# 4th International Olympiad on Astronomy and Astrophysics

**October, 18 – 25, 2025**

**Piatra Neamț - România**

# Theoretical Round

Piatra Neamt
ROMANIA

# Th.1 Binary System (10 Points)

Two stars A and B, that form a non-eclipsing binary system, have peaks of their blackbody radiation spectra at 650 nm and 400 nm, respectively. We also know that the radius of star A is twice the radius of star B.

a) Compute the difference between the absolute magnitudes of the stars A and B.

b) If the combined absolute magnitude of the binary system is 0.00, find the absolute magnitude of each star.

## Marking Scheme

**a)** From Wien's law, we have: $b = T \cdot \lambda$, where $b$ is the Wien's constant, $T$ is the surface temperature of the star, and $\lambda$ is the wavelength corresponding to the maximum peak of radiation. Hence,

$$b = T_A \cdot \lambda_A = T_B \cdot \lambda_B \Rightarrow \frac{T_A}{T_B} = \frac{\lambda_B}{\lambda_A}$$ **2pt**

The luminosity of a star is given by the Stefan-Boltzmann formula:

$$\begin{aligned} L &= 4\pi R^2 \sigma T^4 \Rightarrow \\ \frac{L_A}{L_B} &= \frac{4\pi R_A^2 \sigma T_A^4}{4\pi R_B^2 \sigma T_B^4} \\ &= \left(\frac{R_A}{R_B}\right)^2 \left(\frac{T_A}{T_B}\right)^4 = \left(\frac{R_A}{R_B}\right)^2 \left(\frac{\lambda_B}{\lambda_A}\right)^4 \\ &= \left(\frac{2R_B}{R_B}\right)^2 \left(\frac{400}{650}\right)^4 \\ &\approx 0.57 \end{aligned}$$ **2pt**

Now, we infer the luminosity-absolute magnitude relationship:

$$\begin{aligned} \frac{L_A}{L_B} &= 10^{0.4(M_B - M_A)} \Rightarrow \\ \Delta M = M_A - M_B &= -\frac{1}{0.4} \log_{10}\left(\frac{L_A}{L_B}\right) \\ &= -\frac{1}{0.4} \log_{10} 0.57 \\ &\approx 0.60 \end{aligned}$$ **2pt**

*Note:* Also accept 0.61 (possible due to rounding).

**b)** Denote by $L = L_A + L_B$ the total luminosity of the binary system, and with $M = 0$ the absolute magnitude of the system. We will compare these with the Sun's luminosity $L_\odot$ and its absolute magnitude $M_\odot$. Using the same formula as above, we get:

$$\frac{L}{L_\odot} = 10^{0.4(M_\odot - M)}$$

$$\frac{L}{L_\odot} = \frac{L_A + L_B}{L_\odot} = \frac{L_A}{L_\odot} + \frac{L_B}{L_\odot}$$

$$= 10^{0.4(M_\odot - M_A)} + 10^{0.4(M_\odot - M_B)}$$ **1pt**

$$10^{0.4(M_\odot - M)} = 10^{0.4(M_\odot - M_A)} + 10^{0.4(M_\odot - M_B)}$$

$$10^{-0.4M} = 10^{-0.4M_A} + 10^{-0.4M_B} =$$

$$= 10^{-0.4(M_B + \Delta M)} + 10^{-0.4M_B}$$ **1pt**

$$10^{-0.4M} = 10^{-0.4M_B}(10^{-0.4\Delta M} + 1)$$

$$-0.4M = -0.4M_B + \log_{10}(10^{-0.4\Delta M} + 1)$$

$$M_B = M + \frac{1}{0.4}\log_{10}(10^{-0.4\Delta M} + 1)$$

$$= 0 + \frac{1}{0.4}\log_{10}(10^{-0.4\cdot 0.61} + 1) \approx 0.49$$ **1pt**

$$M_A = M_B + \Delta M = 0.49 + 0.61 = 1.10$$ **1pt**

**Alternative Solution b)**

Denote by $L = L_A + L_B$ the total luminosity of the binary system, and with $M = 0$ the absolute magnitude of the system.

$$M - M_A = -2.5\log\left(\frac{L_A + L_B}{L_A}\right) = -2.5\log\left(1 + \frac{L_B}{L_A}\right)$$ **2pt**

$$0 - M_A = -2.5\log\left(1 + \frac{1}{0.57}\right)$$

$$M_A = 2.5\log(2.74) = 1.10$$ **1pt**

We already calculated the difference in magnitude, which means we can easily calculate $M_B$ now:

$$\Delta M = M_A - M_B \Rightarrow M_B = M_A - \Delta M$$

$$M_B = 1.1 - 0.60 = 0.50$$ **1pt**

## Th.2 Speed of Asteroid (10 Points)

An asteroid moves in an elliptical orbit around the Sun. Its maximum orbital speed is 19.00 km/s and its minimum orbital speed is 14.00 km/s.

a) Compute the eccentricity of the orbit.

b) What is the speed of the asteroid when it is exactly at either of the end points of the minor axis of its orbit?

c) A comet from near the outer edge of our Solar System reaches perihelion with an orbital speed of 90.00 km/s. At this instant, the asteroid is at aphelion and the comet, Sun, and asteroid are aligned (comet-Sun-asteroid). Compute the distance between the comet and the asteroid.

# Marking Scheme

**a)** There are some classical formulas regarding the speed of a celestial object at aphelion and perihelion, respectively ($a$ denotes the semi-major axis of the elliptic orbit, $M_S$ is the mass of the Sun, $G$ is the gravitational constant):

$$v_{max} = \sqrt{\frac{GM_S}{a} \cdot \frac{1+e}{1-e}}; \; v_{min} = \sqrt{\frac{GM_S}{a} \cdot \frac{1-e}{1+e}};$$ **1.5pt**

$$k = \frac{v_{max}}{v_{min}} = \frac{1+e}{1-e}; \; k(1-e) = 1+e;$$

$$e = \frac{k-1}{k+1} = \frac{v_{max} - v_{min}}{v_{max} + v_{min}}$$ **0.5pt**

Numerically, the eccentricity of the asteroid's orbit is:

$$e = (19-14)/(19+14) \approx 0.15$$ **1pt**

**b)** When being at distance $r$ relative to the Sun, the speed of the asteroid is given by the formula:

$$v(r) = \sqrt{GM_S(\frac{2}{r} - \frac{1}{a})}.$$ **1pt**

When being at one of the co-vertices of the orbit, the asteroid is at the intersection between the ellipse and the semi-minor axis, so $r$=$a$. **1pt**
The orbital speed is, in this case:

$$v(a) = \sqrt{\frac{GM_S}{a}} = \sqrt{v_{max} \cdot v_{min}} \approx 16.31\,\text{km/s}.$$ **1pt**

**c)** Because we are talking about a comet from outer edge of Solar System, we can safely assume its orbit is a parabola. Since the total energy on a parabolic orbit is 0, the speed of the comet is calculated via this formula ($d$ is the Sun-comet distance):

$$v = \sqrt{\frac{2GM_\odot}{d}} \Rightarrow d_{min} = \frac{2GM_\odot}{v_{max}^2}$$ **1.5pt**

$$d_{min} = \frac{2 \times 6.673 \times 10^{-11} \times 1.989 \times 10^{30}}{(9 \times 10^4)^2} \approx 0.22\,\text{au}$$ **0.5pt**

Hence, $a = 3.34\,\text{au}$, $r_{max} = a(1+e) = 3.84\,\text{au}$, this is the distance Sun-asteroid at aphelion. **1.5pt**
Now, the distance comet-asteroid, in the given context, is simply:

$$D = d_{min} + r_{max} = 0.22 + 3.84 = 4.06\,\text{au}$$ **0.5pt**

## Th.3 Where is Theodor? (10 Points)

Theodor is in a random city somewhere in the world. Using his knowledge of geography, he narrows down the possible locations to the four cities listed below. Then, on October $20^{\text{th}}$, he measures the time between sunrise and sunset and finds it to be $t = 10^h42^m$.
Can you help him figure out which city he is in? Ignore atmospheric refraction and Sun's apparent size.

a) Compute the Sun's equatorial coordinates on that date. (Consider that the sun moves with constant velocity on the ecliptic from the autumn equinox onward)

b) Based on the measured daytime duration $t$, determine which of the four cities Theodor is in.

c) On this day, exactly at the moment when the Sun sets for Theodor, determine the geographical coordinates of the point on Earth where the Sun is directly overhead.

