Instructions

- **Do not touch envelopes until the start of the examination.**
- The theoretical examination lasts for 5 hours and is worth a total of 250 marks.
- There are **Answer Sheets** for carrying out detailed work and **Working Sheets** for rough work, which are already marked with your student code and question number.
- ***Use only the answer sheets for a particular question for your answer. Please write only on the printed side of the sheet.*** Do not use the reverse side. If you have written something on any sheet which you do not want to be evaluated, cross it out.
- Use as many mathematical expressions as you think may help the evaluator to better understand your solutions. The evaluator may not understand your language. If it is necessary to explain something in words, please use short phrases (if possible in English).
- You are not allowed to leave your work desk without permission. If you need any assistance (malfunctioning calculator, need to visit a restroom, etc.), please draw the attention of the supervisor.
- The beginning and end of the examination will be indicated by the supervisor. The remaining time will be displayed on a clock.
- At the end of the examination you must stop writing immediately. Put everything back in the envelope and leave it on the table.
- Once all envelopes are collected, your student guide will escort you out of the examination room.
- A list of constants and useful relations are included in the envelope.

Table of Constants

Fundamental constants			
Speed of light in vacuum	c	=	2.998×10^{8} m s^{-1}
Planck constant	h	=	6.626×10^{-34} J s
Boltzmann constant	k_B	=	1.381×10^{-23} J K^{-1}
Stefan-Boltzmann constant	σ	=	5.670×10^{-8} W m^{-2} K^{-4}
Elementary charge	e	=	1.602×10^{-19} C
Universal gravitational constant	G	=	6.674×10^{-11} N m^{2} kg^{-2}
Universal electric constant	ϵ_0	=	8.854×10^{-12} m^{-3}kg^{-1}s^{4}A^{2}
Universal gas constant	R	=	8.315 J mol^{-1} K^{-1}
Avogadro constant	N_{A}	=	6.022×10^{23} mol^{-1}
Wien's displacement constant	$b = \lambda_m T$	=	2.898×10^{-3} m K
Mass of electron	m_{e}	=	9.109×10^{-31} kg
Mass of proton	m_{p}	=	1.673×10^{-27} kg
Mass of neutron	m_{n}	=	1.675×10^{-27} kg
Mass of Helium nucleus	m_{He}	=	6.645×10^{-27} kg
Atomic mass unit (a.m.u., Dalton)		=	1.661×10^{-27} kg

Astronomical data			
Hubble constant	H_0	=	70 km s^{-1} Mpc^{-1}
North Ecliptic Pole ($J2000.0$)	$(\alpha_{\mathrm{E}},\ \delta_{\mathrm{E}})$		$(18^{\mathrm{h}}00^{\mathrm{m}}00^{\mathrm{s}}, +66°33'39'')$
North Galactic Pole ($J2000.0$)	$(\alpha_{\mathrm{G}},\ \delta_{\mathrm{G}})$		$(12^{\mathrm{h}}51^{\mathrm{m}}26^{\mathrm{s}}, +27°07'42'')$
1 jansky	1 Jy	=	10^{-26} W m^{-2} Hz^{-1}
1 parsec	1 pc	=	3.086×10^{16} m 206 265 au 3.262 ly
1 astronomical unit (au)	1 au	=	1.496×10^{11} m
1 sidereal day	T_{SD}	=	23.93444 h $23^{\mathrm{h}}56^{\mathrm{m}}04^{\mathrm{s}}$
1 tropical year		=	365.2422 solar days
1 sidereal year		=	365.2564 solar days

Approximations

$(1+x)^n \approx 1 + nx$

$(1+x)(1+y) \approx 1 + x + y$ if $x \ll 1$ and $y \ll 1$

Gauss's formulae

Spherical law of cosines: $\cos a = \cos b \cos c + \sin b \sin c \cos A$

Spherical law of sines: $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$

next page $\Longrightarrow$

The Sun			
Solar luminosity	$L_{\odot}$	$=$	3.826×10^{26} W
Apparent angular diameter of Sun	$\theta_{\odot}$	$=$	32’
Effective temperature of Sun	$T_{\text{eff},\odot}$	$=$	5778 K
Apparent visual magnitude		$=$	-26.75
Absolute visual magnitude		$=$	$+4.82$
Apparent bolometric magnitude		$=$	-26.83
Absolute bolometric magnitude		$=$	$+4.74$
Distance of the Sun from the Galactic centre		$\approx$	8 kpc

