4th Japan Astronomy Olympiad National Finals

Theoretical Solutions

Problem 1

Question 1

(1)

(a) Equation of circular motion for the electron:

$$\frac{m_e v^2}{r} = \frac{k_0 Z e^2}{r^2}$$

(b) From (a):

$$v = \sqrt{\frac{k_0 Z e^2}{m_e r}}$$

Substituting into the de Broglie quantization condition:

$$r_n = \frac{1}{2\pi}\frac{nh}{m_e v} = \frac{1}{2\pi}\frac{nh}{m_e}\sqrt{\frac{m_e}{k_0 r_n Z e^2}}$$

Solving for r_n:

$$\boxed{r_n = \frac{n^2 h^2}{4\pi^2 k_0} \cdot \frac{1}{Z m_e e^2}}$$

(c)

$$E_n = \frac{1}{2}m_e v^2 - \frac{k_0 Z e^2}{r_n} = -\frac{k_0 Z e^2}{2r_n} = -\frac{2k_0^2\pi^2 Z^2 m_e e^4}{n^2 h^2}$$

(d) From $\frac{hc}{\lambda} = E_n - E_l$:

$$R_Z = \frac{2k_0^2\pi^2 Z^2 m_e e^4}{n^2 h^2}$$

(2)

Spectral lines emitted by helium (He) atoms, which have $Z = 2$.

Question 2

(1)

From the equation of motion of Earth around the Sun:

$$\frac{M_\oplus a \left(\frac{2\pi}{T_0}\right)^2}{1} = \frac{GM_\odot M_\oplus}{a^2} \implies T_0 = 2\pi\sqrt{\frac{a^3}{GM_\odot}}$$

(2)

Force balance at L_1:

$$-\frac{GmM_\odot}{(1-x_1)^2a^2}+\frac{GmM_\oplus}{x_1^2a^2}+\frac{m(1-x_1)GM_\odot}{a^2}=0$$

Force balance at L_2:

$$-\frac{GmM_\odot}{(1+x_2)^2a^2}-\frac{GmM_\oplus}{x_2^2a^2}+\frac{m(1+x_2)GM_\odot}{a^2}=0$$

(3)

Expanding the first term at L_1 to first order in x_1:

$$-\frac{GmM_\odot}{a^2}(1+2x_1)+\frac{GmM_\oplus}{x_1^2a^2}+\frac{m(1-x_1)GM_\odot}{a^2}=0$$

Solving:

$$x_1=\sqrt[3]{\frac{M_\oplus}{3M_\odot}}\approx 1\times 10^{-2}$$

Similarly for L_2:

$$x_2=\sqrt[3]{\frac{M_\oplus}{3M_\odot}}\approx 1\times 10^{-2}$$

(4)

Because the radiation sources (the Sun and Earth) always lie in the same direction as seen from L_2, the L_2 point can block radiation most efficiently.

(5)

Both L_1 and L_2 are local maxima (saddle points) of the effective potential, and are therefore unstable equilibrium points.

Question 3

(1)

Spectroscopic observation of a star is used to identify its composition from the intensities of absorption lines characteristic of each element.

(2)

Open clusters are irregular groupings of tens to thousands of young, metal-rich (Population I) stars located in the galactic disk. Globular clusters, by contrast, are tightly packed, roughly spherical collections of more than 10^5 old, metal-poor (Population II) stars, distributed in the galactic halo.

(3)

Assuming all the stars in the cluster formed simultaneously, the age of the cluster equals the main-sequence lifetime of the stars that are just beginning to leave the main sequence (the main-sequence turn-off point).

Question 4

(1)

Adaptive optics: A technology that measures and corrects wavefront distortions caused by atmospheric turbulence in real time, dramatically improving the resolution of telescopes.

(2)

Interstellar reddening: The phenomenon by which starlight appears redder than it actually is because interstellar dust in space scatters and absorbs shorter-wavelength (blue) light more strongly.

(3)

Annual aberration: Due to Earth's orbital motion, light from distant objects appears to arrive from a slightly different direction than their true position; this angular displacement is called annual (stellar) aberration.

