4th Japan Astronomy Olympiad National Finals

Theory Problems

February 16, 2025 13:15–15:45

Instructions

1. Do not open this problem booklet until instructed to begin.
2. This booklet contains 16 pages in total. If any pages are missing, out of order, or have unclear printing, raise your hand and notify the proctor.
3. Use only a black pencil or black mechanical pencil for all answers.
4. On the answer sheet, write not only the final answer but also the calculation process and explanations that lead to it, as necessary.
5. Write your examinee number in the designated field on the answer sheet. Do not write your examinee number anywhere else.
6. Write your answers in the designated answer area for each problem. If it is unclear which sub-question a portion of your answer corresponds to, it may receive no credit. You do not need to answer sub-questions in order, as long as it is clear which sub-question each answer addresses.
7. Do not write any irrelevant characters, symbols, or marks in the answer area.
8. The margins of this booklet may be used for scratch work, but do not tear any pages.
9. Do not take the answer sheet home.
10. After the exam ends, take the problem booklet and calculation paper home.
11. Questions about the problems will not be answered. If you believe there is an error in a problem, note it on the answer sheet; it will be taken into consideration during grading.

Problem 1

Answer the following independent questions (Questions 1–4).

Question 1

Answer the following questions about atomic structure and spectra. You may use the symbols and values of the constants in Table 1–1 in your answers as needed.

Table 1–1: Constants

Constant	Symbol	Value
Electron mass	m_e	9.11×10^{-31} kg
Coulomb's law constant	k_0	$8.99 \times 10^{9}\ \mathrm{N\,m^2\,C^{-2}}$
Elementary charge	e	1.60×10^{-19} C
Planck constant	h	6.63×10^{-34} J s
Speed of light	c	$3.00 \times 10^{8}\ \mathrm{m\,s^{-1}}$
Pi	π	3.14

(1) Consider an electron undergoing circular motion around the nucleus of an atom with atomic number Z. Since the gravitational force is negligible compared to the Coulomb force, assume the electron undergoes uniform circular motion governed solely by the Coulomb force.

(a) Let r be the nucleus–electron distance and v be the orbital speed. Write down the force balance equation for the electron in the radial direction.

(b) The following *de Broglie quantum condition* holds for the electron's motion:

$$2\pi r = \frac{nh}{m_e v} \qquad (n = 1, 2, \ldots)$$

where n is the quantum number of the electron's energy level. Express the orbital radius r_n of an electron at level n using n, Z, and other constants only.

(c) Express the mechanical energy E_n of an electron at level n using n, Z, and other constants only.

(d) Consider an electron transitioning from level n to level l ($n > l$). The energy difference between the levels is emitted as electromagnetic radiation of wavelength λ, satisfying:

$$\frac{1}{\lambda} = R_Z\left(\frac{1}{l^2} - \frac{1}{n^2}\right)$$

Express R_Z using Z and physical constants only.

(2) In 1897, Pickering discovered the following spectral series in the spectrum of a certain astronomical object:

$$\frac{1}{\lambda} = R'\left(\frac{1}{2^2} - \frac{1}{k^2}\right) \qquad (k = 2.5,\ 3,\ 3.5,\ 4,\ \ldots)$$

The value of R' was found to be nearly equal to the Rydberg constant R of the Balmer series

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right) \quad (n = 3, 4, \ldots),$$

but the Pickering series is clearly distinct in that it also allows half-integer values of k (i.e. $k = 2.5, 3.5, \ldots$). Explain briefly (in about one line) what kind of atom produces this spectrum.

Question 2

The James Webb Space Telescope (JWST) is a space telescope launched to perform observations in the infrared. To carry out infrared observations, the telescope must be kept at an extremely low temperature — primarily to suppress thermal noise (radiation from the telescope itself) — and is equipped with a sunshield.

