3rd Japan Astronomy Olympiad National Finals

Theoretical Problems

February 23, 2024 13:15–15:45

Instructions

1. Do not open this problem booklet until the start signal is given.
2. This booklet contains 12 pages in total. If you find any missing, disordered, or illegible pages, raise your hand and notify the proctor.
3. Use only a black pencil or black mechanical pencil for your answers.
4. Write your examinee number in the designated field on the answer sheet. Do not write it anywhere else.
5. Write all answers in the designated spaces on the answer sheet.
6. Do not write any unrelated characters, symbols, or marks in the answer spaces.
7. You may use the margins of this booklet as scratch paper, but do not tear any pages.
8. Do not take the answer sheet home.
9. After the exam, you may take this booklet and any calculation sheets home.
10. No questions about the problems will be answered. If you believe there is an error in a problem, write a note on your answer sheet; it will be considered during grading.

Problem 1

Answer the following independent sub-questions (Questions 1–5) about various topics related to the solar system. On the answer sheet, provide not only the final answer but also equations, reasoning, and explanations showing how you arrived at your answer.

Question 1. Formation and Evolution of the Sun

Answer the following sub-questions (1) and (2) about the formation and evolution of the Sun. Where necessary, you may use the current solar mass $M_\odot = 2.0 \times 10^{30}$ kg, current solar radius $R_\odot = 7.0 \times 10^5$ km, current solar luminosity $L_\odot = 3.8 \times 10^{26}$ W, current solar surface temperature $T_\odot = 5800$ K, and gravitational constant $G = 6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$.

(1) It is now understood that approximately 4.6 billion years ago, interstellar gas contracted, and the interior of the gas sphere was heated by the release of gravitational energy, causing the proto-Sun to begin shining. Given that the proto-Sun had a surface temperature of 3000 K and a luminosity of $1000\,L_\odot$, determine the radius of the proto-Sun as a multiple of the current solar radius. Give your answer to 2 significant figures.

(2) In the latter half of the 19th century, Kelvin and Helmholtz proposed the following hypothesis for the mechanism by which the Sun shines: "Initially, a very low-density gas of total mass $1.0\,M_\odot$, uniformly distributed within an infinitely large sphere, falls toward the center and releases gravitational energy, which is converted into radiant energy." Assume this hypothesis is correct and that all the energy needed to power the Sun comes from the release of gravitational energy. For simplicity, assume the Sun is a uniform-density gas sphere and that the solar luminosity has always been constant at its current value. Determine how many years it would take for the gas to contract from infinite radius to the current solar radius. Give your answer to 2 significant figures, and briefly discuss whether the result is physically reasonable. Note that the gravitational energy of a uniform-density gas sphere of total mass M and radius R is given by:

$$U = -\frac{3}{5}\frac{GM^2}{R}.$$

Question 2. Minimum Initial Speed to Reach the Moon

Find, to 2 significant figures, the minimum initial speed required for a rocket launched from Earth's surface to reach the lunar surface. Assume the rocket is accelerated only at the instant of launch, and that the mass distributions of Earth and the Moon are both spherically symmetric. Use $G = 6.67\times10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$, Earth's mass $M = 5.97\times10^{24}$ kg, Earth's radius $R = 6.38 \times 10^3$ km, the Moon's mass $m = \frac{1}{80}M$, and the center-to-center distance between Earth and the Moon $a = 60R$.

Question 3. Mars Composed Entirely of Basalt

Assume that Mars is composed entirely of basalt, and answer sub-questions (1) and (2) below.

(1) Using the fact that the gravitational acceleration at the Martian surface is $3.71\ \mathrm{m\,s^{-2}}$, find the radius of Mars in km to 2 significant figures. The density of basalt is $3.10\ \mathrm{g\,cm^{-3}}$. You may use $G = 6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$.

(2) Find, in km to 2 significant figures, the maximum height of a cylindrical mountain composed entirely of basalt that could exist on Mars. The maximum compressive stress that basalt can withstand is 200 MPa.

Question 4. Asteroid Obeying Kepler's Laws

Consider an asteroid moving in accordance with Kepler's laws, and answer sub-questions (1) and (2) below.