(4)

Recombination (decoupling): An event approximately 380,000 years after the Big Bang, when the temperature of the universe had dropped enough for electrons and protons to combine into neutral hydrogen, allowing photons to travel freely in straight lines for the first time.

Problem 2

(1)(a)

$$v = \frac{2\pi r}{P} = \frac{2 \times 3.1 \times 7.8 \times 10^8 \text{ m}}{3.7 \times 10^8 \text{ s}} \approx 13 \text{ m s}^{-1}$$

(1)(b)

$$\frac{a_{\text{Peg}}}{a_J} = \left(\frac{P_{\text{Peg}}}{P_J}\right)^{2/3} \approx 10^{-2}$$

Using $v \approx \dfrac{2\pi}{P}\dfrac{m}{M}a$:

$$m = \frac{M}{2\pi}\frac{Pv}{a} \implies \frac{m_{\text{Peg}}}{m_J} = \frac{P_{\text{Peg}}}{P_J} \cdot \frac{v_{\text{Peg}}}{v_J} \cdot \left(\frac{a_{\text{Peg}}}{a_J}\right)^{-1} \approx 0.43$$

(2)(a)

Minimum inclination factor: 0.5,   Maximum: 0, 1

(2)(b)

$$\frac{1}{\pi} \times 56 \text{ m s}^{-1} \times 3.7 \times 10^5 \text{ s} \approx 6.6 \times 10^3 \text{ km}$$

(3)

The amplitude of the radial displacement is:

$$\Delta R = \frac{v_0 T_{\text{pul}}}{2\pi}$$

The constraint $\Delta R \leq R$ gives:

$$v_0 T_{\text{pul}} \leq 2\pi \times 7.0 \times 10^8 \text{ m} \approx 4.4 \times 10^9 \text{ m}$$

Plot this boundary on a log-log graph. From the resulting figure, the possibility that 51 Peg is a Cepheid variable *cannot* be ruled out.

(4)

Since luminosity $\propto R^2$, the brightness constraint is:

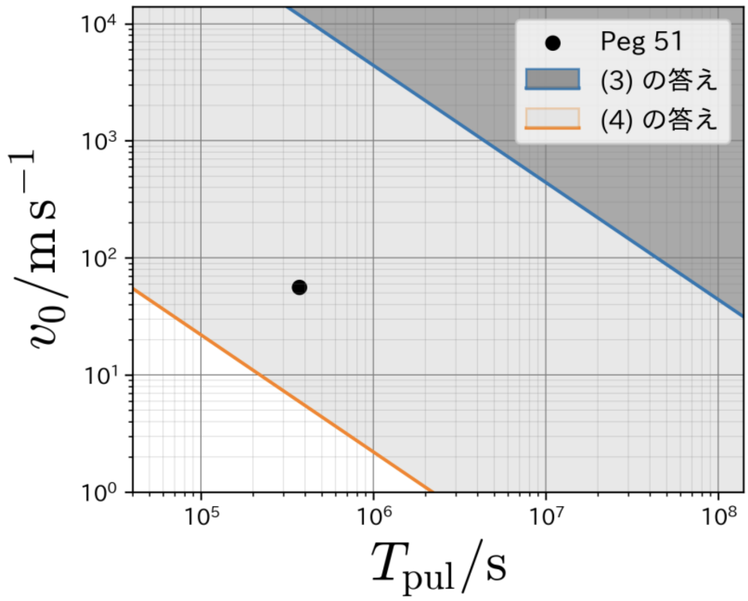
$$\left(1 + \frac{\Delta R}{R}\right)^2 \leq 1.001$$

For small $\Delta R/R$:

$$\Delta R \leq 0.0005\, R$$

$$v_0 T_{\text{pul}} \leq 0.0005 \times 2\pi \times 7.0 \times 10^8 \text{ m} \approx 2.2 \times 10^6 \text{ m}$$

From the figure, the Cepheid hypothesis for 51 Peg *can* be ruled out.