Here we consider JWST's orbit. We wish to insert JWST into an orbit that revolves around the Sun while remaining within the non-negligible gravitational influence of Earth. In such an orbit, Kepler's laws do not hold. Considering the rotating reference frame that co-revolves with JWST around the Sun, JWST experiences three forces: Earth's gravity, the Sun's gravity, and the centrifugal force. In this frame, there are two points along the Sun–Earth line where these three forces balance. These are called *Lagrange points*. The one inside Earth's orbit is called L_1, and the one outside is called L_2 (see Figure 1–1). An object at a Lagrange point revolves around the Sun with the same period T_0 as Earth's orbital period, while remaining near that Lagrange point.

Answer the following questions. You may use the symbols and values of the constants in Table 1–2 as needed.

Table 1–2: Constants

Constant	Symbol	Value
Solar mass	$M_\odot$	1.99×10^{30} kg
Earth mass	$M_\oplus$	5.97×10^{24} kg
Earth's orbital semi-major axis	a	1.50×10^{11} m
Gravitational constant	G	$6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

(1) By considering the force balance on Earth, express Earth's orbital period T_0 (for a circular orbit of radius a) using a and other constants.

(2) Let the distances from Earth to L_1 and L_2 be $x_1 a$ and $x_2 a$ respectively, and let the mass of JWST be m. Write down the radial force-balance equations for JWST located at L_1 and at L_2. Your final answers must not contain T_0.

(3) Using the results of (1) and (2), find x_1 and x_2, expressing each in terms of appropriate physical quantities. Also compute numerical values of x_1 and x_2 to 2 significant figures. Since Earth's mass is much less than the Sun's, you may assume $|x_1| \ll 1$ and $|x_2| \ll 1$.

(4) Which of L_1 or L_2 would more efficiently block radiation from both Earth and the Sun if JWST were placed there? Give your answer in about one line, with reasoning.

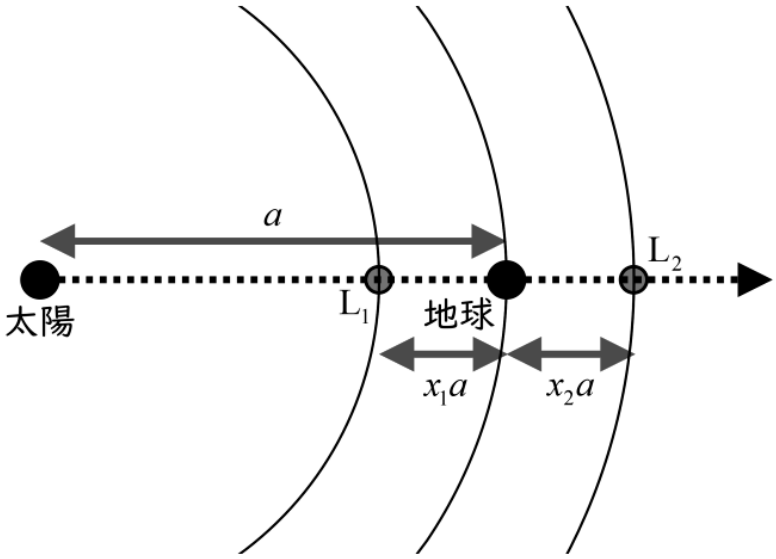

Figure 1–1: Positions of the Sun, Earth, L_1, and L_2. The Sun is at the left; Earth is at the right end of the arrow of length a (the Earth–Sun distance). L_1 is at distance x_1a inside Earth's orbit, and L_2 is at distance x_2a outside.

(5) In practice, JWST is not placed exactly at L_1 or L_2 but orbits around one of them. Consider why it is impossible to place JWST precisely at L_1 or L_2, and answer in about one line.

Question 3

Answer the following questions about star clusters and stars.

(1) Describe in about two lines the observational method used to investigate the chemical composition of a star's atmosphere.

(2) Describe in about three lines the differences between open clusters and globular clusters, focusing on the characteristics and number of their constituent stars and their spatial distributions.

(3) Figure 1–2 is a colour–magnitude diagram (CMD) of a certain star cluster. Describe in about two lines how this diagram can be used to estimate the age of the cluster (the time elapsed since its formation).

