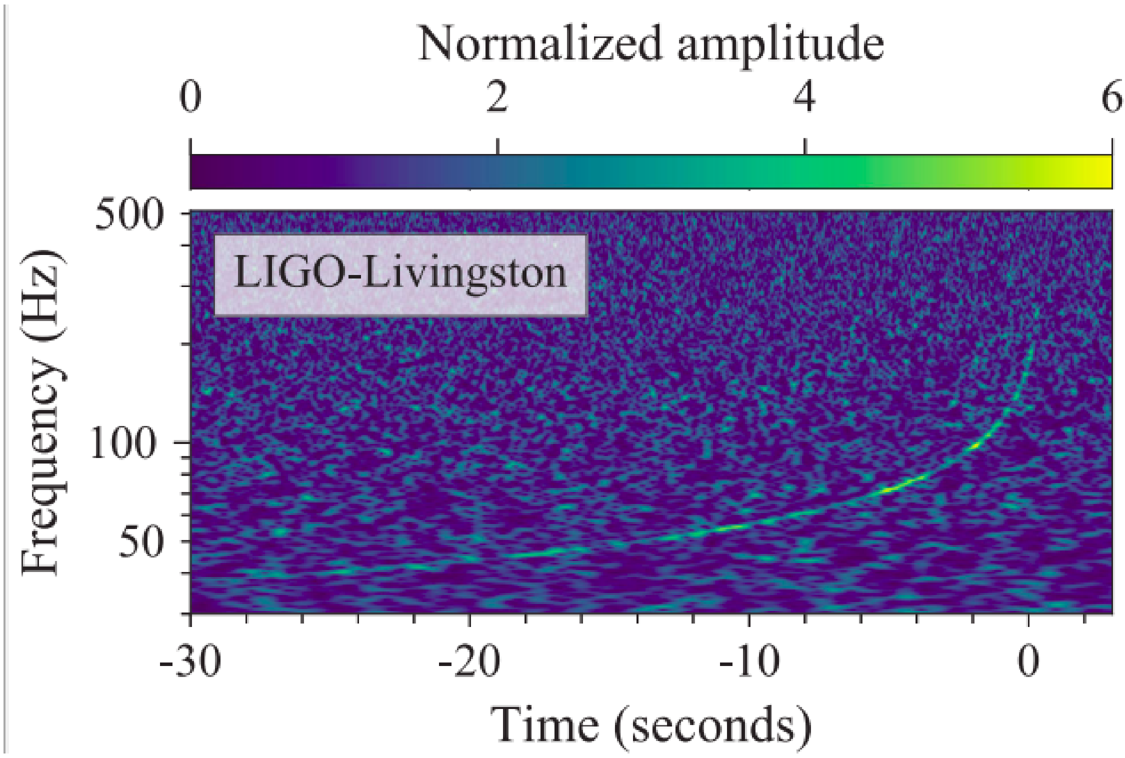
In this problem, we will analyze Time-frequency representations of data containing the gravitational-wave event GW170817, Which was the first observation by LIGO of a merger of a binary neutron star system.



1 Data from the graph

(1.1) You are given semi-logarithmic graph, from which you need to extract values of frequency and time. Find a linear expression to obtain actual time t from the measured horizontal axis coordinate x. **(2pt)**

To make the scaling as accurate as possible, we take the time values far apart. When $x = 0.0\,\text{cm}$ $t = -30\,\text{s}$, using the ruler we can also measure that at $t = 0\,\text{s}$ we have $x = 11.1\,\text{cm}$. **1.0**

This leads to

$$
\begin{aligned}
0.0\,\text{cm} &= (-30\,\text{s})A + B \\
11.1\,\text{cm} &= (0\,\text{s})A + B = B \\
A &= \frac{11.1\,\text{cm}}{30\,\text{s}} = 0.37\,\text{cm/s} \\
\therefore t &= \frac{x - 11.1}{0.37}\text{s}
\end{aligned}
$$

1.0

(1.2) Similarly, find an expression for the frequency f as a function of the measured coordinate y. **(3pt)**

Analogues to the previous scaling,

$$\begin{aligned} 0.0\,\mathrm{cm} &= C\log(30\,\mathrm{Hz}) + D \\ 5.3\,\mathrm{cm} &= C\log(500\,\mathrm{Hz}) + D \qquad \mathbf{1.0} \\ 5.3\,\mathrm{cm} &= C\log\left(\frac{500}{30}\right) \\ C &= 4.34\,\mathrm{cm} \\ D &= -6.41\,\mathrm{cm} \\ \therefore f &= 10^{\left(\frac{y+6.41}{4.34}\right)}\mathrm{Hz} \qquad \mathbf{2.0} \end{aligned}$$