1)Kutaisi (φ=42.27° N, L=42.66° E)

2)Vassouras (φ=22.41° S,L=43.66° W )

3)Mumbai (φ=18.96° N, L=72.83° E)

4)Chorzów (φ= 50.31° N, L=18.97° E)

## Marking Scheme

**a)** Compute the Sun's equatorial coordinates on that date. (We consider that the Sun moves with constant velocity on the ecliptic).
On $22^{nd}$ September, we can consider $\lambda_S = 180°$ (also $23^{rd}$ September)

$$\lambda_S = 0°/180° + \frac{D}{365.25} \times 360° \qquad \textbf{0.5pt}$$

where $D$ is the number of days between the chosen point and $20^{th}$ of October.

$$\lambda_S = 207.6° \qquad 206.6° \qquad \textbf{0.5pt}$$

$$\sin\delta = \sin\beta\cos\varepsilon + \cos\beta\sin\varepsilon\cos(90° - \lambda)$$

$$\sin\delta = \sin\varepsilon\sin\lambda \qquad \text{(Since } \beta = 0°\text{)} \qquad \textbf{1pt}$$

$$\delta = -10°38' \qquad -10°16' \qquad \textbf{0.5pt}$$

$$\cos\alpha = \frac{\cos\lambda}{\cos\delta} \qquad \textbf{0.5pt}$$

$$\alpha = 13^h42^m \qquad 13^h38^m \qquad \textbf{1pt}$$

*Note:* We would give 75% for students computing $\alpha$ directly, considering constant velocity on the equator.
*Note:* We would give 25% for students using Euclidian geometry instead of spherical trigonometry.

**b)** Based on the measured daytime duration $t$, determine which of the four cities Theodor is in.

$$\cos(90^\circ - h) = \cos(90^\circ - \varphi)\cos(90^\circ - \delta) + \sin(90^\circ - \varphi)\sin(90^\circ - \delta)\cos H$$
$$\therefore 0 = \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos H$$
$$\cos H = -\tan\varphi\tan\delta$$ **2pt**

*Note:* If students write the last formula directly, they receive the maximum.

$$t_l = H + 12h + n - L + \eta + D.S.T.$$

but all are constants (we can consider $\eta$ constant for a day), so

$$\Delta t_l \approx \Delta H$$ **0.5pt**

Considering that the two solutions for $\cos x = a$ in $[0, 2\pi]$ are $x = \arccos a$ and $x = 2\pi - \arccos a$, that means $t = 2\arccos(-\tan\varphi\tan\delta)$ (after conversion to hours)

$$t_1 = 10^h 42^m,\ t_2 = 12^h 35^m,\ t_3 = 11^h 31^m,\ t_4 = 10^h 16^m,$$ **1pt**

**Theodor is in city 1: *Kutaisi*** **0.5pt**

*Note:* We award full points for the calculation of both solutions for $H$ for each city and for calculating the time as their difference.

**c)** For the sun to be viewed as directly above ($h_S = 90^\circ$), the condition is $\varphi = \delta$ **0.5pt**

$$\therefore \varphi = -10^\circ 38'$$ **0.5pt**

$$L = 12h - T_{GMT}$$ **0.5pt**

$$T_{GMT} = H + 12h - L$$

$$L = 37^\circ 47'\,W\ or\ 37^\circ 1'\,W \text{ if using 23 sept.}$$ **0.5pt**

# Th.4 Exploring NEAs (10 Points)

Near-Earth Asteroids (NEAs) are small rocky bodies orbiting the Sun in the vicinity of Earth's orbit. A science mission designed to study a NEA target is operating a spacecraft that will make its closest approach to the target at a distance of $\Delta_c = 100\,\mathrm{m}$. One of the instruments aboard the spacecraft is a CCD camera called the Asteroid Framer (AF), which can acquire images in the V-band at an effective wavelength of $\lambda = 550\,\mathrm{nm}$. After the mission launched, the operations team conducted initial tests by acquiring images of the Earth using the AF camera with an aperture diameter of $D = 10\,\mathrm{mm}$. At that time, the spacecraft was at a distance of $d = 1.5 \times 10^6\,\mathrm{km}$ from Earth. The Earth was observed on the images within a photometric aperture of $\alpha_\oplus = 60$ pixels in diameter.

a) Compute the pixel scale in arcseconds per pixel.

b) Estimate the linear diameter of the smallest morphological feature that can be observed on the surface of the asteroid at the closest approach.

The saturation test on initial images showed Earth was just over-exposed at 10 s exposure time and just under-exposed at 0.5 s (under-exposed means signal-to-noise ratio $< 5$). Using this data, the operations team can determine the exposure time interval allowed to observe the NEA target, which has the expected albedo of $A = 0.10$. Ignore phase angle.

***Note:*** Assume that the detected signal is proportional to the object's surface brightness and the exposure time, and that the surface brightness is proportional to the albedo divided by the square of its distance from the Sun.

c) Determine the corresponding exposure time interval for which the planned observations of the NEA target at the closest approach will be successful without any under–exposure or over–exposure. At that time, the target will be located at $r_{NEA} = 1.20\,\mathrm{au}$ from the Sun.

## Marking Scheme

**a)** The pixel scale can be calculated as the ratio between the angular dimension and the corresponding number of pixels within which it is observed,

$$s = \frac{2R_\oplus/d}{\alpha_\oplus} \qquad \textbf{1pt}$$

Numerically,

$$s = \frac{2 \times 6.378 \times 10^3}{1.50 \times 10^6} \times \frac{180 \times 3600}{\pi} \times \frac{1}{60} = 29\,\mathrm{arcsec/pix} \qquad \textbf{1pt}$$

**b)** First, we apply the Rayleigh criterion to quantify the diffraction effects,

$$\theta = 1.22\frac{\lambda}{D} = \frac{1.22 \times 550 \times 10^{-9}\ \mathrm{m}}{0.01\ \mathrm{m}} \times \frac{180 \times 3600}{\pi}$$
$$= 13.8\,\mathrm{arcsec} \qquad \textbf{0.5pt}$$

The minimum angular separation calculated above is less than the angular resolution of the camera:

$$\theta < s \times (1\ \mathrm{pix}) \qquad \textbf{0.5pt}$$

Thus, the minimum size of a detectable feature should be comparable to the linear projection of a single pixel:

$$l = s \times (1\ \mathrm{pix}) \times \Delta_c \qquad \textbf{1pt}$$

Note for graders: Solutions using 2 or 3 pixels for the estimations will receive full scores.
Numerically,

$$l = 29\ \mathrm{arcsec} \times \frac{\pi}{180 \times 3600} \times 100\ \mathrm{m} = 1.4\ \mathrm{cm} \qquad \textbf{1pt}$$

**c)** The detected signal per pixel is proportional to the product of the surface brightness of the object, measured in W/(m$^2$·sr), and the exposure time:

$$S \propto Bt \qquad \textbf{1pt}$$

The surface brightness is due to the sunlight reflected by the object, so it is proportional to the albedo and the inverse of the squared heliocentric distance:

$$B \propto \frac{A}{r^2} \qquad \textbf{1pt}$$

Comparing the surface brightnesses of Earth and the NEA target, one gets:

$$\frac{B_{\mathrm{NEA}}}{B_{\oplus}} = \frac{A_{\mathrm{NEA}} r_{\oplus}^2}{A_{\oplus} r_{\mathrm{NEA}}^2} \qquad \textbf{1pt}$$

**Remark.**The surface brightness is independent of the distance between the sensor and the extended target. Both Earth and the NEA target are extended objects (not point-like sources).

Both objects should be observed under the same conditions (appropriate exposure times), so one can consider similar signal levels recorded for both objects. Thus, considering the first equation:

$$t_{\mathrm{NEA}} = t_{\mathrm{Earth}} \times \frac{A_{\oplus} r_{\mathrm{NEA}}^2}{A_{\mathrm{NEA}} r_{\oplus}^2} \qquad \textbf{1pt}$$

Numerically,

$$t_{\mathrm{NEA}} = t_{\mathrm{Earth}} \times \frac{0.30}{0.10} \times \left(\frac{1.20}{1.00}\right)^2 = 4.32\, t_{\mathrm{Earth}} \qquad \textbf{0.5pt}$$

The exposure time interval will be:

$$2.16 \text{ s} - 43.2 \text{ s} \qquad \textbf{0.5pt}$$

## Th.5 Ceahlău Mountain (15 Points)

Ceahlău Mountain is located in Neamț County, not far from the city of Piatra Neamț. An observer on the top of the mountain ($\varphi = 46°53'57'' \, N, L = 26°24'41'' \, E$, altitude $h = 1907$ m above sea level), looks at the starry sky, making observations of the star Almach in the constellation Andromeda.

a) Determine whether, from the top of Mount Ceahlău, the star Almach ($\gamma$ AND, $\alpha = 02^h03^m54^s$, $\delta = 42°19'47''$) is circumpolar. The radius of the Earth is known to be $R_\oplus = 6378$ km.

b) In 1778, it was discovered that $\gamma$ AND is a double star. The two stars are separated by an angle of about $10''$. What should be the diameter of the objective lens of a telescope to see the two components distinctly (the wavelength is known to be $\lambda = 550$ nm)?

c) It was later discovered that the Almach star is a multiple star system consisting of the primary component $\gamma_1$ (A) with apparent magnitude $m_1 = 2.27$ and a subsystem $\gamma_2$ with apparent magnitude $m_2 = 4.84$. Calculate the apparent magnitude of the system consisting of the star $\gamma_1$ and the subsystem $\gamma_2$.

d) What is the absolute magnitude of the star, knowing that it is approximately 355 light-years away?