(1) Consider an asteroid with an elliptical orbit around the Sun as a focus. Let the semi-major axis be a [m] and the orbital eccentricity be e. Also let the gravitational constant be G [$\mathrm{m^3\,kg^{-1}\,s^{-2}}$] and the solar mass be M [kg]. Express the orbital speed v at perihelion in terms of a, e, G, and M. Note that the area of an ellipse with semi-major axis a and eccentricity e is $\pi\sqrt{1-e^2}\,a^2$. You may also assume the asteroid's mass is negligible compared to the Sun's mass.

(2) An asteroid is observed to follow an elliptical orbit with semi-major axis 1.5 au and eccentricity 0.20. Find the orbital speed of this asteroid at perihelion in $\mathrm{km\,s^{-1}}$, to 1 significant figure. Use 1 au $= 1.5 \times 10^{11}$ m, 1 yr $= 3.1 \times 10^{7}$ s, $\pi = 3.1$.

Question 5. Habitable Zone

Liquid water is essential for life to exist. The theoretical region around a star where liquid water can exist on a planetary surface is called the habitable zone. Answer sub-questions (1) and (2) below. Assume the habitable zone is determined solely by the equilibrium (radiative equilibrium) temperature.

(1) In the present solar system, Earth is the only planet in the habitable zone. However, it is believed that Venus was also in the habitable zone just after the solar system formed. Discuss and briefly explain why Venus has since moved outside the habitable zone. Assume that the orbital semi-major axes of the planets have not changed significantly since the solar system formed.

(2) In the present solar system, there are bodies that are outside the conventional habitable zone yet are believed to harbor liquid water. Name such a body specifically and explain why liquid water can exist there despite being outside the habitable zone.

Problem 2

Answer the following questions (Questions 1 and 2) related to the end states of stars. On the answer sheet, provide not only the final answer but also equations, reasoning, and explanations showing how you arrived at your answer.

Question 1. Stellar End States by Mass

The final fate of a star depends on its mass (on the main sequence). Answer the following questions.

(1) Stars with mass up to approximately 8 $M_\odot$ eject their outer layers in the final stage of evolution, forming planetary nebulae. The compact object left behind at the center is a white dwarf. On the HR diagram, white dwarfs are distributed only within a narrow, elongated region, as shown in Figure 2–1.

(a) Using Figure 2–1, determine how many times larger the upper limit of the white dwarf radius is compared to the lower limit. Give your answer to 2 significant figures.

(b) As seen in Figure 2–1, the distribution of white dwarfs is confined to the high-temperature side of a certain surface temperature. Briefly explain this in connection with the history of the universe.

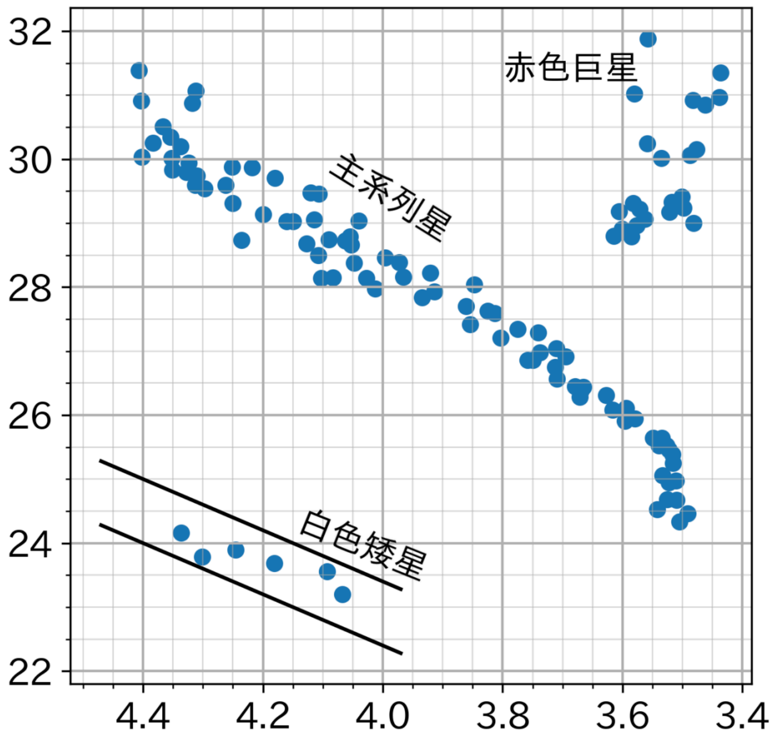

Figure 2–1: HR diagram. The y-axis shows $\log_{10}$(Luminosity/W) and the x-axis shows $\log_{10}$(Surface temperature/K). Labeled regions include red giants (赤色巨星), main sequence (主系列星), and white dwarfs (白色矮星).