Question 4

Briefly explain each of the following terms in about two lines each.

(1) Adaptive optics

(2) Interstellar reddening

(3) Annual aberration of light

(4) Recombination epoch ("clearing of the universe")

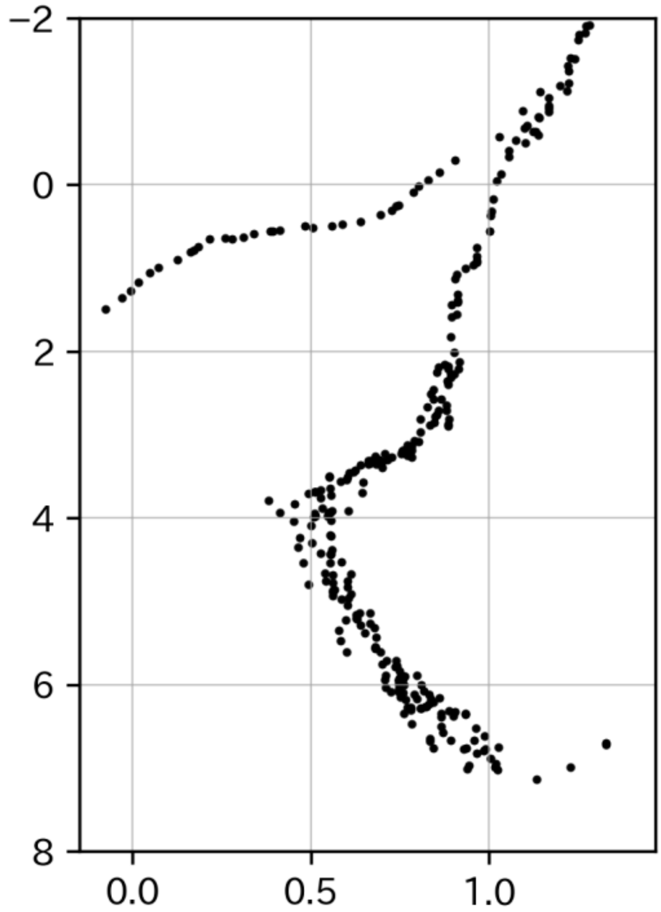

Figure 1–2: Colour–magnitude diagram of a star cluster. Horizontal axis: colour index (0.0–1.2); Vertical axis: absolute magnitude (−2 to 8). The main sequence and a turnoff point are visible.

Problem 2

Various methods have been developed to detect planets orbiting stars outside our solar system. When a planet orbits a star, the star also wobbles in a small orbit around the common centre of mass. When the star is observed spectroscopically from Earth, this causes a periodic variation in the observed radial velocity. Detecting this variation reveals the existence of an exoplanet — this is the *radial velocity method*.

Figure 2–1 shows the radial velocity variation of "51 Peg", the first star confirmed to host an exoplanet via the radial velocity method. The period of the radial velocity variation was $4.23\,\text{days} = 3.7 \times 10^5\,\text{s}$.

In the following problems, assume the radius and mass of 51 Peg are both equal to those of the Sun.

(1) Answer the following questions about the wobble of the star. For simplicity, assume the planet's orbital eccentricity is zero, and the planet's orbital plane is parallel to the line of sight. Also assume that all of 51 Peg's radial velocity variation is caused by the single planet "51 Peg b", with no other contributions.