The Earth and Moon			
Obliquity of the ecliptic (Earth)	ϵ	$=$	$23.5'$
Platonic year (period of precession of Earth’s axis)		$=$	25 765 years
Apparent visual magnitude of full Moon		$=$	-12.74
Apparent angular diameter of Moon	θ_{L}	$=$	$31'$
Inclination of the lunar orbit to the ecliptic		$=$	$05°08'43''$
Inclination of the lunar equator to its orbital plane		$=$	$6.687°$
Lunar sidereal month	T_{SL}	$=$	27.321661 d 655.71986 h
Synodic month		$=$	$29.530589d$
Tropical month		$=$	$27.321582d$
Anomalistic month (the average time between two successive Moon’s perigees)		$=$	$27.554550d$
Draconic month (time between two successive transits of the Moon through the same node)		$=$	$27.212221d$

The Solar System

Object	**Mean radius** [km]	**Mass** [kg]	**Semimajor axis** [au]	**Eccentricity**
Sun	695 700	1.988×10^{30}	—	—
Mercury	2 440	3.301×10^{23}	0.387	0.206
Venus	6 052	4.867×10^{24}	0.723	0.007
Earth	6 378	5.972×10^{24}	1.000	0.016 710
Moon	1 737	7.346×10^{22}	3.844×10^{5} km	0.054 900 (range $0.026 - 0.077$)
Mars	3 390	6.417×10^{23}	1.524	0.093
Jupiter	69 911	1.898×10^{27}	5.203	0.048
Saturn	58 232	5.683×10^{26}	9.537	0.054
Uranus	25 362	8.681×10^{25}	19.189	0.047
Neptune	24 622	1.024×10^{26}	30.070	0.009

Neptune (5 points)

Given that Neptune will be at opposition on 21 September 2024, calculate in which year Neptune was last at opposition near the time of the northern-hemisphere spring equinox. Assume that the orbits of Earth and Neptune are circular.

Magnetic Field (5 points)

An emission line of wavelength $\lambda = 600\,\mathrm{nm}$ was observed in the spectrum of a white dwarf. Assuming that it originates from the interaction of a free non-relativistic electron with a magnetic field,

(a) calculate the magnetic flux density of the field;

(b) estimate the wavelength of another spectral line, the discovery of which could confirm that the lines originate from particles of a plasma interacting with the magnetic field.

Microlensing (5 points)

A faint subdwarf star ($I = 20.4\,\mathrm{mag}$) in the Galactic bulge was observed to brighten to $I' = 15.2\,\mathrm{mag}$ as a result of gravitational microlensing, allowing a high-resolution spectrum to be obtained with the UVES spectrograph on the Very Large Telescope (mirror diameter $D = 8.2\,\mathrm{m}$).

Estimate the diameter of the telescope needed to obtain a spectrum of the same quality with the same instrument and exposure time for this star at its normal apparent brightness. The fiber aperture is small enough so that the sky background is negligible.

Europa (10 points)

(a) Assuming that the ice covering the ocean on Europa is $6\,\mathrm{km}$ thick, that the surface temperature on the night side of Europa is $100\,\mathrm{K}$ and that the temperature at the ice-water boundary is $273\,\mathrm{K}$, calculate the total power corresponding to the heat emitted from the interior of this moon.

(b) On Earth, the mean geothermal heat flux measured at the continental surface is $70 \times 10^{-3}\,\mathrm{Wm^{-2}}$ and originates mainly from radioactive decay. Is the heat emanating from the interior of Europa more likely to come from radioactive decay or tidal forces? Assume that Earth and Europa have a similar isotopic composition. (Select the correct answer on the answer sheet and show your working.)

Notes: the heat passing through a wall with a surface area S and thickness d in time t is described by the formula:

$$Q = \lambda S \Delta T t / d,$$

where λ stands for thermal conductivity and ΔT for the temperature difference.