## Marking Scheme

**a)** A circumpolar star is a star that, viewed from a certain latitude on Earth, never sets below the horizon, so when the minimum altitude is *0*. It follows that to be circumpolar, the minimum declination

$$\delta_m = 90° - \varphi \Rightarrow \delta_m = 90° - 46°53'57'' = 43°6'3'' \text{ at sea level.}$$ **1pt**

Considering atmospheric refraction $\alpha = 34'$, $\delta_m = 42°32'3''$, therefore, at the given latitude, the Almach star is not circumpolar. **1pt**

At an altitude of 1907 meters, the horizon appears to drop.

$$\cos\theta = \frac{R}{R+h} \Rightarrow \arccos\frac{R}{R+h} \approx 1.4°$$ **2pt**

$$\delta_m = 90° - \varphi - \alpha = 41°8'$$

so from Mount Ceahlău, the star Almach is circumpolar. **1pt**

**b)**

$$\theta = \frac{1.22\lambda}{D} \Rightarrow D = \frac{1.22\lambda}{\theta}$$ **2pt**

$$\theta = 10'' = 4.84 \cdot 10^{-5} \, rad$$

$$D \approx 14 \, mm$$ **1pt**

**c)** From Pogson's law $\frac{E_1}{E_2} = 10^{0.4\cdot(m_2 - m_1)}$

$$E_1 = 10.537 \cdot E_2$$ **2pt**

Total brightness (amount of energy per unit time per unit area): $E = E_1 + E_2$

$$E = 11.537 \cdot E_2$$ **1pt**

$$\frac{E_1}{E} = 2.5^{m - m_1}$$

$$m = 2.17$$ **1pt**

**d)**

$$M = m + 5 - 5 \cdot log_{10}(d)$$ **1pt**

$$d = 355 \, ly = 108.896 \, Pc$$ **1pt**

$$M = -3.04$$ **1pt**

## Th.6 The Flying Aquila (15 Points)

Radu is observing the night sky from Aveiro ($\varphi = 40.67°N$; $L = 8.65°W$ in the year 2024.
During his observations, he sees Altair ($\alpha_{Altair} - 19^h51^m$; $\delta_{Altair} = 8°55'50''$) rising at local solar time $T_{Local,1}$=22:39:43. Later, at local solar time $T_{Local,2}$=02:42:58, he observes that Altair and Arcturus ($\delta_{Arcturus} = 19°33'6''$) are at the same altitude. The observations are made from sea level. Ignore atmospheric effects.

a) Determine the sidereal time at the moment Altair rises.

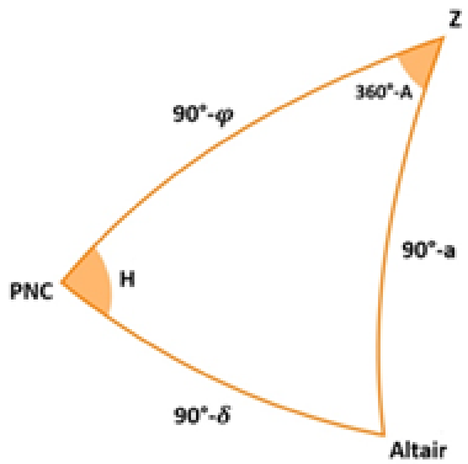
b) Determine the current date (at the moment Altair rises). Ignore the equation of time, and consider that the right ascension of the Sun is increasing uniformly through the year, at a rate of $3^m56^s$ per day. The spring equinox occurs on March 21st.

c) Determine the right ascension of Arcturus. Take into account that Arcturus is in the Western hemisphere of the sky at $T_{Local,2}$.

*Note:* In your solution, please use the following notation: $a$ - altitude; $H$ - hour angle; δ- declination; α- right ascension

## Marking Scheme

**a)** Drawing of the spherical triangle: North Celestial Pole – Zenith – Star




Using the spherical law of cosines in the previous triangle:

$$\sin a_{Altair} = \sin\varphi \sin\delta_{Altair} + \cos\varphi \cos\delta_{Altair} \cos H_{Altair}$$

At its rising, neglecting horizon effects and refraction, the altitude of Altair is $a_{Altair} = 0°$. Therefore $sin a_{Altair} = 0$ and

$$\sin\varphi \sin\delta_{Altair} + \cos\varphi \cos\delta_{Altair} \cos H_{Altair} = 0$$

This allows us to calculate Altair's hour angle

$$cos H_{Altair} = -\tan\varphi \tan\delta_{Altair}$$

$$H_{Altair} = \pm 6h31m$$ **2pt**

i.e. it will rise *6h31m* before crossing meridian When it is at meridian $T_{sideral} = \alpha_{Altair} = 19h51m$ **1pt** **1pt**

$$T_{sideral} = 19h51m - 6h31m = 13h20m$$ **1pt**

**b)** When Altair rises, $T_{local} = 22:39:43$, $H_{Sun} = 10:39:43$ **1pt**

As sideral time is known $H_\gamma = 13h20m$ **0.5pt**

$\alpha_{Sun} = 2h40m17s$ **1pt**

40.75 days after $\gamma$ **0.5pt**

$1st$ May **1pt**

**c)** Since the change in $\alpha_{Sun}$ in the span of a few hours is negligible, we can claim that

$$\Delta T_{Local} \approx \Delta T_{Sideral}$$

Therefore, when the stars are at the same altitude:

$$\begin{aligned} T_{Siderasl,2} &\approx T_{Siderasl,1} + T_{Loccal,2} - T_{Loccal,1} \\ &= 13h20m + 2h43m - 22h40m \\ &= 17h23m \end{aligned}$$ **1pt**

The same reasoning as in part a) applied to both Arcturus and Altair allows us to write

$$\sin a_{Altair} = \sin\varphi \sin\delta_{Altair} + \cos\varphi \cos\delta_{Altair} \cos H_{Altair}$$

$$\sin a_{Arcturus} = \sin\varphi \sin\delta_{Arcturus} + \cos\varphi \cos\delta_{Arcturus} \cos H_{Arcturus}$$ **1pt**

Since $a_{Altair} = a_{Arcturus}$, we can equate the two expressions and solve for the hour angle of Arcturus:

$$\cos H_{Arcturus} = \frac{\sin\varphi \sin\delta_{Altair} + \cos\varphi \cos\delta_{Altair} \cos H'_{Altair} - \sin\varphi \sin\delta_{Arcturus}}{\cos\varphi \cos\delta_{Arcturus}}$$

Solving this equation, we obtain:

$$H_{Arcturus} \in \{3h10m36s;\ 20h49m30s\}$$

Since Arcturus is in the western hemisphere of the sky, the valid solution is $H_{Arcturus} = 3h10m36s$ **3pt**

**Results between *3h9m* and *3h12m* should receive full marks.** Since the sidereal time and the hour angle of Arcturus are both known, we can now calculate its right ascension:

$$\alpha_{Arcturus} = T_{Sideral,2} - H_{Arcturus}$$

$$\alpha_{Arcturus} \approx 14h12m$$ **1pt**

**Results between *14h10m* and 14h18m should receive full marks.**

## Th.7 Lunar Eclipse (30 Points)

On September 7th 2025, there was a total lunar eclipse, visible from Australia, Asia, Africa and Europe. A detailed lunar eclipse figure, obtained from the website eclipsewise.com of eclipse predictions by Fred Espenak, is shown below. Based on the figure, respond to the following items:

a) Measure the inclination between the orbit of the Moon around the Earth and the Ecliptic plane. The orbit of the Moon is assumed to be circular.

b) During the total eclipse phase, the moon appears red in colour. Which of the following factors are necessary to explain this red colour? Write their item letters in the aswersheet.

