= 10^{\frac{4.75-(-3.5)}{2.5}}(3.85 \times 10^{26}\ m)$ $= \mathbf{7.68 \times 10^{29}\ W}$	**A1**