(a) In the solar system, the Sun–Jupiter common centre of mass does not coincide with the solar centre, so the Sun also wobbles as Jupiter orbits. Find the

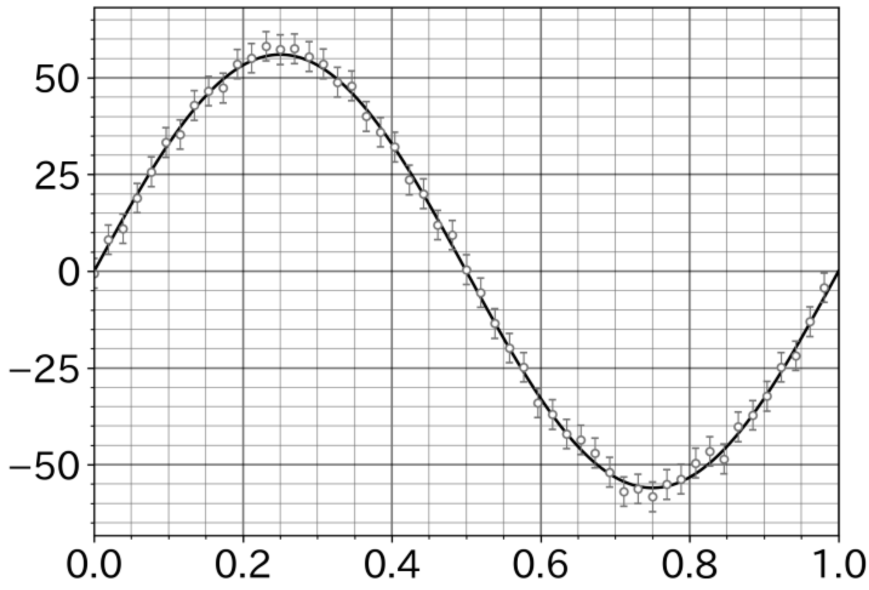

Figure 2–1: Radial velocity variation of 51 Peg as a function of time. The horizontal axis is normalised so that one orbital period equals 1. Vertical axis: radial velocity / $\mathrm{m\,s^{-1}}$ (range approximately -55 to $+55$). The curve is a smooth sinusoid with amplitude $\approx 50\,\mathrm{m\,s^{-1}}$.

maximum radial velocity of the Sun in $\mathrm{m\,s^{-1}}$ to 2 significant figures. Assume Jupiter's mass is 1/1000 of the Sun's mass, Jupiter's orbital semi-major axis is 7.8×10^8 km, and Jupiter's orbital period is $11.9\,\mathrm{yr} = 3.7 \times 10^8$ s.

(b) By what factor does the 51 Peg – 51 Peg b distance exceed the Sun–Jupiter distance? By what factor does 51 Peg b's mass exceed Jupiter's mass? Read the necessary values from Figure 2–1.

(2) Astronomer A observed 51 Peg and obtained the results in Figure 2–1, concluding that 51 Peg hosts a planet. However, teacher Nodoka pointed out: "The radial velocity variation of this star might not be caused by the gravitational wobble from a planet; it could instead be caused by 51 Peg being a *pulsating variable star*, i.e. the star's surface periodically expands and contracts (pulsates)."

In the following, assume 51 Peg has no exoplanet, and the radial velocity variation is caused only by pulsation. For simplicity, assume the star is always spherical with its radius varying sinusoidally in time, with a mean equal to the solar radius. The measured velocity is taken to be the velocity of the point on the stellar surface with the largest velocity variation as seen by the observer.

(a) In Figure 2–1, at what times (within the range shown) is the radius of 51 Peg at its maximum and minimum? Express your answers in units of the period, listing all possibilities within the figure.

(b) Find the difference between the maximum and minimum radii of 51 Peg in km. You may use (without proof) the fact that for the sinusoidal wave $y = A\sin\left(\frac{2\pi}{T}x\right)$ over $0 \le x \le T/2$, the area of the positive region is $\dfrac{AT}{\pi}$.

(3) The amplitude of the radius change found in (2)(b) must not exceed the stellar ra-

dius; i.e. if half the computed radius change exceeds the stellar radius, the pulsating-variable-star hypothesis for 51 Peg can be rejected.

Consider a star with the same radius and mass as 51 Peg showing a radial velocity variation analogous to that of 51 Peg. On the log–log graph paper provided, plot the region in which the pulsating-variable-star hypothesis is *rejected*, with the period of the radial velocity variation (s) on the horizontal axis and the amplitude of the radial velocity ($\mathrm{m\,s^{-1}}$) on the vertical axis. Also mark the position of 51 Peg on the graph. Take the radius of 51 Peg to be 7.0×10^5 km.