(2) Stars with mass exceeding approximately 8 $M_\odot$ undergo a (core-collapse) supernova explosion in their final stage of evolution.

(a) Name two examples of high-density compact objects left behind after a supernova explosion.

(b) The early universe contained almost only hydrogen and helium. Yet Population I stars contain a variety of elements. Explain the role that supernova explosions have played in realizing the high metallicity observed in Population I stars.

(c) In spectroscopic imaging observations with a telescope, Doppler shifts of known spectral lines can be used to measure not only the projected spatial distribution of objects on the celestial sphere, but also their line-of-sight motion. Consider observing the [O III] emission line (rest wavelength 5007 Å) at points A through G placed along the diameter of the spherical supernova remnant shown in Figure 2–2. Assuming the remnant does not rotate at all and expands isotropically, select the figure from options ①–④ below that most appropriately shows the observed [O III] emission line spectra arranged in order, and briefly state your reasoning. The horizontal axis units in options ①–④ are all in Å. Note that wavelength shifts due to proper motion have been corrected, and the effects of interstellar absorption etc. are ignored.

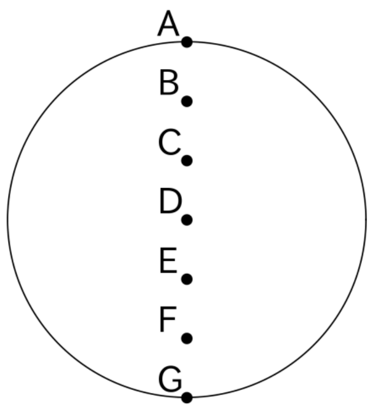

Figure 2–2: Schematic diagram of the supernova remnant. Points A (top) through G (bottom) are placed along the vertical diameter of the spherical shell.

