3rd Japan Astronomy Olympiad
National Finals
Theoretical Solutions

Note: The answers shown here are representative solutions. In actual grading, solutions other than those listed below may also have received credit.

Problem 1

Question 1.

(1) From

$$\frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T}{T_\odot}\right)^4,$$

we obtain

$$\frac{R}{R_\odot} = \left(\frac{L}{L_\odot}\right)^{1/2} \left(\frac{T}{T_\odot}\right)^{-2} = \sqrt{1000} \cdot \frac{5800\,\mathrm{K}}{3000\,\mathrm{K}} \approx 1.2 \times 10^2.$$

Hence the radius is approximately $\mathbf{1.2 \times 10^2}$ times the solar radius.

(2) The energy radiated away from the start of contraction to the present is

$$0 - \left(-\frac{3}{5}\frac{GM_\odot^2}{R_\odot}\right) = \frac{3}{5}\frac{GM_\odot^2}{R_\odot}.$$

Dividing by $L_\odot$ gives the Kelvin–Helmholtz timescale:

$$\frac{3}{5}\frac{GM_\odot^2}{R_\odot} \cdot \frac{1}{L_\odot} \approx 6.02 \times 10^{14}\,\mathrm{s} \approx 1.9 \times 10^7\,\mathrm{yr},$$

i.e. approximately **19 million years**. This is not a physically reasonable age for the Sun, because radiometric dating reveals an abundance of fossils and rocks older than 19 million years, establishing that Earth (and by extension the Sun) is far older.

Question 2.

The gravitational potential at a point on the Earth–Moon line at distance r from the Earth's centre ($0 < r < a$, where a is the Earth–Moon distance, M is Earth's mass, and m is the Moon's mass) is

$$U(r) = -\frac{GM}{r} - \frac{Gm}{a-r}.$$

Setting $dU/dr = 0$:

$$\frac{dU}{dr} = \frac{GM}{r^2} - \frac{Gm}{(a-r)^2} = 0 \implies r_0 = \frac{\sqrt{M}}{\sqrt{m} + \sqrt{M}}\,a.$$

To reach the Moon, a projectile launched from Earth's surface with speed v_0 must reach the potential maximum at $r = r_0$. Energy conservation gives

$$U(R) + \tfrac{1}{2}v_0^2 = U(r_0),$$

where R is Earth's radius. Substituting numerical values yields

$$\boxed{v_0 \approx 11.1\,\mathrm{km\,s^{-1}}}.$$

Question 3.

(1) Using $g = GM/R^2$ and $M = \frac{4}{3}\pi\rho_0 R^3$:

$$R = \frac{3g}{4\pi G\rho_0} \approx 4.3 \times 10^3\,\mathrm{km}.$$

(2) The pressure at the base of a mountain of height H is $\rho g H$. Requiring this not to exceed 200 MPa:

$$H_{\max} \approx 1.7 \times 10^1\,\mathrm{km}.$$

Question 4.

(1) By Kepler's Second Law, the perihelion velocity v satisfies

$$\frac{1}{2}(1-e)\,a\,v = \frac{\pi\sqrt{1-e^2}\,a^2}{P}.$$

By Kepler's Third Law,

$$\frac{a^3}{P^2} = \frac{GM}{4\pi^2}.$$

Solving simultaneously:

$$\boxed{v = \sqrt{\frac{1+e}{1-e}\cdot\frac{GM}{a}}}.$$

(2) From Kepler's Third Law:

$$GM = 4 \times (3.1)^2 \times \frac{(1.5 \times 10^{11}\,\mathrm{m})^3}{(3.1 \times 10^7\,\mathrm{s})^2}.$$

Substituting into the result of (1):

$$v \approx 3 \times 10^1\,\mathrm{km\,s^{-1}}.$$

Question 5.

(1) The current solar luminosity is greater than it was immediately after the Sun's formation, so the habitable zone has shifted outward over time; Mars may have been within the habitable zone in the past.

(2) **Examples:** Europa (moon of Jupiter), Enceladus (moon of Saturn), etc.