(4) When a star pulsates, a brightness variation accompanying the pulsation can be observed. Continuous observation of 51 Peg revealed a brightness variation of no more than 0.1% of the mean luminosity. Using this information, an even stronger constraint can be placed on the pulsating-variable-star hypothesis. For simplicity, assume the surface temperature of the pulsating variable star does not change due to pulsation. Reject the pulsating-variable-star hypothesis when the computed brightness variation (from the radial velocity period and amplitude) is $\geq 0.1\%$. Overlay the rejected region on the graph drawn in (3), clearly distinguishing it from the region drawn in (3).

(5) In problems (1)–(4) above, various simplifying assumptions were made. Choose one of these assumptions and discuss qualitatively how the results of (3) and/or (4) would change if that assumption were not satisfied.

Problem 3

Answer the following questions on cosmology. You may use $1\,\mathrm{pc} = 3.1 \times 10^{13}\,\mathrm{km}$ and $c = 3.0 \times 10^{5}\,\mathrm{km\,s^{-1}}$ as needed.

Question 1

In 1929, Hubble discovered that the recession speed of a galaxy is proportional to its distance, proposing the Hubble–Lemaître law. Figure 3–1 reproduces Hubble's original scatter plot of galaxy distance vs. recession speed, with a regression line overlaid.

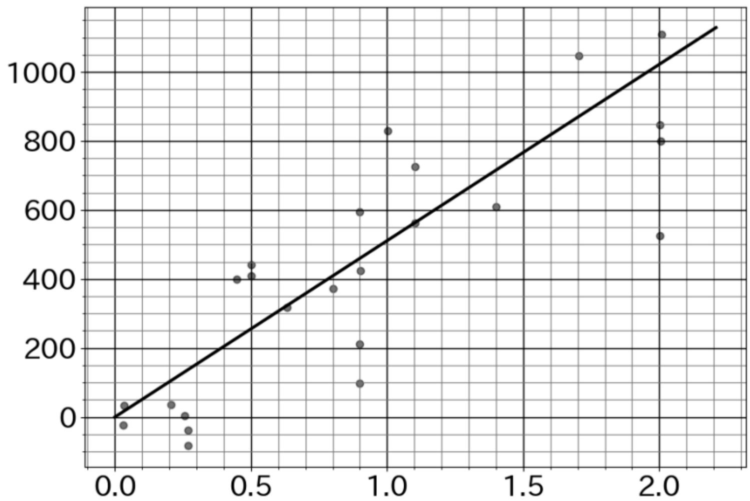

Figure 3–1: Scatter plot of galaxy distance vs. recession speed. Horizontal axis: distance / Mpc (0.0–2.0); Vertical axis: recession speed / $\mathrm{km\,s^{-1}}$ (0–1000). A straight regression line passes through the origin.

(1) The Hubble–Lemaître law is explained by the expansion of the universe.

(a) By reading Figure 3–1, find the Hubble constant to 1 significant figure.

(b) With an appropriate assumption, use the Hubble constant found in (a) to estimate the age of the universe to 1 significant figure, in years. State clearly the assumption you are using.

One alternative explanation for the Hubble–Lemaître law is the *tired light hypothesis*: light loses energy somehow as it propagates through space. Specifically, if $E(r)$ is the energy of light after travelling a distance r through space, then

$$E(r) = E(0)\, e^{-kr},$$

where $k > 0$ is a constant. Under the tired light hypothesis, the universe is not expanding, so no cosmological distance effects are considered.

(2) Confirm that the tired light hypothesis can reproduce the Hubble–Lemaître law even without cosmic expansion, and determine the value of k that satisfies the Hubble constant found in (1)(a). Since we are considering the nearby universe where the law holds, take $|z| \ll 1$. You may use the approximation $\log_e(1+x) \simeq x$ for $|x| \ll 1$.