The thermal conductivity of ice $\lambda = 3\,\mathrm{Wm^{-1}K^{-1}}$. The mass and radius of Europa are $4.8 \times 10^{22}\,\mathrm{kg}$ and $1561\,\mathrm{km}$.

Dark Energy (12 points)

Observations indicate that the expansion of the Universe is accelerating. Fluctuations of the cosmic microwave background favour a flat (Euclidean) geometry, in which the total mass density (i.e. the sum of density of matter and equivalent mass density of all forms of energy) should be equal to the so-called critical density:

$$\rho_{\mathrm{cr}} = \frac{3\,H_0^2}{8\,\pi\,G},$$

where H_0 is the present value of the Hubble constant. However, the total density of matter (luminous and dark) is estimated at

$$\rho_{\mathrm{m},0} \approx 2.8 \times 10^{-27}\,\mathrm{kg\,m^{-3}}\,.$$

To resolve this discrepancy, the standard cosmological model assumes that the Universe is filled with a mysterious 'dark energy' of constant energy density ε_Λ.

Determine the value of ε_Λ and calculate for which redshift in the past the energy density equivalent to matter was equal to the density of dark energy. Neglect the contribution of electromagnetic radiation.

Bolometer (13 points)

The entrance cavity of a particular bolometer is a cone with an opening angle of 30°, the surface of which has an energy absorption coefficient of $a = 0.99$. Assume that there is no scattering of the incident radiation on the walls of the cavity, only multiple mirror-like (specular) reflections. The bolometer is connected to a cooler which keeps the bolometer cavity surface at practically $0\,\mathrm{K}$ temperature. The instrument is orbiting at $2\,\mathrm{au}$ from the Sun and is pointed directly at the centre of the Solar disk.

Calculate the temperature of a black body which would radiate the same amount of energy as the entire bolometer opening does per unit surface area.

Note: the opening angle is defined as twice the angle between the axis of the cone and its generatrix.

Libration (20 points)

As a result of libration, studied among others by Johannes Hevelius, more than half of the Moon's surface can be observed from Earth. Assume that the observer is geocentric.

(a) Estimate ϕ_B, the maximum angle of libration in latitude. The axial tilt (obliquity) of the Moon with respect to its orbital plane is $\alpha = 6°41'$.

(b) Estimate ϕ_L, the maximum angle of libration in longitude. Assume that the Moon is always aligned with the same side facing towards the second focus F2 of its orbit, and that the eccentricity of the Moon's orbit e changes between 0.044 and 0.064 on a timescale of several months.

(c) Estimate the fraction of the Moon's surface which can be seen from Earth.

(d) Calculate how many months (lunations) are needed for an observer to see the Moon's surface determined in part (c).

Neutrinos (20 points)

In a simplified model of a supernova explosion, the core of a star, composed of pure iron $^{56}_{26}$Fe nuclei with a total mass of 1 $M_{\odot}$, changes into a neutron star composed of individual electrons, protons and neutrons in numerical proportions of 1:1:8. This process is called ‘neutronization’ and results in the emission of a large number of neutrinos.

Calculate the solar neutrino flux on Earth. How much larger would the flux of neutrinos reaching the Earth from the supernova be than the steady neutrino emission of the Sun, if the supernova exploded in the centre of the Galaxy and the process of neutronization of the core took about $0.01\,\mathrm{s}$? Give an order-of-magnitude answer.

Second eclipse (20 points)

For each of two eclipsing binary systems, Bolek and Lolek, the primary eclipses were observed with very high cadence as depicted below:

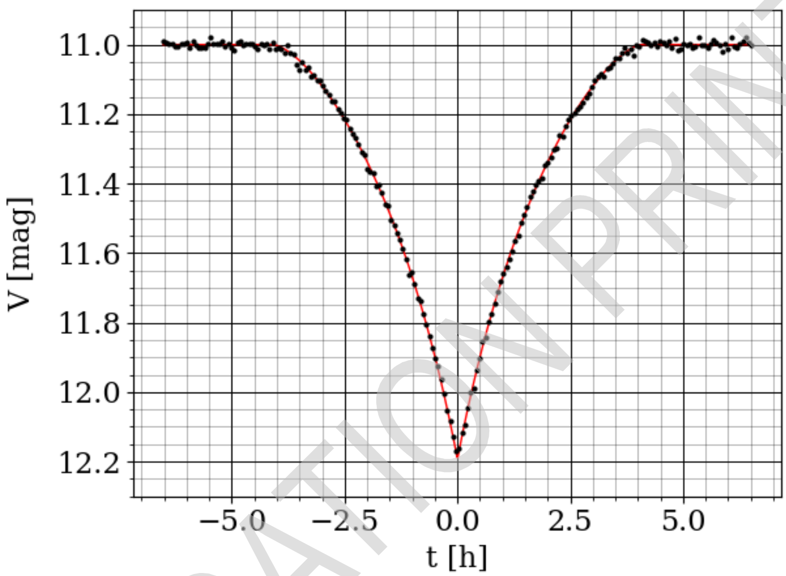


Figure 1. Observed lightcurve for system Bolek.

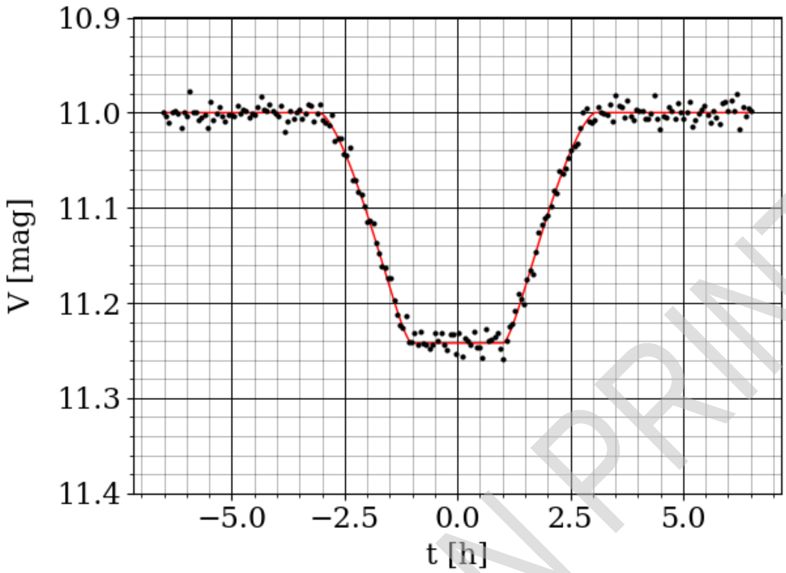


Figure 2. Observed lightcurve for system Lolek.

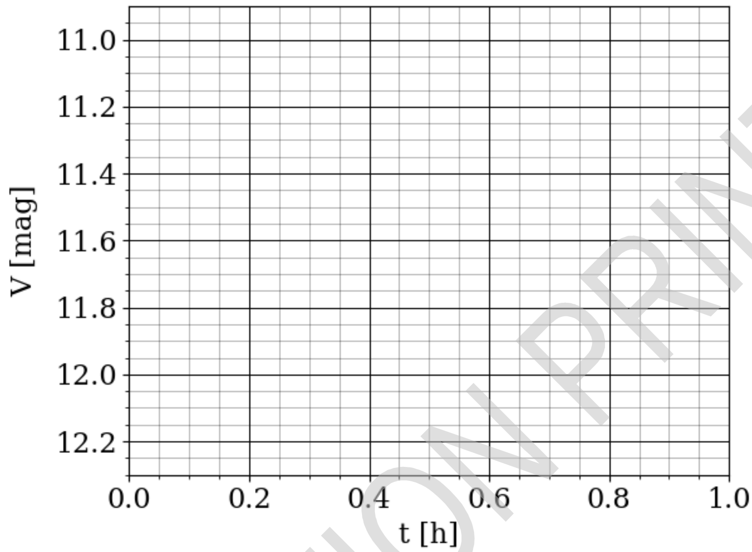
In the figures, t is the time in hours relative to the moment of minimum and V is the brightness in the V (visible) band in magnitudes. The points are the measurements and the line is the fitted model of the shape of the eclipse.