   A) Density of atmosphere
   B) Albedo of Moon
   C) Orbital radius of Moon
   D) Albedo of Earth
   E) Effective temperature of the Sun

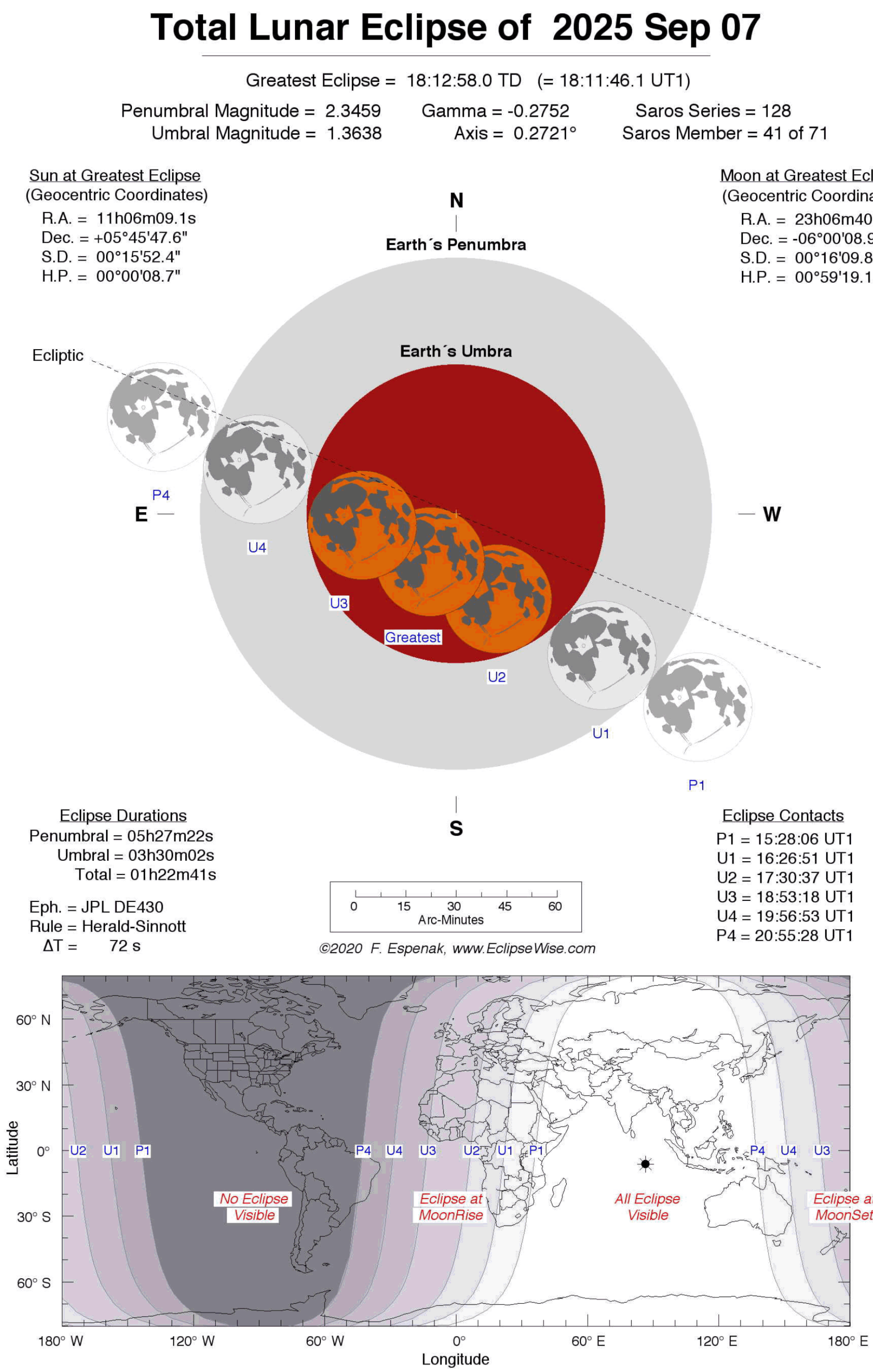
c) Measure the angular diameter of the Moon. Knowing the value of the radius of the Moon, calculate the Earth - Moon distance during the eclipse, in kilometres.

d) Measure the angular diameter of Earth's umbra and the angular diameter of Earth's penumbra.

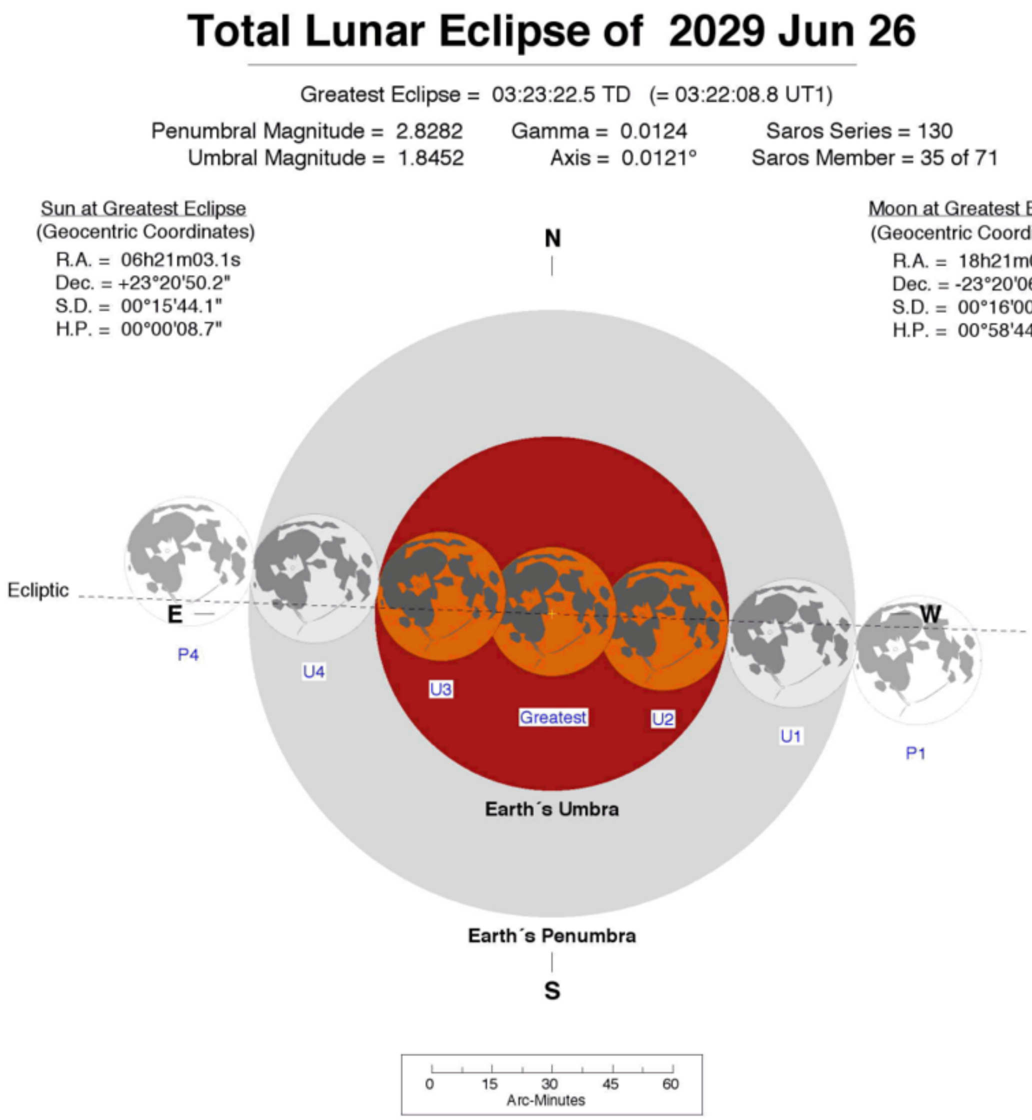
e) What is the instantaneous angular velocity of the Moon relative to the observer, during the total eclipse? Give the answer in arcminutes per hour ('/h).

f) On 26 June 2029, a central total lunar eclipse will take place. This means that the centre of the disk of the Moon will pass through the axis of Earth's umbral shadow (see figure below). If the angular velocity is assumed to be the same for the two eclipses, by what percentage will the 2029 total lunar eclipse last longer than the one in 2025 (percentage to be calculated of the totality duration in 2025)?

g) Calculate the total duration of the eclipse in 2029 (from Penumbral first contact to Penumbral last contact).



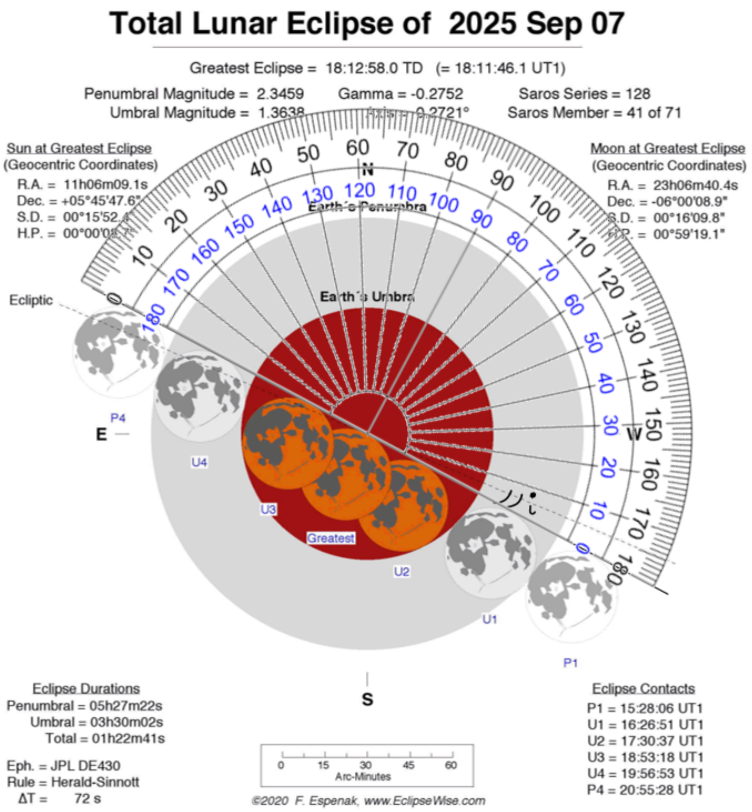

COURTESY OF ***21st CENTURY CANON OF Lunar ECLIPSES***, FRED ESPENAK, ASTROPIXELS PUBLISHING, 2020




# Marking Scheme

a) The angle of inclination measured is about $5° - 6°$. **4pt**

b) A) and C), each correct option **+2pt**, each wrong option **-1pt**
Note for graders: the minimum score on this part is 0, no negative score can be obtained. **4pt**




c) The scale of the image is given at the bottom of the page. From there, we find the correspondence 2.5 cm – 60 arc-minutes.