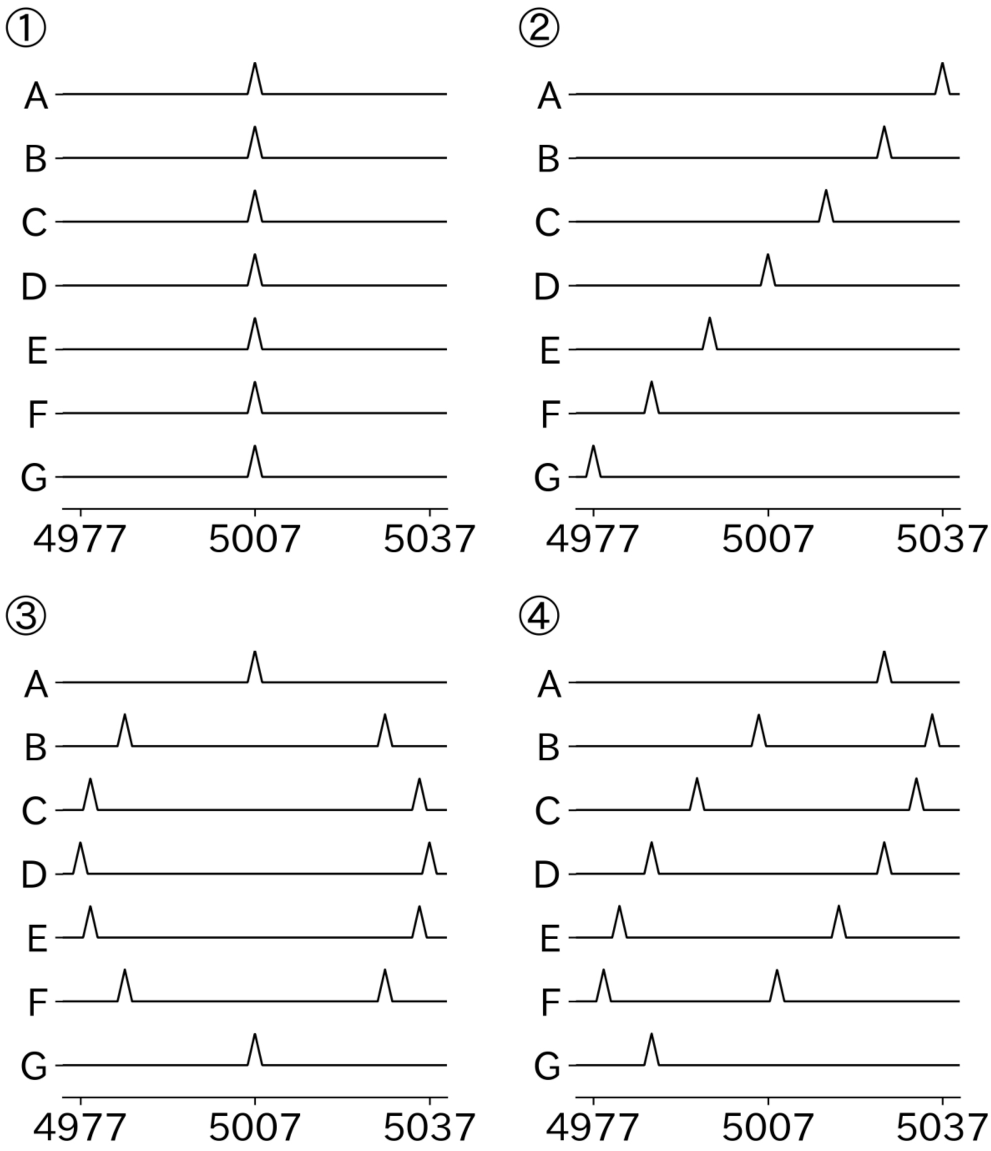

Figure 2–3: Options ①–④ showing [O III] spectral line profiles at positions A–G. The horizontal axis runs from 4977 Å to 5037 Å in each panel.

Question 2. Point-Source Explosion (Sedov–Taylor Blast Wave)

Answer the following questions about a point-source explosion. In your answers, clearly distinguish between ρ (density) and p (pressure).

After a supernova explosion, a shock wave propagates through the interstellar medium (ISM) of uniform density. In the rest frame of the shock front, let v_1, ρ_1, p_1 be the fluid velocity, mass density, and pressure upstream of the shock, and v_2, ρ_2, p_2 be those downstream ($v_1 > 0$, $v_2 > 0$). In the strong shock limit, the Rankine–Hugoniot relations hold:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{\gamma+1}{\gamma-1}, \qquad \frac{p_2}{p_1} = \frac{2\gamma M_1^2}{\gamma+1},$$

where γ is the adiabatic index, $M_1 = v_1/c_1$ is the Mach number of the shock speed in the upstream medium, and $c_1 = (\gamma p_1/\rho_1)^{1/2}$ is the upstream sound speed.

We consider a spherically symmetric explosion. The explosion occurs at time $t = 0$, and the shock front spreads into the interstellar space. Let $R(t)$ be the position of the shock front at time t and $V_{\rm sh} = dR/dt$ be the propagation speed of the shock front (set $R(t=0)=0$). The ISM is completely at rest with uniform density ρ_0. Only the material originally in the ISM inside the shock front is considered; material ejected by the supernova itself is negligible. The physical quantities immediately before and after the shock front satisfy the Rankine–Hugoniot relations. Thus $v_1 = V_{\rm sh}$ and $\rho_1 = \rho_0$. Radiative cooling etc. are neglected; all swept-up ISM material is treated as having a single uniform density, pressure, and velocity.

(1) Since the ISM density is uniform, the total mass is obtained by multiplying the volume by the density. Express the total mass $M_{\rm mat}(t)$ of ISM swept up by the shock front up to time t in terms of R and ρ_0.