Question 2

To observationally test whether the expanding universe model or the tired light hypothesis is correct, we study the dependence of a galaxy's *surface brightness* (energy received per unit area, per unit time, per unit solid angle by the observer) on redshift z. This is called the *Tolman surface brightness test*. Here, z may be taken to be small. For simplicity, ignore the effects of interstellar extinction.

(1) First, consider the tired light hypothesis.

  (a) Show that, in the absence of cosmic expansion and photon energy loss, surface brightness is independent of distance to the object. Assume the spatial curvature of the universe is zero.

  (b) Show that, under the tired light hypothesis, surface brightness is inversely proportional to $(1 + z)$.

(2) Next, consider the expanding universe model. Consider a galaxy of luminosity L at distance D. Suppose light emitted by the galaxy at times t_1 and $t_1 + \Delta t_1$ reaches the observer at times t_0 and $t_0 + \Delta t_0$ respectively, with

$$\frac{\Delta t_0}{\Delta t_1} = 1 + z.$$

  (a) Show that the observed energy flux is

$$f = \frac{L}{4\pi D^2 (1 + z)^2}.$$

  (b) The angle θ subtended by a galaxy of diameter l is

$$\theta = \frac{l}{D}(1 + z).$$

  Using this, show that in the expanding universe, surface brightness is inversely proportional to $(1 + z)^4$.

(3) A galaxy at $z \simeq 0$ has surface brightness $19.0\,\mathrm{mag\,arcsec^{-2}}$, and a galaxy at $z \simeq 0.4$ has surface brightness $20.5\,\mathrm{mag\,arcsec^{-2}}$. Assuming these galaxies have the same intrinsic luminosity, discuss which of the two models — expanding universe or tired light — is supported.

Problem 4

It is believed that nearly all galaxies contain a supermassive black hole (SMBH) with mass exceeding $10^6\ M_\odot$ at their centres. Answer the following questions about black holes. You may use the constants in Table 4–1 as needed.

Table 4–1: Constants

Constant	Symbol	Value
Pi	π	3.14
Gravitational constant	G	$6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Speed of light	c	$3.00 \times 10^{8}\ \mathrm{m\,s^{-1}}$
Proton mass	m_p	$1.67 \times 10^{-27}\ \mathrm{kg}$
Thomson scattering cross-section (electron)	σ_T	$6.65 \times 10^{-29}\ \mathrm{m^2}$
Astronomical unit	au	$1.50 \times 10^{11}\ \mathrm{m}$
Solar mass	$M_\odot$	$1.99 \times 10^{30}\ \mathrm{kg}$
Solar luminosity	$L_\odot$	$3.83 \times 10^{26}\ \mathrm{W}$

Question 1

It is believed that black holes with masses of a few to a few tens of $M_\odot$ are formed at the final evolutionary stage of certain stars. Outline in 7–8 lines the mass of stars capable of forming black holes and the evolutionary process from birth as a main-sequence star to the formation of a black hole.

Question 2

At the centre of the Milky Way, there exists the radio source Sagittarius A* (Sgr A*), believed to be an SMBH. Very close to Sgr A*, the star S2 orbits around Sgr A* on an elliptical orbit with Sgr A* at one focus. Observations show that S2 has an orbital period of 16 years, a semi-major axis of 1×10^3 au, and an eccentricity of 0.88. Find the black hole mass M_{BH} in units of $M_\odot$ to 1 significant figure. Relativistic effects may be ignored here.

Question 3

Some SMBHs in external galaxies shine brightly as gravitational energy is released by infalling material. Objects in which a very compact region at the galaxy centre emits strong electromagnetic radiation across many wavelengths through this process are called *active galactic nuclei* (AGN). As in Question 2, direct kinematic measurements of nearby stars or gas can be used to estimate M_{BH} only for the Milky Way and nearby galaxies with current observational technology. However, observations of AGN allow M_{BH} to be estimated even for SMBHs in distant galaxies.