(5)

(Example answer) In part (1), we assumed the planet's orbital plane is parallel to the line of sight. If instead the plane is inclined, the true orbital speed of the planet would be larger than what was derived. Consequently, the region in which the Cepheid hypothesis can be rejected (derived in parts (3) and (4)) shifts toward the upper right and becomes narrower.

Problem 3

Question 1

(1)(a)

$$H_0 \approx 5 \times 10^2 \ \mathrm{km\,s^{-1}\,Mpc^{-1}}$$

(1)(b)

Assuming the universe began as a point and all galaxies have receded at constant speed, the age of the universe equals the Hubble time:

$$\frac{1}{H_0} = \frac{1}{5 \times 10^2 \ \mathrm{km\,s^{-1}\,Mpc^{-1}}} = \frac{1}{5 \times 10^2 \ \mathrm{km\,s^{-1}\,Mpc^{-1}}} \times \frac{1 \ \mathrm{yr}}{3.1 \times 10^7 \ \mathrm{s}} \times \frac{3.1 \times 10^{19} \ \mathrm{km}}{1 \ \mathrm{Mpc}} = 2 \times 10^9 \ \mathrm{yr}$$

(2)

Let $\lambda(r)$ be the wavelength of light after traveling distance r. From $E = hc/\lambda$:

$$\frac{\lambda(r)}{\lambda(0)} = e^{kr}$$

Taking the logarithm:

$$kr = \ln\left(\frac{\lambda(r)}{\lambda(0)}\right) = \ln(1+z) \approx z$$

Therefore:

$$k = \frac{z}{r} = \frac{H_0}{c} = \frac{500 \ \mathrm{km\,s^{-1}\,Mpc^{-1}}}{3 \times 10^5 \ \mathrm{km\,s^{-1}}} = 1.7 \times 10^{-3} \ \mathrm{Mpc^{-1}}$$

Question 2

(1)(a)

The flux at distances d_1 and d_2 from a source of luminosity L:

$$f_1 = \frac{L}{4\pi d_1^2}, \quad f_2 = \frac{L}{4\pi d_2^2}$$

The solid angle subtended (radius r):

$$\Omega_1 = \frac{\pi r^2}{4\pi d_1^2}, \quad \Omega_2 = \frac{\pi r^2}{4\pi d_2^2}$$

Surface brightness:

$$S_1 = \frac{f_1}{\Omega_1} = \frac{L}{\pi r^2}, \quad S_2 = \frac{f_2}{\Omega_2} = \frac{L}{\pi r^2}$$

Hence the surface brightness is *independent of distance*.

(1)(b)

Under the tired-light hypothesis, from Question 1(2), the photon frequency decreases by a factor of $(1+z)$, so each photon's energy decreases by $(1+z)$, and thus the surface brightness decreases as $(1+z)^{-1}$.

(2)(a)

Redshift stretches the photon wavelength by $(1+z)$, reducing each photon's energy by $(1+z)^{-1}$. Additionally, cosmic expansion stretches the time interval between photon arrivals by $(1+z)$, reducing the energy received per unit time by another factor of $(1+z)^{-1}$. Therefore:

$$f = \frac{L}{4\pi D^2 (1+z)^2}$$

(2)(b)

The solid angle $\Omega \propto \theta^2 \propto (1+z)^2$. Hence:

$$S = \frac{f}{\Omega} \propto (1+z)^{-4}$$

(3)

Expanding universe: Surface brightness $\propto (1+z)^{-4}$, so at $z \approx 0.4$:

$$20.5 - \frac{5}{2}\log(1.4)^4 \approx 19.0 \text{ mag}$$

Tired-light: Surface brightness $\propto (1+z)^{-1}$:

$$20.5 - \frac{5}{2}\log(1.4) \approx 20.1 \text{ mag}$$

The data support the **expanding universe** model.

Problem 4

Question 1

Stars born with mass $\geq 8\,M_\odot$ fuse hydrogen in their cores, forming a helium core that pushes hydrogen burning outward. The star expands into a red giant. The core then undergoes helium fusion, followed by fusion of progressively heavier elements (C, O, ...) until an iron core forms. Iron cannot release energy through fusion, so radiation pressure can no longer support the core against gravity; the core undergoes gravitational collapse and the star explodes as a core-collapse supernova, leaving behind a neutron star. For sufficiently massive progenitors, the neutron star mass exceeds the maximum stable mass and collapses further into a stellar-mass black hole.