(2) The material swept up inside the shock front is heated by the shock. Using the fact that the internal energy per unit mass of the gas is $\varepsilon_U = \frac{1}{\gamma-1}\frac{p}{\rho}$, express the total internal energy $U(t)$ of the material inside the shock front at time t in terms of ρ_0, R, p_2, ρ_2, and γ. Then, using the Rankine–Hugoniot relations, eliminate p_2 and ρ_2 from $U(t)$ and express it in terms of ρ_0, R, $V_{\rm sh}$, and γ.

(3) Express the expansion velocity $V_{\rm mat}$ of the material inside the shock front relative to the ISM rest frame in terms of $V_{\rm sh}$ and v_2. Then, using the Rankine–Hugoniot relations, eliminate v_2 and express $V_{\rm mat}$ in terms of $V_{\rm sh}$ and γ.

(4) Express the total kinetic energy $K(t)$ of the material inside the shock front at time t in terms of ρ_0, R, $V_{\rm sh}$, and γ, and show that $K(t)$ equals $U(t)$.

(5) Because radiative cooling is negligible, the sum of internal energy and kinetic energy is conserved and equals the explosion energy E_0. Using this fact, express $V_{\rm sh}$ in terms of ρ_0, E_0, R, and γ.

(6) Since $V_{\rm sh} = dR/dt$, the equation found in (5) is a differential equation for R. Assuming $R(t) = ct^\alpha$ for some constant c and real number $\alpha > 0$, solve for R and express it in terms of ρ_0, E_0, γ, and t. You may use the differentiation formula $\frac{d}{dt}(ct^\alpha) = c\alpha t^{\alpha-1}$.

(7) Given explosion energy $E_0 = 10^{51}$ erg and interstellar hydrogen number density $n = 1\ \mathrm{cm}^{-3}$, find the position of the shock wave at $t = 10^4$ yr in units of pc. A scientific calculator may be used; errors up to a factor of 10 are acceptable. Assume the ISM consists of hydrogen only and use $\gamma = \frac{5}{3}$. If needed, use proton mass $m_p = 1.7\times10^{-24}$ g and 1 pc $= 3.1\times10^{18}$ cm.

Problem 3

Stars form inside molecular clouds, regions of the ISM where matter is densely concentrated. The hydrogen molecule (H_2), the most abundant molecule in molecular clouds, has a symmetric molecular structure. Therefore, except under limited conditions such as very high temperatures and densities, it is difficult to observe H_2 emission lines from molecular clouds. Instead, emission lines from carbon monoxide (CO), the second most abundant molecule in molecular clouds after H_2, are commonly used to observe the gas in molecular clouds.

Question 1. Interstellar Medium, Molecular Clouds, and Cosmic Composition

Figure 3–1 shows the distribution of temperatures and number densities that interstellar matter can take. The ISM can be broadly classified into 5 types: (a)–(e). Answer the following questions about molecular clouds and the composition of the universe.

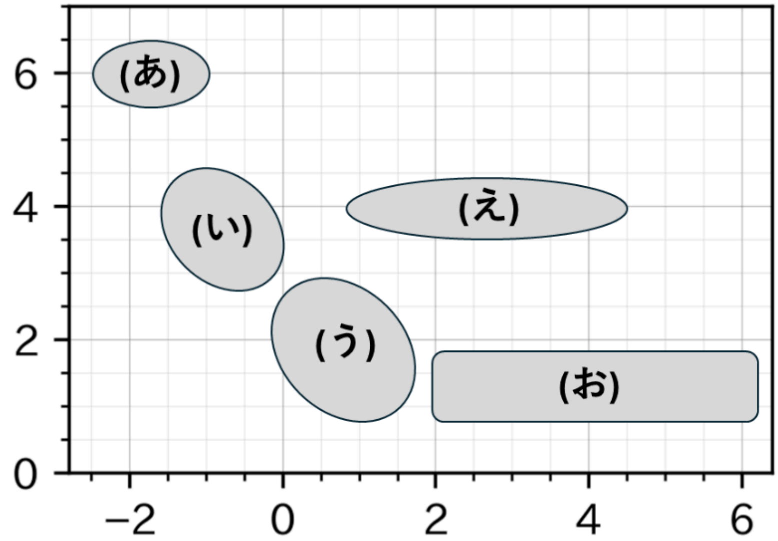

Figure 3–1: Distribution of temperature and number density of interstellar matter. The y-axis is $\log_{10}$(Temperature/K) and the x-axis is $\log_{10}$(hydrogen number density/cm^{-3}). Regions labeled (a) [あ] through (e) [お] correspond to different ISM phases.