(1) Around the SMBH in an AGN, there exists a region filled with ionised gas that is gravitationally bound to the SMBH and moves at high speed. As a result, spectroscopic observations of AGN can reveal broad emission lines from this region (broadened by the Doppler effect). This region is called the *broad line region* (BLR).

An AGN was repeatedly observed over time. The continuum component from the AGN centre showed luminosity variability. The luminosity of the broad emission lines from the BLR varied in response, but with a time delay relative to the continuum.

(a) Analysis of the time delay revealed that the light-crossing timescale for the distance from the AGN centre to the BLR ($d_{\rm BLR}$) is approximately 50 days. Find $d_{\rm BLR}$ in metres to 1 significant figure.

(b) Spectroscopic data show that the BLR velocity in this AGN is $v = 2 \times 10^3\,\mathrm{km\,s^{-1}}$. Assuming the BLR motion is gravitationally bound to the SMBH and follows Keplerian motion, estimate $M_{\rm BH}$ in units of $M_\odot$ to 1 significant figure.

(2) The method in (1) requires repeated spectroscopic observations of the same AGN over a long period, making it very observationally expensive. Applying this method to many AGN revealed a strong positive correlation between $d_{\rm BLR}$ and the continuum luminosity $L_{\rm cont}$. Using this relation, a relatively low-cost $M_{\rm BH}$ estimation method was developed that uses a single spectroscopic observation of an AGN:

$$\log_{10}\left(\frac{M_{\rm BH}}{M_\odot}\right) = \alpha \log_{10}\left(\frac{L_{\rm cont}}{L_\odot}\right) + \beta \log_{10}\left(\frac{v}{\mathrm{km\,s^{-1}}}\right) + \gamma,$$

where α, β, γ are constants.

(a) If $L_{\rm cont}$ is proportional to $d_{\rm BLR}^2$, find the values of α and β. As in (1)(b), assume the BLR is gravitationally bound to the SMBH and follows Keplerian motion.

(b) An AGN with $M_{\rm BH} = 4.0 \times 10^8\,M_\odot$ was observed to have $v = 3.1 \times 10^3\,\mathrm{km\,s^{-1}}$ and $L_{\rm cont} = 7.8 \times 10^{11}\,L_\odot$. Use this observation to find γ to 2 significant figures, using the values of α and β obtained in (a).

(c) Another AGN was observed with $v = 1.3 \times 10^3\,\mathrm{km\,s^{-1}}$ and $L_{\rm cont} = 8.3 \times 10^{10}\,L_\odot$. Using the values of α, β, γ from (a) and (b), estimate $M_{\rm BH}$ for this AGN in units of $M_\odot$ to 1 significant figure.

Question 4

Recent observations have revealed that SMBHs exist even in the universe only a few hundred million years after the Big Bang. Here we use a simple model to investigate how such SMBHs could have formed in such a short time.

(1) Consider a situation in which material accretes onto a compact object. The accreting material experiences an outward radiation pressure force from the radiation emitted by the object. If the luminosity becomes too large, radiation pressure exceeds gravity and accretion is halted. Therefore, there exists an upper limit to the luminosity that allows accretion to be maintained. This luminosity limit, determined by the balance between radiation force and gravity, is called the *Eddington luminosity*. Fill in the blanks **(A)** through **(E)** in the following derivation. For blank **(D)**, choose either $\leq$ or $\geq$.

Consider a mass M object onto which material is accreting isotropically. The object radiates isotropically with luminosity L. By considering the force balance on a proton–electron pair at distance r from the object, we derive the Eddington luminosity $L_{\rm Edd}$.