***Note for graders:*** **Distances in cm may change due to scaling, but answers in arcmin should remain the same.**

Measuring the diameter of the Moon on the figure, we get $1.6cm$.

Applying a simple proportion, we determine the angular diameter of the Moon to be:

$$d_{Moon} = \frac{1.3\ cm \times 60'}{2.5\ cm} = 31.2'$$ **4pt**

Answers between $31'$ and $32'$ are accepted (lack of precision due to ruler markings).
The angular radius of the moon will be:

$$r_{Moon} = \frac{d_{Moon}}{2} = 15.6' \approx 16'$$ **1pt**

The linear value will be:

$$R_{Moon} = D_{Earth-Moon} \times \sin(r_{Moon}) = 1737km$$

$$D_{Earth-Moon} = \frac{R_{Moon}}{\sin(r_{Moon})} = \frac{1737}{\sin(16')} \approx 373200\ km$$ **2pt**

Answers between $1730km$ and $1790km$ are accepted (corresponding to $15.5'$ and $16'$).

d) Similar to point c, we measure the diameter of Earth's umbra – 3.7 cm – and the diameter of Earth's penumbra – 6.3 cm.

The angular diameters will be:

$$d_{umbra} = \frac{3.7cm \times 60'}{2.5cm} = 89' = 1°29'$$

(88' - 90')

$$d_{penumbra} = \frac{6.3cm \times 60'}{2.5cm} = 151' = 2°31'$$ **6pt**

(150' - 153')

e) To determine the velocity, we choose a time interval (for example, between U2 and U3) and we measure on the figure what angular distance the moon traveled along its trajectory during that time.

$$\Delta t_{U3-U2} = t_{U3} - t_{U2} = 18^h 53^m 18^s - 17^h 30^m 37^s$$
$$\Delta t_{U3-U2} = 1^h 22^m 41^s$$ **0.5pt**

The linear distance travelled by the Moon between these two times is $\Delta l_{U3-U2} = 1.9cm$, which corresponds to:

$$\frac{1.9cm \times 60'}{2.5cm} = 45.6'$$ **0.5pt**

(45' - 47') The angular velocity will then be:

$$\omega_{Moon} = \frac{\Delta l_{U3-U2}}{\Delta t_{U3-U2}} = \frac{45.6'}{1^h 22^m 41^s} \approx 33.1'/h$$ **1pt**

(32.5'/h - 34.1'/h)

f) Since for this point we have the angular velocity and we want to determine the duration of the total eclipse, we need to measure the distance that the Moon travelled during the totality. First, we need to establish the scale of the new image. Here, 2.7 cm correspond to the $60'$.

Now, by measuring the path of the Moon, we get 2.5 cm.

The angular path will then be:

$$\Delta l_{totality} = \frac{2.5cm \times 60}{2.7cm} = 55.56'$$ **1.5pt**

(55' - 57')

The duration will be:

$$\Delta t_{totality} = \frac{\Delta l_{totality}}{\omega_{Moon}} = \frac{55.56'}{33.1'/h} = 1.69\ hours \approx 1^h 41^m$$ **0.5pt**

(1.6 - 1.75 hours), (1h36m - 1h75m) The difference between the totality durations will be:

$$\Delta t_{totality2029} - \Delta t_{totality2025} = 1^h 41^m - 1^h 22^m 41^s = 18^m 19^s$$ **0.5pt**

Expressing it as a percentage of the 2025 duration:

$$\frac{\Delta t_{totality2029} - \Delta t_{totality2025}}{\Delta t_{totality2025}} = \frac{18^m 19^s}{1^h 22^m 41^s} \approx 22.15\%$$ **0.5pt**

(15.7% - 26.6%)

g) To calculate the duration of the eclipse, we need to first measure the distance the Moon travels between $P_1$ and $P_4$: 8.2 cm.

$$\Delta l_{eclipse} = \frac{8.2cm \times 60'}{2.7cm} = 3.037° \approx 3°2'13''$$ **2pt**

(3 - 3.1)

$$\Delta t_{eclipse} = \frac{\Delta l_{eclipse}}{\omega_{Moon}} = \frac{3.037 \times 60'}{33.1'/h} \approx 5.5\ hours \approx 5^h 30^m$$ **2pt**

(5h17m - 5h42m)

# Th.8 Roche Limit, Hill Sphere, Lagrange Points (40 points)

The following astronomical data are given:

- The masses of *Mars* $M_M$ and the *Sun* $M_\odot$ are expressed relative to *Earth's* mass $M_\oplus$: $M_M = 0.107 M_\oplus$ and $M_\odot = 333054 M_\oplus$.
- The mean *Mars-Sun* distance is $r_{MS} = 1.524\,\text{au}$ and we consider that *Mars* has a circular orbit.
- The mean *Earth-Sun* distance is $r_{ES} = 1\,\text{au} = 1.496 \times 10^{11}\,\text{m}$ and we consider that *Earth* has a circular orbit.
- *Mars's* moon *Phobos* has a mass equal to $M_{Phobos} = 1.5 \times 10^{-8} M_M$;
- *Earth's* sidereal period is $P_{Sid\oplus} = 1\,\text{year} = 365.2564\,\text{days}$;
- The average radius of the satellite Phobos is approximately $R_{Phobos} = 12.5\,\text{km}$;
- The radius of its orbit around *Mars* is $d_{PhobosM} = 9380\,\text{km}$;
- Phobos is spiraling inward towards *Mars* at the constant rate of $1.8\,\text{cm/yr}$.
- You can use the following approximation:

$$(1+x)^n \approx 1 + nx, \text{when } x << 1.$$

- Roche limit is the boundary inside which the tidal force of the planet on a small mass is more than its self-gravity.

a) Derive the formula for the $L_1$ Lagrange point (distance between secondary body and the $L_1$ point) and calculate its numerical value for the *Sun-Mars* system. Find the ratio of Phobos's orbital radius to this distance.

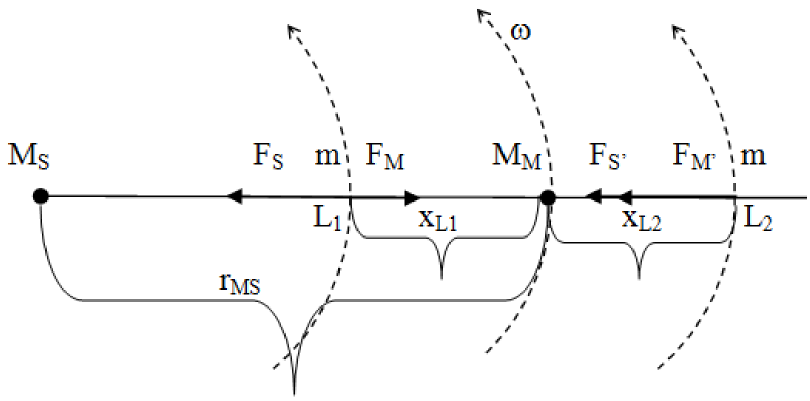
b) Derive the formula for the $L_2$ Lagrange point (distance between secondary body and the $L_2$ point) and calculate its numerical value for the the *Mars-Phobos* system.

c) Derive the formula for the *Roche* limit of the *Mars-Phobos* system. How long would it take Phobos to reach this limit? It is known that it approaches the surface of the planet during its rotation and will either collide with it or begin to disintegrate, forming a ring. The satellite is considered to be solid.

d) Calculate the distance $l$ between the *Lagrange* points $L_4$ and $L_5$ for Sun - Mars system.