(1) From (a)–(e), select the one corresponding to a molecular cloud.

(2) From (a)–(e), select the one that primarily constitutes an H II region. Also, briefly explain in about one line what an H II region is, including its formation mechanism.

(3) Individual hydrogen atoms are almost never stable under natural conditions on Earth, yet they exist stably in outer space. Briefly describe in about one line the reason for this.

(4) In outer space, approximately how long after the birth of the universe did neutral hydrogen atoms first become stably present? Explain in about two lines, together with the reason why hydrogen atoms became able to exist stably.

Question 2. Emission Lines from Rotational Transitions of CO

The CO molecule rotates about an axis perpendicular to its bond axis. Because the energy is quantized, emission lines are produced corresponding to the energy differences between rotational states (rotational transitions). Answer the following questions about CO rotational transition emission lines.

(1) The Balmer series emission lines from hydrogen atoms, observed in star-forming regions and active galactic nuclei, are produced by a different emission mechanism from the rotational transition emission lines of CO. Briefly explain in about one line the emission mechanism of the Balmer series.

(2) Let us find the wavelength of CO rotational transition emission lines. As noted above, the rotational energy of CO is quantized. This discrete energy is expressed using the rotational quantum number J. The energy E_J of the state with rotational quantum number J is expressed using the moment of inertia I and angular momentum p_θ as:

$$E_J = \frac{p_\theta^2}{2I}.$$

(a) The angular momentum p_θ is expressed using Planck's constant h as:

$$p_\theta = \frac{h}{2\pi}\sqrt{J(J+1)}.$$

Express the energy level E_J for rotational quantum number J in terms of I and J.

(b) The moment of inertia I of the CO molecule is written using the mass of the C atom m_C, the mass of the O atom m_O, and the interatomic distance r in CO as:

$$I = \frac{m_C m_O}{m_C + m_O} r^2.$$

Calculate the moment of inertia I of CO in SI units to 3 significant figures. Use $m_C = 1.993 \times 10^{-26}$ kg, $m_O = 2.657 \times 10^{-26}$ kg, $r = 1.128$ Å.

(c) Find, in SI units to 3 significant figures, the energy released when the CO molecule transitions from $J = 1$ to $J = 0$, i.e., the energy level difference ΔE between $J = 1$ and $J = 0$. Use $h = 6.626 \times 10^{-34}$ J s and $\pi = 3.142$.

(d) By considering the wavelength of a photon with energy ΔE, find the wavelength of the emission line released when CO transitions from $J = 1$ to $J = 0$ (hereafter called the CO$(1-0)$ line) to 3 significant figures. The energy E [J] of a photon with frequency ν [Hz] is $E = h\nu$. Use $c = 2.998 \times 10^8$ m s^{-1}. Also, select from options ①–④ below the telescope capable of observing this wavelength of electromagnetic radiation. (Ignore redshift effects.)