Reason: These bodies orbit massive planets and therefore experience strong tidal forces; tidal heating can maintain liquid water beneath their icy surfaces even far from the Sun.

Problem 2

Question 1.

(1-a) For example, let R_1 and R_2 be the upper and lower bounds of the radius at $\log(T/\mathrm{K}) = 4.15$, with corresponding luminosities L_1 and L_2. From $L = 4\pi\sigma R^2 T^4$:

$$\log \frac{R_1}{R_2} = \frac{1}{2} \log \frac{L_1}{L_2},$$

so

$$\frac{R_1}{R_2} = 10^{\frac{1}{2}(\log L_1 - \log L_2)} = 10^{10/2} \approx 3.2.$$

Hence the ratio of radii is approximately **3.2**.

(1-b) White dwarfs generate no energy; they only cool. Because there is a finite upper bound on the age of the Universe, there is a lower bound on their surface temperature (the coolest white dwarfs cannot have been cooling for longer than the age of the Universe).

(2-a) Black holes and neutron stars.

(2-b) Core-collapse supernovae synthesise heavy elements by nuclear fusion in the stellar interior and then disperse those elements into the interstellar medium.

(2-c) The shifts in emission lines are caused by the Doppler effect. Because the nebula expands isotropically, the line-of-sight velocity is zero at positions A and G (on the plane of the sky) and maximum at position D (along the line of sight). Only diagram **3** represents this behaviour.

Question 2.

(1) Multiplying volume by density:

$$M_{\mathrm{mat}}(t) = \frac{4}{3}\pi\rho_0 R^3.$$

(2) Multiplying the given expression for specific internal energy by the mass:

$$U(t) = M_{\mathrm{mat}}(t) \cdot \epsilon_U = \frac{4}{3}\pi\rho_0 R^3 \cdot \frac{1}{\gamma - 1}\frac{p_2}{\rho_2^2} \cdot \rho_2 = \frac{4\pi\rho_0}{3(\gamma-1)}\frac{p_2^2}{\rho_2^2}\, R^3.$$

From the Rankine–Hugoniot relations:

$$\rho_2 = \frac{\gamma+1}{\gamma-1}\rho_0, \qquad p_2 = \frac{2\gamma}{\gamma+1}\mathcal{M}_1^2 p_1 = \frac{2}{\gamma+1}\rho_0 V_{\mathrm{sh}}^2.$$

Substituting:

$$\boxed{U(t) = \frac{8\pi\rho_0}{3(\gamma+1)^2}\, V_{\mathrm{sh}}^2\, R^3.}$$

(3) In the rest frame of the interstellar medium, the shock front moves at $V_{\rm sh}$. In the rest frame of the shock front, swept-up material flows inward at v_2. Hence the swept-up material moves at

$$V_{\rm mat} = V_{\rm sh} - v_2$$

relative to the interstellar medium. From the Rankine–Hugoniot relations:

$$v_2 = \frac{\gamma-1}{\gamma+1}v_1 = \frac{\gamma-1}{\gamma+1}V_{\rm sh},$$

therefore

$$\boxed{V_{\rm mat} = \frac{2}{\gamma+1}V_{\rm sh}.}$$

(4) Explicitly computing $K(t)$ and verifying $K(t) = U(t)$:

$$K(t) = \frac{1}{2}\rho_0 V_{\rm mat}^2 \cdot \frac{4}{3}\pi R^3 = \frac{1}{2}\rho_0\left(\frac{2}{\gamma+1}\right)^2 V_{\rm sh}^2 \cdot \frac{4}{3}\pi R^3 = \frac{8\pi\rho_0}{3(\gamma+1)^2}\,V_{\rm sh}^2\,R^3 = U(t). \quad \checkmark$$