## Marking Scheme

a) For the Lagrange points $L_1$ and $L_2$:



For Mars in orbit around the Sun:

$$F_{cf} = F_{SM}$$

$$M_M\omega^2 r_{MS} = G\frac{M_M M_\odot}{r_{MS}^2}$$ **2pt**

$$\omega = \sqrt{\frac{GM_\odot}{r_{MS}^3}}$$ **1pt**

For a body of mass $m$ at $L_1$:

$$m\omega^2(r_{MS} - x_{L1}) = G\frac{mM_\odot}{(r_{MS} - x_{L1})^2} - G\frac{mM_M}{x_{L1}^2}$$ **2pt**

$$\frac{M_\odot}{r_{MS}^3}(r_{MS} - x_{L1}) = \frac{M_\odot}{(r_{MS} - x_{L1})^2} - \frac{M_M}{x_{L1}^2}$$

$$\frac{M_\odot}{r_{MS}^3}(r_{MS} - x_{L1})^3 = M_\odot - \frac{M_M(r_{MS} - x_{L1})^2}{x_{L1}^2}$$

$$M_\odot\left(1 - \frac{x_{L1}}{r_{MS}}\right)^3 = M_\odot - M_M\left(\frac{r_{MS}}{x_{L1}} - 1\right)^2$$

Now, $x_{L1} << r_{MS}$, which means that $\frac{x_{L1}}{r_{MS}} << 1$ and $\frac{r_{MS}}{x_{L1}} >> 1$. Thus we can use the following approximations:

$$\left(1 - \frac{x_{L1}}{r_{MS}}\right)^3 \approx 1 - 3\frac{x_{L1}}{r_{MS}}$$ **2pt**

and

$$\frac{r_{MS}}{x_{L1}} - 1 \approx \frac{r_{MS}}{x_{L1}}$$ **1pt**

$$M_\odot\left(1 - 3\frac{x_{L1}}{r_{MS}}\right) = M_\odot - M_M\left(\frac{r_{MS}}{x_{L1}}\right)^2$$

$$3M_\odot\frac{x_{L1}}{r_{MS}} = M_M\left(\frac{r_{MS}}{x_{L1}}\right)^2$$

$$x_{L1}^3 = \frac{M_M}{3M_\odot}r_{MS}^3$$

$$x_{L1} = r_{MS}\sqrt[3]{\frac{M_M}{3M_\odot}}$$ **2pt**

Numerical value for *Sun-Mars* system:

$$x_{L1} = 1.524 \times 1.496 \times 10^{11} \times \sqrt[3]{\frac{0.107M_\oplus}{3 \times 333054M_\oplus}}$$

$$x_{L1} = 1.08 \times 10^9 m = 1.08 \times 10^6 km$$ **2pt**

Comparing this with Phobos's orbit, $d_{PhobosM} = 9380\ km$, we obtain: $\frac{d_{PhobosM}}{x_{L1}} \approx 9 \times 10^{-3}$. **1pt**

b) For a body of mass $m$ at $L_2$:

$$m\omega^2(r_{MS} + x_{L2}) = G\frac{mM_\odot}{(r_{MS} + x_{L2})^2} + G\frac{mM_M}{x_{L2}^2}$$ **2pt**

$$\frac{M_\odot}{r_{MS}^3}(r_{MS} + x_{L2}) = \frac{M_\odot}{(r_{MS} + x_{L2})^2} + \frac{M_M}{x_{L2}^2}$$

$$\frac{M_\odot}{r_{MS}^3}(r_{MS}+x_{L2})^3 = M_\odot + \frac{M_M(r_{MS}+x_{L2})^2}{x_{L2}^2}$$

$$M_\odot\left(1+\frac{x_{L2}}{r_{MS}}\right)^3 = M_\odot + M_M\left(\frac{r_{MS}}{x_{L2}}+1\right)^2$$

Now, $x_{L2} << r_{MS}$, which means that $\frac{x_{L2}}{r_{MS}} << 1$ and $\frac{r_{MS}}{x_{L2}} >> 1$. Thus we can use the following approximations:

$$\left(1+\frac{x_{L2}}{r_{MS}}\right)^3 \approx 1+3\frac{x_{L2}}{r_{MS}}$$ **2pt**

and

$$\frac{r_{MS}}{x_{L2}}+1 \approx \frac{r_{MS}}{x_{L2}}$$ **1pt**

$$M_\odot\left(1+3\frac{x_{L2}}{r_{MS}}\right) = M_\odot + M_M\left(\frac{r_{MS}}{x_{L2}}\right)^2$$

$$3M_\odot\frac{x_{L2}}{r_{MS}} = M_M\left(\frac{r_{MS}}{x_{L2}}\right)^2$$

$$x_{L2}^3 = \frac{M_M}{3M_\odot}r_{MS}^3$$

$$x_{L2} = r_{MS}\sqrt[3]{\frac{M_M}{3M_\odot}}$$ **2pt**

The position of the $L_2$ point for the Mars-Phobos system will be at a distance equal to:

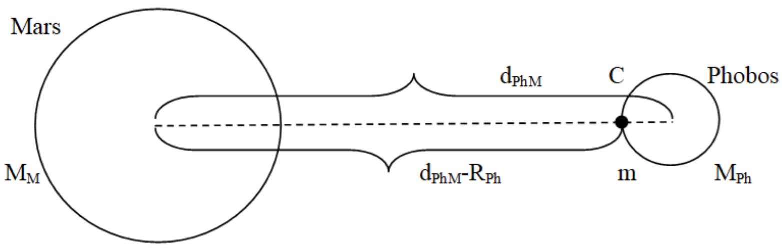
$$x_{L2} = d_{PhobosM}\sqrt[3]{\frac{M_{Phobos}}{3M_M}}$$

$$x_{L2} = 9380\times\sqrt[3]{\frac{1.5\times10^{-8}M_M}{3M_M}}$$

$$x_{L2} \approx 16km$$ **2pt**

The $L_2$ point is close to Phobos's surface $R_{Phobos} = 12.5km$.

c) For the Roche limit:




We calculate the tidal force experienced by Phobos due to Mars's gravitational influence. That is, the difference in gravitational attractions exerted by Mars on a material point of mass m, located on the surface of the satellite, and the same mass m located at the center of Phobos:

$$F_{tidal} = G\frac{mM_M}{(d_{PhobosM}-R_{Phobos})^2} - G\frac{mM_M}{d_{PhobosM}^2}$$ **4pt**

$$F_{tidal} = G\frac{mM_M}{d_{PhobosM}^2\left(1-\frac{R_{Phobos}}{d_{PhobosM}}\right)^2} - G\frac{mM_M}{d_{PhobosM}^2}$$

We take into account the relationship $R_{Phobos} << d_{PhobosM}$, which means $\frac{R_{Phobos}}{d_{PhobosM}} << 1$ and we can use the approximation given in the problem:

$$\left(1 - \frac{R_{Phobos}}{d_{PhobosM}}\right)^{-2} \approx 1 + 2\frac{R_{Phobos}}{d_{PhobosM}}$$ **2pt**

$$F_{tidal} = G\frac{mM_M}{d^2_{PhobosM}}\left(1 + 2\frac{R_{Phobos}}{d_{PhobosM}} - 1\right)$$

$$F_{tidal} = 2G\frac{mM_M R_{Phobos}}{d^3_{PhobosM}}$$ **1pt**

The gravitational attraction felt by the point of mass m due to Phobos's influence is:

$$F_{grav} = G\frac{mM_{Phobos}}{R^2_{Phobos}}$$ **1pt**

The Roche limit corresponds to the equality of the two forces:

$$F_{tidal} = F_{grav}$$ **1pt**

$$G\frac{mM_{Phobos}}{R^2_{Phobos}} = 2G\frac{mM_M R_{Phobos}}{d^3_{PhobosM\_Roche}}$$

$$d^3_{PhobosM\_Roche} = 2\frac{M_M}{M_{Phobos}}R^3_{Phobos}$$

$$d_{PhobosM\_Roche} = R_{Phobos}\sqrt[3]{2\frac{M_M}{M_{Phobos}}}$$ **2pt**

$$d_{PhobosM\_Roche} = 12.5\sqrt[3]{2\frac{M_M}{1.5 \times 10^{-8} M_M}}$$

$$d_{PhobosM\_Roche} = 6386 km$$ **2pt**

To calculate the time it would take Phobos to reach that radius around Mars, we need to compute the difference in radii:

$$\Delta d = d_{PhobosM} - d_{PhobosM\_Roche} = 9380 - 6386 = 2994 km$$ **1pt**

$$\Delta t = \frac{\Delta d}{v} = \frac{2994000}{1.8 \times 10^{-2}}\ years = 1.67 \times 10^8\ years$$ **1pt**

**Alternative solution, which will receive the maximum number of points (15 points):**
Students might calculate the Roche limit by modeling the satellite as two spheres of mass $\frac{M_{Phobos}}{2}$, with density equal to Phobos's density. **3pt**
The radius $r$ of such a sphere will be:

$$2 \times \frac{4\pi r^3}{3} = \frac{4\pi R^3_{Phobos}}{3}$$

$$r = \frac{R_{Phobos}}{\sqrt[3]{2}}$$ **1pt**

The gravitational force between the two bodies will be:

$$F_{grav} = G\frac{M^2_{Phobos}}{4(2r)^2}$$ **2pt**

The difference in the gravitational force due to Mars will be:

$$\Delta F = G\frac{M_M M_{Phobos}}{2(d_{PhobosM} - r)^2} - G\frac{M_M M_{Phobos}}{2(d_{PhobosM} + r)^2}$$ **2pt**

$$\Delta F = G\frac{M_M M_{Phobos}}{2(d_{PhobosM})^2}\left(\frac{1}{\left(1-\frac{r}{d_{PhobosM}}\right)^2} - \frac{1}{\left(1+\frac{r}{d_{PhobosM}}\right)^2}\right)$$