You can assume that in both cases the eclipses are central ($i = 90^\circ$) and last for a very small fraction of the orbital period, limb darkening is negligible, and the orbits have low eccentricity.

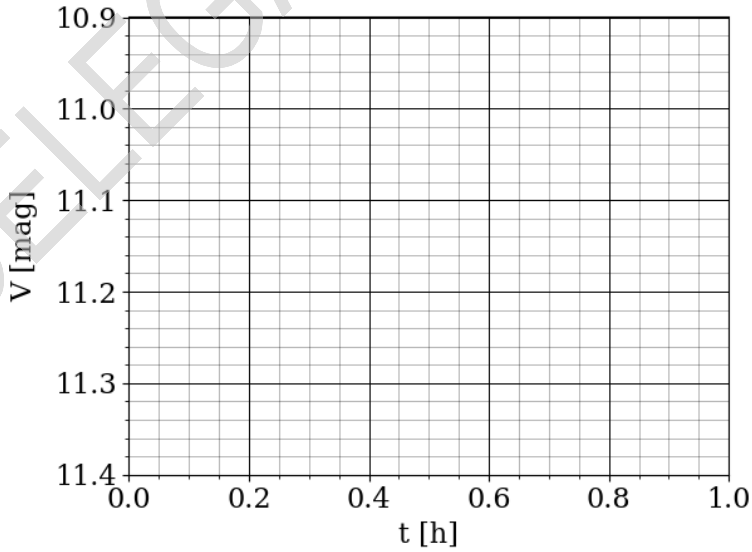
On the Answer Sheet, draw the predicted shape of the light curve for each of the secondary eclipses. Write down the equations and calculations leading to your predictions.

Вторинні затемнення

T.9 (20 pt)



Передбачувана крива блиску для системи Болек



Передбачувана крива блиску для системи Льолек

Aldebaran (25 points)

On 9 March 1497, Nicolaus Copernicus observed the occultation of Aldebaran by the Moon from Bologna. In his work *De revolutionibus orbium cœlestium* (Book VI, Chapter 27) Copernicus described the event: ”I saw the star touching the dark edge of the Moon and disappearing at the end of the 5th hour of the night between the horns of the Moon, closer to the south horn by a third of the Moon’s diameter.”

Assuming that the occultation was observed on the local meridian, that at maximum occultation Aldebaran was $0.32'$ above the southern edge of the Moon, and that the apparent angular diameter of the Moon as seen from Bologna was $31.5'$, solve the following tasks:

(a) Find the latitude φ_1 of a place with the same longitude as Bologna, from which Aldebaran would have appeared to pass behind the centre of the Moon. (6 points)

(b) Find the duration of the occultation as seen from latitude φ_1 if Aldebaran appeared to pass along the diameter of the lunar disk. For simplicity, also assume that the Moon and the observer are moving linearly at constant speed, that the Moon’s orbit is circular and that the declination of the Moon does not change during the occultation. (5 points)

(c) Find the topocentric angular velocity of the Moon against the background stars (for an observer on the surface of the Earth) during the occultation for an observer at latitude φ_1, in arcmin/hour, applying the same assumptions as in part (b). (7 points)

(d) Estimate the range of the Moon’s topocentric angular velocities (against the background stars for an observer on the surface of the Earth) in arcmin/hour at latitude φ_1, assuming a circular orbit. Show how this result can be justified by expressing the relative velocity of the Moon and observer in terms of their velocity vectors. (7 points)

The declination of Aldebaran was $\delta_A = 15.37^\circ$ in 1497 (due to precession), and the latitude of Bologna is $\varphi_B = 44.44^\circ$ N.

X-ray emission from galaxy clusters (30 points)

Clusters of galaxies are strong X-ray sources. It has been established that the emission mechanism is thermal bremsstrahlung (free-free radiation) from a hot hydrogen and helium plasma inside the cluster. The luminosity L_X (in Watts) of each component of the plasma is described by the formula:

$$L_X = 6 \times 10^{-41} N_e N_X T^{\frac{1}{2}} V Z_X^2,$$

where the symbols represent:

- X– Hydrogen (H) or Helium (He),
- N_e– number density of electrons [m^{-3}],
- N_X– number density of ions X [m^{-3}],
- Z_X– atomic number of ion X,
- T– temperature of the plasma [K],
- V– volume occupied by the plasma [m^3].