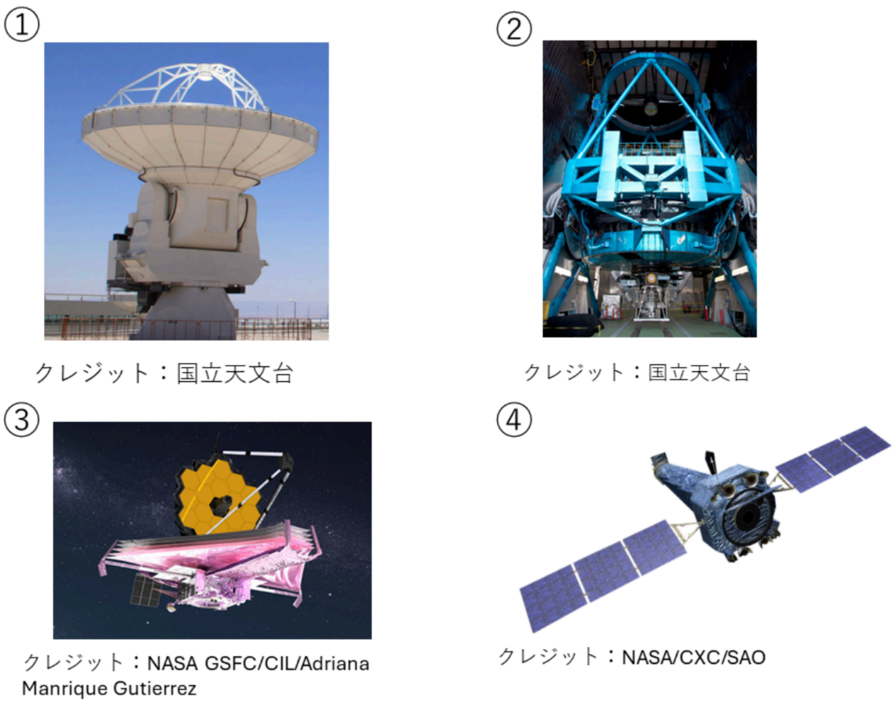

Figure 3–2: Options ①–④: four different telescopes. ① Radio telescope (credit: NAOJ). ② Optical/infrared telescope (credit: NAOJ). ③ James Webb Space Telescope (credit: NASA GSFC/CIL/Adriana Manrique Gutierrez). ④ X-ray satellite (credit: NASA/CXC/SAO).

(3) The CO$(1-0)$ line can be used to estimate the mass of gas in a galaxy. If the luminosity of the CO$(1-0)$ line is $L_{\rm CO}$ [$\rm K\,km\,s^{-1}\,pc^2$], then using the conversion factor $X_{\rm CO}$ [$M_\odot\,(\rm K\,km\,s^{-1}\,pc^2)^{-1}$], the molecular gas mass $M_{\rm mol}$ [$M_\odot$] is:

$$M_{\rm mol} = X_{\rm CO} \times L_{\rm CO(1-0)}.$$

(a) The CO$(1-0)$ line of galaxy G is observed at a central frequency of 107.7 GHz. Find the redshift z of galaxy G and the distance to galaxy G, both to 2 significant figures. Use the Hubble constant $H = 70\ \rm km\,s^{-1}\,Mpc^{-1}$ and assume cosmological corrections are negligibly small.

(b) The CO$(1-0)$ luminosity $L_{\rm CO}$ is related to the CO$(1-0)$ flux $F_{\rm CO}$ [$\rm Jy\,km\,s^{-1}$], distance D [Mpc], and rest-frame frequency $\nu_{\rm rest}$ [GHz] by:

$$\left(\frac{L_{\rm CO}}{\rm K\,km\,s^{-1}\,pc^2}\right) = 3.25\times10^7 \left(\frac{F_{\rm CO}}{\rm Jy\,km\,s^{-1}}\right)\left(\frac{D}{\rm Mpc}\right)^2\left(\frac{\nu_{\rm rest}}{\rm GHz}\right)^{-2}(1+z)^{-1}.$$

Given that the CO$(1-0)$ flux of galaxy G is $F_{\rm CO} = 2.0\times10\ \rm Jy\,km\,s^{-1}$, find $L_{\rm CO}$ in units of $\rm K\,km\,s^{-1}\,pc^2$ to 2 significant figures.
(Note: $1\ \rm Jy = 10^{-26}\ W\,m^{-2}\,Hz^{-1}$.)

(c) Observations of nearby galaxies suggest $X_{\rm CO} = 0.8\,M_\odot\,(\rm K\,km\,s^{-1}\,pc^2)^{-1}$. Using this value, find the molecular gas mass $M_{\rm mol}$ of galaxy G in units of $M_\odot$ to 1 significant figure.

(d) Observations of galaxy G at other wavelengths show that $1\times10^2\,M_\odot$ of stars are formed per year in galaxy G. If galaxy G converts molecular gas into stars at this rate, how many years would it take to consume all the molecular gas in galaxy G? Give your answer in years to 1 significant figure. Assume no other processes increase or decrease the molecular gas mass in galaxy G.