Question 2

$$M_{\rm BH} = \left(\frac{1 \times 10^3 \text{ au}}{1 \text{ au}}\right)^3 \left(\frac{16 \text{ yr}}{1 \text{ yr}}\right)^{-2} M_\odot \approx 4 \times 10^6\, M_\odot$$

Question 3

(1)(a)

$$d_{\rm BLR} = c\tau = 3.00 \times 10^8 \text{ m s}^{-1} \times (50 \times 24 \times 60 \times 60) \text{ s} \approx 1 \times 10^{15} \text{ m}$$

(1)(b)

Balancing gravity and centripetal force for gas in the BLR (SMBH mass M):

$$\frac{GmM}{d_{\rm BLR}^2} = \frac{mv^2}{d_{\rm BLR}} \implies M = \frac{d_{\rm BLR}v^2}{G} = \frac{1.3 \times 10^{15} \text{ m} \times (2 \times 10^6 \text{ m s}^{-1})^2}{6.67 \times 10^{-11} \text{ m}^3\,\text{kg}^{-1}\,\text{s}^{-2}} \approx 4 \times 10^7\, M_\odot$$

(2)(a)

Let $L_{\rm cont} = C\, d_{\rm BLR}^2$. Then:

$$\log L_{\rm cont} = \log C + 2 \log d_{\rm BLR} \implies \log d_{\rm BLR} = \tfrac{1}{2} \log L_{\rm cont} + \text{const}$$

From $M \propto d_{\rm BLR} v^2$:

$$\log M_{\rm BH} = \log d_{\rm BLR} + 2 \log v + \text{const} = \frac{1}{2} \log L_{\rm cont} + 2 \log v + \text{const}$$

Therefore $\alpha = 0.5$, $\beta = 2$.

(2)(b)

$$\log(4.0 \times 10^8) = 0.5 \log(7.8 \times 10^{11}) + 2 \log(3.1 \times 10^3) + \gamma \implies \gamma \approx -4.3$$

(2)(c)

$$\log\left(\frac{M_{\rm BH}}{M_\odot}\right) = 0.5 \log(8.3 \times 10^{10}) + 2 \log(1.3 \times 10^3) - 4.3 \approx 7.36 \implies M_{\rm BH} \approx 2 \times 10^7\, M_\odot$$

Question 4

(1)

(a) $\dfrac{L}{4\pi r^2}$

(b) $\dfrac{\sigma_T}{4\pi c r^2}$

(c) $\dfrac{G m_p M}{r^2}$

(d) $\leq$

(e) $\dfrac{4\pi G c m_p}{\sigma_T} M$

(2)

$$L_{\text{Edd}} = \eta \dot{M}_{\text{Edd}} c^2 = \frac{4\pi G c m_p M}{\sigma_T} \implies \dot{M}_{\text{Edd}} = \frac{4\pi G m_p}{\eta c \sigma_T} M \approx 2\, M_\odot\, \text{yr}^{-1}$$

(3)

Solving the given differential equation with initial mass M_0 at $t = 0$:

$$M_{\text{BH}}(t) = M_0 \exp\left(\frac{(1-\eta)\, 4\pi G m_p}{\eta c \sigma_T}\, t \right)$$

Time to grow by a factor of 10:

$$t_{10} = \ln 10 \times \left(\frac{(1-\eta)\, 4\pi G m_p}{\eta c \sigma_T} \right)^{-1} \approx 1 \times 10^8 \text{ yr}$$

(4)

Using log(Age) values of -0.3 (at $z = 9$) and -1.0 (at $z = 30$):

$$M_{\text{BH}}(z = 30) = 10^8 \times 10^{-\dfrac{10^{-0.3} - 10^{-1.0}}{1.15 \times 10^8 \times 10^4}} \approx 3 \times 10^4\, M_\odot$$