The radiative energy flux F at distance r is, since the luminosity L spreads over a sphere of radius r:

$$F = \textbf{(A)}$$

The radiation pressure P is $P = F/c$. Since the radiation pressure on an electron is much larger than on a proton, the outward radiation force on a single proton–electron pair, using the Thomson scattering cross-section σ_T, is $F_{\rm rad} = \sigma_T P$. Hence, in terms of L and r:

$$F_{\rm rad} = \textbf{(B)}$$

Since the proton mass is much larger than the electron mass, the inward gravitational force on the proton–electron pair, treating only the proton of mass m_p at distance r from the object of mass M, is:

$$F_{\rm grav} = \textbf{(C)}$$

For accretion to be maintained, the inequality $F_{\rm rad}$ **(D)** $F_{\rm grav}$ must hold. From this inequality, the Eddington luminosity (the upper limit on luminosity for sustained accretion) is:

$$L_{\rm Edd} = \textbf{(E)}$$

(2) In an AGN, of the mass ΔM accreted per unit time, a fraction η (i.e. a mass $\eta\Delta M$) is converted into radiated energy released as electromagnetic radiation of luminosity L. Consider an AGN with an SMBH of mass $M_{\rm BH}$ shining at the Eddington luminosity. Express the mass accreted per unit time $\Delta M_{\rm Edd}$ using η and $M_{\rm BH}$. Also, for $M_{\rm BH} = 1 \times 10^8\, M_\odot$ and $\eta = 0.1$, calculate $\Delta M_{\rm Edd}$ in units of $M_\odot\,{\rm yr}^{-1}$ to 1 significant figure.

(3) A black hole accreting at the Eddington limit gains mass at the rate of the non-radiated fraction of $\Delta M_{\rm Edd}$, i.e. $(1-\eta)\Delta M_{\rm Edd}$ corresponds to the time derivative of $M_{\rm BH}$:

$$\frac{dM_{\rm BH}}{dt} = (1-\eta)\,\Delta M_{\rm Edd}.$$

Solve this differential equation to find $M_{\rm BH}$ as a function of t, with $M_{\rm BH}(0) = M_0$. Also, with $\eta = 0.1$ constant, find the time t_{10} for $M_{\rm BH}$ to reach $10M_0$, in years to 1 significant figure.

(4) Recent observations have discovered an SMBH with $M_{\rm BH} = 1 \times 10^8\, M_\odot$ at redshift $z = 9$. Assuming this SMBH has grown at the Eddington limit, find $M_{\rm BH}$ at $z = 30$. Refer to Figure 4–1 for the correspondence between redshift z and cosmic age. Also state in about one line whether this mass is larger or smaller compared to the mass of black holes formed through stellar evolution as considered in Question 1.

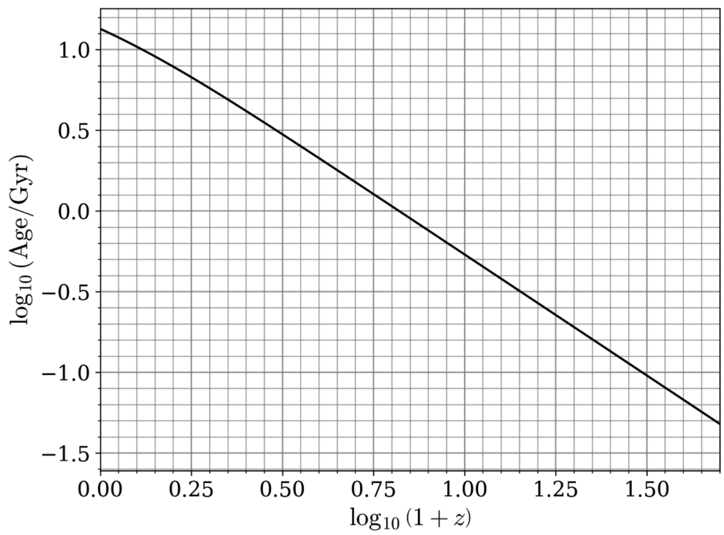

Figure 4–1: Correspondence between redshift z and cosmic age. Note that the horizontal axis is $\log_{10}(1+z)$. Horizontal axis: $\log_{10}(1+z)$ (0.00–1.50); Vertical axis: $\log_{10}(\mathrm{Age/Gyr})$ (−1.5 to 1.0). The curve is a straight line on this log–log plot.