Using the approximation given in the problem:

$$\Delta F \approx G\frac{M_M M_{Phobos}}{2(d_{PhobosM})^2}\left(1+\frac{2r}{d_{PhobosM}} - 1 + \frac{2r}{d_{PhobosM}}\right)$$

$$\Delta F \approx G\frac{M_M M_{Phobos}}{2(d_{PhobosM})^2} \times \frac{4r}{d_{PhobosM}}$$

$$\Delta F \approx G\frac{M_M M_{Phobos} \times 2r}{(d_{PhobosM})^3}$$ **2pt**

The Roche limit corresponds to the equality of these two forces:

$$G\frac{M_M M_{Phobos} \times 2r}{(d_{PhobosM_Roche})^3} = G\frac{M^2_{Phobos}}{4(2r)^2}$$ **2pt**

$$\frac{32r^3}{(d_{PhobosM\ \ Roche})^3} = \frac{M_{Phobos}}{M_M}$$

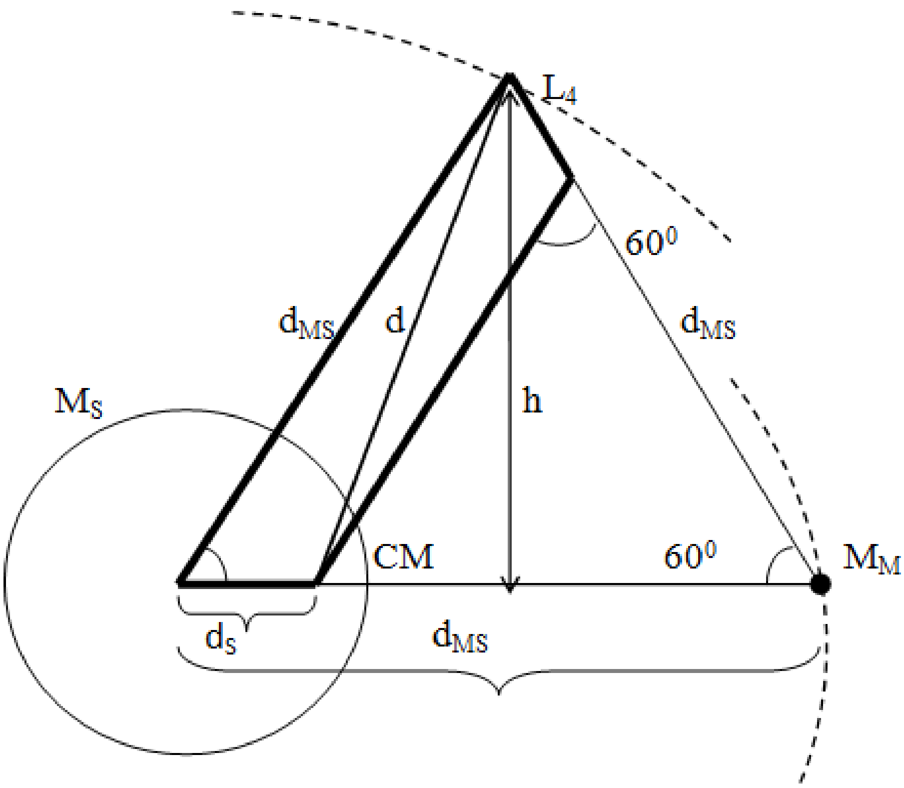
$$\frac{32R^3_{Phobos}}{2(d_{PhobosM\ \ Roche})^3} = \frac{M_{Phobos}}{M_M}$$

$$d_{PhobosM\ \ Roche} = 2R_{Phobos}\sqrt[3]{2\frac{M_M}{M_{Phobos}}}$$ **1pt**

$$d_{PhobosM\ \ Roche} = 12772km$$ **1pt**

This means that Phobos is already closer to Mars than the Roche limit. **1pt**

d) Determining the CM of the Mars-Sun system: Let $d_S$ be the distance from the center of the Sun to the center of mass CM, and $d$ the distance from the CM to the Lagrange point $L_4$.



Distance $l$ calculation between Lagrange points $L_4$ and $L_5$:

$$h^2 = d_{MS}^2 - \frac{d_{MS}^2}{4} \Rightarrow h = 1.32AU$$

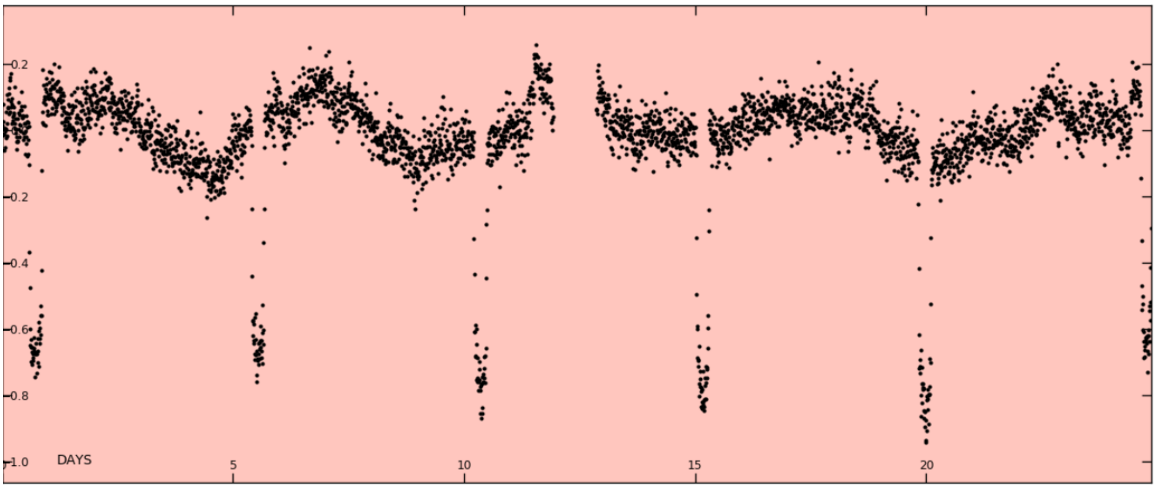
$$l = 2h = 2.64AU$$

**3pt**

## Th.9 Lightcurves (40 points)

### (A) Transiting Exoplanet

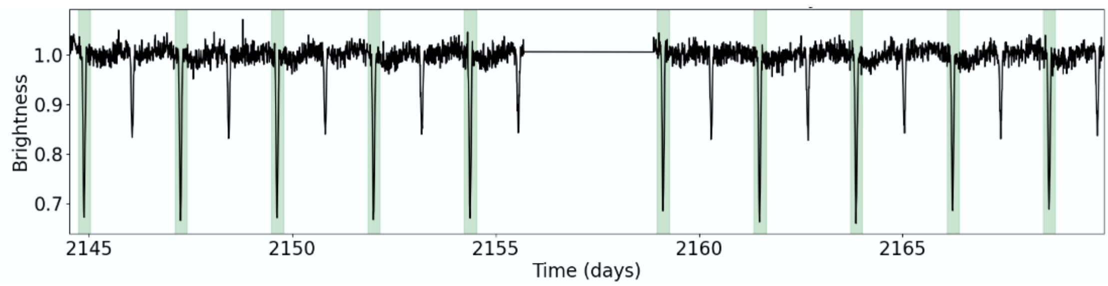
The figure below presents the light curve of an exoplanet transit. On the X-axis are represented the days elapsed since the first observation, and on the Y-axis is shown the percentage of the average luminosity by which the brightness of the system increases or decreases relative to the average brightness of the star, that is: $100 \times \frac{L(t) - L_{average}}{L_{average}}$. Determine the period $P$ of the planet and the ratio $\frac{Planet\ Radius}{Star\ Radius}$.




*Note:* On Y-axis *0.2* means *0.2%* and *1.0* means *1%*

### (B) Eclipsing Binary Star System

Another type of object with an interesting light curve is an eclipsing binary star system. The lightcurve of such an object is presented below, with the X-axis representing days since the start of the observing mission, and the Y-axis representing the relative brightness of the system, that is: $\frac{L(t)}{L_{average}}$. We know that the smaller star is hotter than the larger star. Calculate the period P of the system and the ratio $\frac{Small\ Star\ Radius}{Big\ Star\ Radius}$.