(a) Determine the total mass (in solar masses) of the plasma which emits the X-rays, assuming that:

- the plasma is fully ionized with 1 helium ion for every 10 hydrogen ions,
- $L_{\text{total}} = 1.0 \times 10^{37}$ W,
- $T = 80 \times 10^6$ K,
- the plasma is uniformly distributed in a sphere of radius $R = 500$ kpc,
- self-absorption is negligible.

(16 points)

The photons of the cosmic microwave background (CMB) interact with plasma in a process known as inverse Compton scattering. The CMB normally has a thermal blackbody spectrum at a temperature of 2.73 K. However, interaction with the plasma leads to distortion of the CMB spectrum (known as the Sunyaev-Zeldovich effect).

(b) Estimate the mean free path of CMB photons in the plasma, i.e. the average distance travelled by a photon before interacting with an electron. Express it in Mpc. The effective cross section for photon-electron interactions is $\sigma = 6.65 \times 10^{-29}\,\mathrm{m}^2$.

(5 points)

(c) Estimate the typical energy of CMB photons.

(3 points)

(d) The energy of CMB photons can be increased by a factor of up to $(1+\beta)/(1-\beta)$ due to the inverse Compton scattering, where $v = \beta c$ is the velocity of electrons. Estimate the energy of scattered CMB photons.

(6 points)

DART (40 points)

The Double Asteroid Redirection Test (DART) was a NASA mission to evaluate a method of planetary defense against near-Earth objects. The spacecraft hit Dimorphos, a moon of the asteroid Didymos, to study how the impact affected its orbit.

(a) Calculate the expected orbital period change (in minutes), assuming that the collision was head-on, central, and perfectly inelastic.

Assume that before the impact Dimorphos orbited Didymos on a circular orbit with a period of $P = 11.92\,\mathrm{h}$. The masses of Dimorphos and Didymos are $m = 4.3 \times 10^9\,\mathrm{kg}$ and $M = 5.6 \times 10^{11}\,\mathrm{kg}$, respectively. The mass and speed of the DART spacecraft relative to Dimorphos at a moment of impact were $m_\mathrm{s} = 580\,\mathrm{kg}$ and $v_\mathrm{s} = 6.1\,\mathrm{km\,s^{-1}}$. Neglect the gravitational influence of other bodies.

(20 points)

(b) In reality, the orbital period of Dimorphos was observed to be changed by $\Delta P_0 = -33\,\mathrm{min}$. This is due to the momentum transfer associated with the recoil of the ejected debris: the spacecraft was absorbed by the asteroid, but the impact excavated some material from the asteroid and ejected it into space. Calculate the momentum of the ejected debris and express it as a fraction of the momentum of Dimorphos before the collision. You can assume that the mass of the ejected material is much smaller than the mass of Dimorphos.

(15 points)

(c) Calculate the velocity change (in $\mathrm{mm\,s^{-1}}$) of Dimorphos as a result of the impact, taking into account the effect of the ejected debris.

(5 points)

LISA (45 points)

The Laser Interferometer Space Antenna (LISA) is a proposed experiment to detect low-frequency gravitational waves. It consists of three spacecraft arranged in an equilateral triangle. A passing gravitational wave changes the distance between the spacecraft, which can be precisely measured (more details are given in the notes below).

One of the sources of low-frequency gravitational waves are compact binary star systems, for example binary white dwarfs. Such a system was recently discovered at a distance of $2.34\,\mathrm{kpc}$ from the Sun. The orbital period of the binary was found to be $414.79\,\mathrm{s}$ and is changing at a rate of -7.49×10^{-4} s yr^{-1} due to the emission of gravitational waves.

(a) Check if this binary system can be detected by LISA.

(25 points)

(b) Calculate the chirp mass.

(5 points)

(c) Determine the masses of both components knowing that the ratio between the radius of one of the components to the semi-major axis of the orbit is 0.139, and assuming both components follow the mass-radius relation for white dwarfs given in the table below.