(5) By energy conservation $E_0 = K(t) + U(t) = 2U(t)$:

$$E_0 = \frac{16\pi\rho_0}{3(\gamma+1)^2}\,V_{\rm sh}^2\,R^3,$$

so

$$\boxed{V_{\rm sh} = \sqrt{\frac{3E_0(\gamma+1)^2}{16\pi\rho_0}}\,R^{-3/2}.}$$

(6) Substituting $R = ct^\alpha$ and $\dot{R} = c\alpha t^{\alpha-1}$ into the above:

$$c\alpha t^{\alpha-1} = \sqrt{\frac{3E_0(\gamma+1)^2}{16\pi\rho_0}}\,c^{-3/2}\,t^{-3\alpha/2}.$$

For this to be an identity in t we need $\alpha - 1 = -3\alpha/2$, giving $\alpha = 2/5$, and then solving for c:

$$\boxed{R(t) = \left(\frac{75\,E_0(\gamma+1)^2}{64\pi\rho_0}\right)^{1/5} t^{2/5}.}$$

(7) Substituting numerical values:

$$R \approx 14.0\,\text{pc}.$$

Problem 3

Question 1.

(1) Answer: **(e)**.

(2) Answer: **(d)**. This is a star-forming region in which the surrounding hydrogen is ionised and made to glow by ultraviolet radiation from hot, newly formed stars embedded within it.

(3) Interstellar space is at far lower pressure and density than the surface of the Earth, so molecules that would quickly collide and react on Earth can persist for astronomically long times in space.

(4) Approximately 380,000 years after the Big Bang, the Universe cooled enough for electrons to be captured by atomic nuclei (the epoch of recombination, or "cosmic dawn"), allowing neutral hydrogen atoms to exist stably for the first time.

Question 2.

(1) When an electron transitions from a higher energy level to a lower one, it releases energy in the form of electromagnetic radiation, producing an emission line at the corresponding frequency.

(2-a) Eliminating p_θ from the expression for E_J:

$$E_J = \frac{p_\theta^2}{2I} = \frac{h^2}{8\pi^2}\frac{J(J+1)}{I}.$$

(2-b) Substituting numerical values:

$$I \approx 1.45 \times 10^{-46}\,\mathrm{kg\,m^2}.$$

(2-c) The energy difference between the $J = 1$ and $J = 0$ levels:

$$\Delta E = \frac{h^2}{8\pi^2 I}(1 \cdot 2 - 0 \cdot 1) \approx 7.68 \times 10^{-23}\,\mathrm{J}.$$

(2-d) Using $E = h\nu$ and $\lambda\nu = c$:

$$\lambda = \frac{hc}{\Delta E} \approx 2.59 \times 10^{-3}\,\mathrm{m}.$$

This falls in the **millimetre-wave** band, so the correct telescope is ① **ALMA**.

Question 3.

(3-a) Converting the rest frequency 107.7 GHz to a wavelength:

$$\lambda_{\mathrm{rest}} = \frac{2.998 \times 10^8\,\mathrm{m\,s^{-1}}}{107.7\,\mathrm{GHz}} \approx 2.78 \times 10^{-3}\,\mathrm{m}.$$

The redshift is

$$z = \frac{\lambda_{\mathrm{obs}} - \lambda_{\mathrm{rest}}}{\lambda_{\mathrm{rest}}} = \frac{2.78 \times 10^{-3} - 2.59 \times 10^{-3}}{2.59 \times 10^{-3}} \approx 0.073.$$

Using $v = cz$ and $v = Hr$:

$$r = \frac{cz}{H} \approx 3.1 \times 10^2\,\mathrm{Mpc}.$$

(3-b) Using $\nu_{\mathrm{rest}} = (1+z)\nu_{\mathrm{obs}}$:

$$L_{\mathrm{CO}} \approx 4.4 \times 10^9\,\mathrm{K\,km\,s^{-1}\,pc^2}.$$

(3-c) Multiplying the CO luminosity by the conversion factor X_{CO}:

$$M_{\mathrm{mol}} \approx 4 \times 10^9\, M_\odot.$$

(3-d) Dividing the molecular gas mass by the star formation rate:

$$\frac{4 \times 10^9\, M_\odot}{1 \times 10^2\, M_\odot\, \mathrm{yr}^{-1}} \approx 4 \times 10^7\, \mathrm{yr}.$$