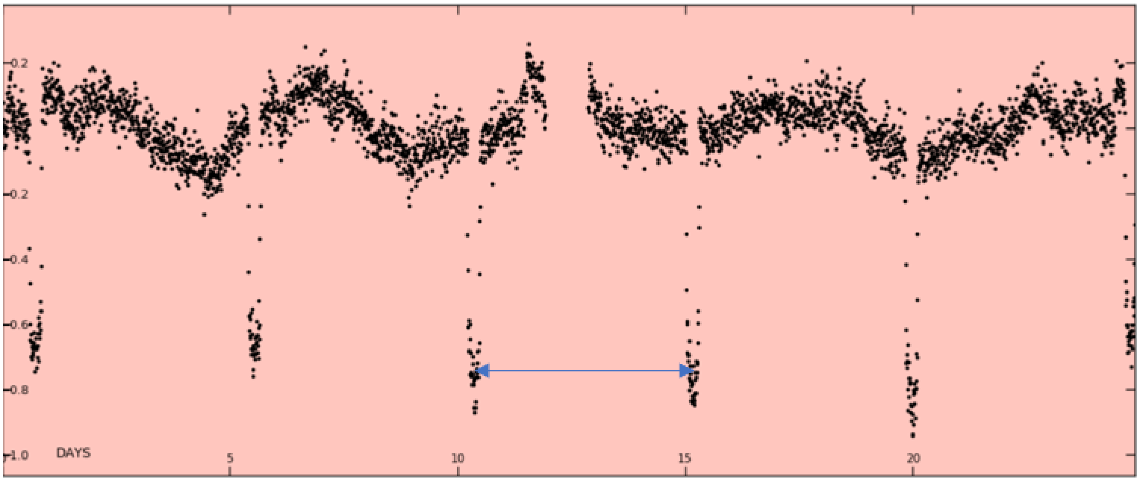



*Note:* On Y-axis *0.7* means *70%* and *1.0* means *100%*

# Marking Scheme

### (A) Transiting Exoplanet (15 pt)

To calculate the period, we can proceed in different ways. The most straightforward way is to measure the distance between two consecutive minima in cm in the figure. Then, by measuring the distance between the 5 and 10 day marks, we can compute the scale of the figure on the OX axis. We then use this scale to calculate the period of the planet.




5 days ... 2.6 cm.

The scale will be 1.92 days/cm. The distance between two consecutive minima is 2.5 cm.

The period will then be:

$$P = 2.5 \times \frac{5}{2.6} \approx 4.81 \; days$$

**5pt**

**Note for graders: The calculated scale might be different due to printing/zoom figure changes from the initial problem writing to the final version. However, the value of the period will not be affected by this. Answers within $\pm 0.1$ days will receive full points**

For a lower error in determining the period, we can measure longer distances on the paper:

- To determine the scale, we measure between the 5 and 20 day marks: 15 days ... 7.8 cm
- To determine the period, we measure the distance between the first and the one before last minima (the last one is not fully visible) and then we divide the distance by the number of periods: 10 cm divided by 4

$$P = \frac{10}{4} \times \frac{15}{7.8} \approx 4.81 \; days$$

**Note for graders: Both methods will receive the maximum point score.**

To calculate the ratio of the radii, we start by noting the minimum value in percent as $n\%$. From the description of the OY axis, we can write:

$$100 \times \frac{L_{transit} - L_{average}}{L_{average}} = n$$

This is just the formula given in the problem, written explicitly for a transiting moment of time.

$$L_{transit} - L_{average} = \frac{n}{100} \times L_{average}$$

where $L_{average}$ is the average luminosity of the system (the luminosity of the star without transit), and $L_{transit}$ is the luminosity of the system at the minimum caused by the transit. $L_{transit}$ can be written as:

$$L_{transit} = \frac{L_{average}}{\pi R^2} \left( \pi R^2 - \pi r^2 \right)$$

Here $R$ is the radius of the start and $r$ is the radius of the planet. The calculation of $n$ then becomes:

$$\frac{L_{average}}{\pi R^2} \left( \pi R^2 - \pi r^2 \right) - L_{average} = \frac{n}{100} \times L_{average}$$

$$\left(\frac{r}{R}\right)^2 = -\frac{n}{100} \Rightarrow \frac{r}{R} = \sqrt{\frac{|n|}{100}}$$ **5pt**

Now we need to determine the scale of the figure on the OY axis. Measuring the distance between −0.2% and −0.8%, we get 2.2 cm. Thus:

0.6% … 2.2 cm.

The depth of a minimum varies between 2.8 and 3.2 cm, depending on which minimum is chosen for the analysis. For the purpose of this problem, any value in the 2.8 – 3.1 interval will receive the maximum point score, as well as measuring all minima and taking an average of the depth of the transit (a more accurate method). The numerical value here will be calculated for 2.9 cm.

$$|n| = 2.9 \times \frac{0.6}{2.2} \approx 0.79\%$$ **3pt**

$$\frac{r}{R} = \frac{Planet\ Radius}{Star\ Radius} \approx 0.09$$ **2pt**

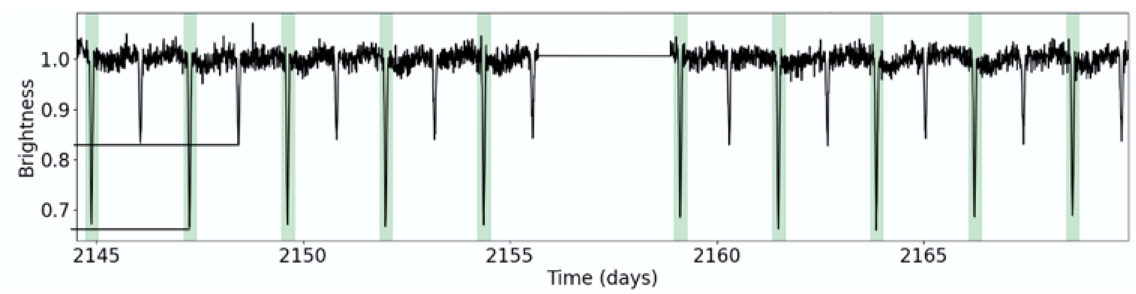
For the extremes values we would get:

$$\frac{r}{R} = \frac{Planet\ Radius}{Star\ Radius} \approx 0.087 \approx 0.09$$

$$\frac{r}{R} = \frac{Planet\ Radius}{Star\ Radius} \approx 0.093 \approx 0.09$$

By rounding the answer to the correct number of significant digits, we obtain the same result.

**(B) Eclipsing Binary Star System (25 pt)**




The scale of this figure on the OX axis is: 20 days … 9.1 cm The measured value corresponding to the period is 1.1 cm.

We then get the period:

$$P = 1.1 \times \frac{20}{9.1} \approx 2.42\ days$$ **6pt**

**Note for graders: The calculated scale might be different due to printing/zoom figure changes from initial problem writing to final version. However, the value of the period will not be affected by this. Answers within ±0.1 days will receive full points**

The average luminosity for a binary star system is:

$$L_{average} = L_R + L_r$$

where $L_R$ is the luminosity of the large, cooler star of radius $R$ and $L_r$ is the luminosity of the smaller, hotter star of radius $r$. During the secondary minima (smaller, hotter star transiting the bigger, cooler star), the luminosity of the system will be:

$$L_{Secondary} = L_r + \frac{L_R}{\pi R^2}\left(\pi R^2 - \pi r^2\right)$$

During the principal minima (smaller, hotter star completely eclipsed by the bigger, cooler star), the luminosity of the system will be:

$$L_{Primary} = L_R$$ **6pt**

The corresponding luminosity ratios are:

$$n_{Primary} = \frac{L_{Primary}}{L_{average}} = \frac{L_R}{L_R + L_r}$$

$$n_{Secondary} = \frac{L_{Secondary}}{L_{average}} = \frac{L_r + \frac{L_R}{\pi R^2}\left(\pi R^2 - \pi r^2\right)}{L_R + L_r}$$

$$n_{Secondary} = \frac{L_R + L_r - L_R \times \frac{r^2}{R^2}}{L_R + L_r} = 1 - \frac{L_R}{L_R + L_r} \times \frac{r^2}{R^2}$$

$$n_{Secondary} = 1 - n_{Primary} \times \frac{r^2}{R^2}$$

$$\frac{r}{R} = \sqrt{\frac{1 - n_{Secondary}}{n_{Primary}}}$$ **5pt**

From the figure, we measure the Y-scale: 0.2 ... 1.1 cm Then, $1 - n_{Primary}$ will be, in cm: 1.8 cm.

$$n_{Primary} = 1 - 1.8 \times \frac{0.2}{1.1} = 1 - 0.327 \approx 0.67$$ **3pt**

$1 - n_{Secondary}$ will be, in cm: 0.9 cm.

$$n_{Secondary} = 1 - 0.9 \times \frac{0.2}{1.1} = 1 - 0.16 \approx 0.84$$ **3pt**

$$\frac{r}{R} = \sqrt{\frac{1 - 0.84}{0.67}} \approx 0.49$$ **2pt**

$\pm 0.02$ values accepted for full marks