(15 points)

Notes:

1. A binary star system with an orbital period P emits gravitational waves with a frequency of $f = 2/P$.

2. LISA measures a dimensionless quantity called the characteristic strain amplitude, S, given by

   $S = h\sqrt{fT_{\text{obs}}}$,

   where $T_{\text{obs}} = 4$ yr is the expected duration of the mission. h is the gravitational wave strain, given by:

   $h = \frac{2(G\mathcal{M})^{5/3}(\pi f)^{2/3}}{c^4 D}$,

   where $\mathcal{M}$ is the so-called chirp mass, f is the frequency of the gravitational wave and D is the distance to the system. If we denote the masses of the components of the binary as M_1 and M_2, then the chirp mass is given by:

$$\mathcal{M} = \frac{(M_1 M_2)^{3/5}}{(M_1+M_2)^{1/5}}.$$

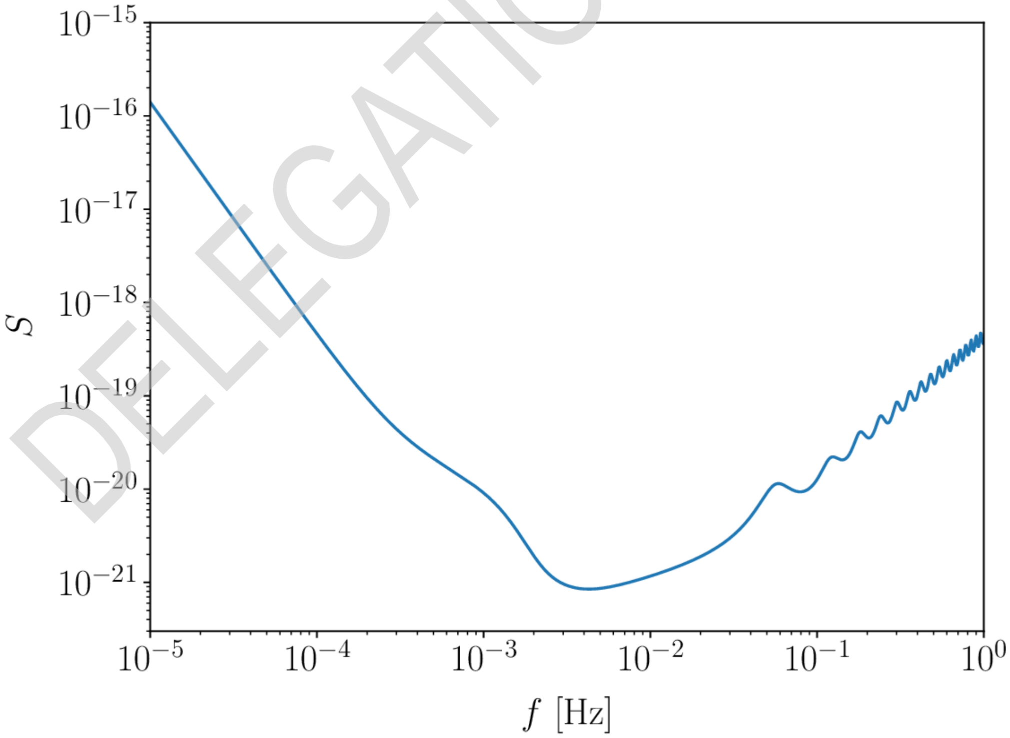
The expected sensitivity of LISA as a function of a gravitational wave frequency is presented on the figure below.

3. The semi-major axis a of the binary system changes due to the emission of gravitational waves at a rate:

$$\frac{\Delta a}{\Delta t} = -\frac{64}{5}\frac{G^3}{c^5}\frac{M_1 M_2 (M_1+M_2)}{a^3}.$$

M ($M_\odot$)	R ($R_\odot$)
0.48	0.0144
0.50	0.0147
0.52	0.0150
0.54	0.0153
0.56	0.0156
0.58	0.0159
0.60	0.0162
0.62	0.0165
0.64	0.0168

Mass-radius relation for white dwarfs based on theoretical models of Althaus et al. (2013) for white dwarfs of $\log g = 7.7$.



The expected sensitivity of LISA as a function of gravitational wave frequency.