(1.3) Using the relations that you obtained, extract at least 12 values of time and corresponding frequency from the given graph. At least one of the values should correspond to a frequency more than 100 Hz. **(6pt)**

It is not necessary to use X-scaling as students can choose convenient points from the graph.

x(cm)	y(cm)	t(s)	f(Hz)	$f^{-8/3}(Hz^{-8/3})$
-	0.5	-28.0	39.1	5.67E-05
-	0.5	-26.0	39.1	5.67E-05
-	0.6	-24.0	41.3	4.92E-05
-	0.6	-22.0	41.3	4.92E-05
-	0.7	-20.0	43.5	4.27E-05
-	0.8	-18.0	45.9	3.71E-05
-	0.9	-16.0	48.4	3.22E-05
-	1.0	-14.0	51.0	2.79E-05
-	1.1	-12.0	53.8	2.43E-05
-	1.2	-10.0	56.7	2.11E-05
-	1.4	-8.0	63.1	1.59E-05
-	1.6	-6.0	70.1	1.20E-05
-	1.8	-4.0	78.0	0.90E-05
-	2.3	-2.0	101.7	0.43E-05
-	3.4	0.0	182.4	0.09E-05

0.5 points for each row (except last column). Maximum 6 points.

2 Calculate system parameters

The most plausible explanation for this evolution of frequency is the in-spiralling of two orbiting masses, m_1 and m_2, due to gravitational-wave emission. At the lower frequencies, such evolution is characterized by the chirp mass

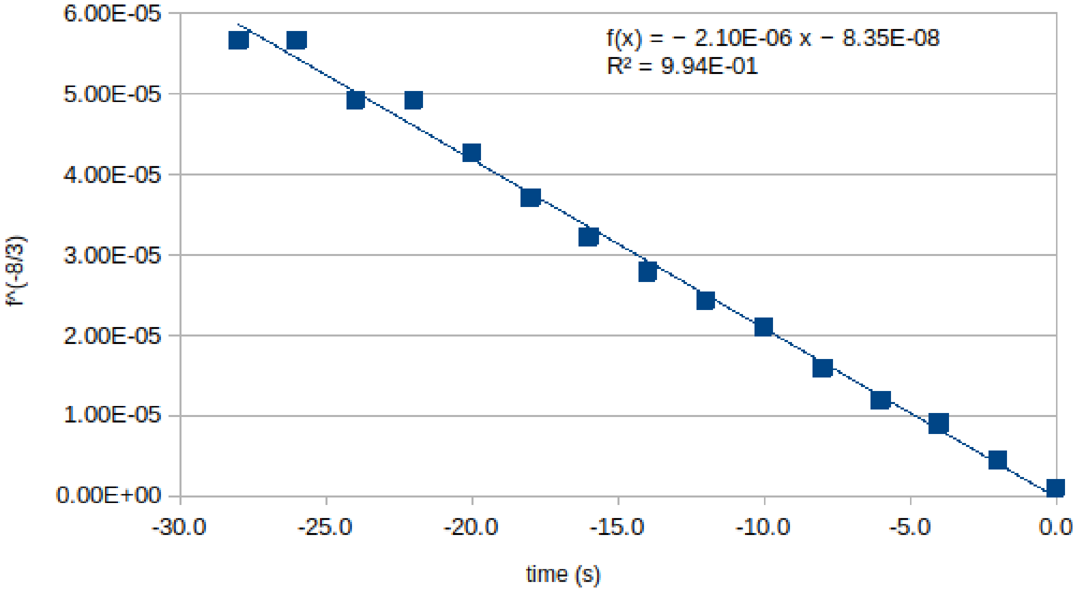
$$M_{chirp} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} = \frac{c^3}{G}\left[\frac{5}{96}\pi^{-8/3}f^{-11/3}\dot{f}\right]^{3/5}$$

where f and $\dot{f}$ are the observed frequency and its time derivative and G and c are the gravitational constant and speed of light.
(2.1) Linearize the equation given above and obtain the frequency dependence on time. **(3pt)**
Note: If $x^n\dot{x} = k$, then $\frac{x^{n+1}}{(n+1)} = kt + C$, where k, n and C are all some constants.

$$\frac{96}{5}\pi^{8/3}\left(\frac{GM_{chirp}}{c^3}\right)^{5/3} = f^{-11/3}\dot{f}$$

$$\therefore \left[\frac{96}{5}\pi^{8/3}\left(\frac{GM_{chirp}}{c^3}\right)^{5/3}\right] t + \left[\frac{3}{8}\right] f^{-8/3} + C = 0$$ **3.0**

(2.2) Find chirp mass in terms of solar masses and its uncertainty, by using a millimeter paper. **(15pt)**



Calculation of $f^{-8/3}$ values **3.0**
Properly drawn graph **6.0**

$$\text{slope} = \frac{8}{3}\left[\frac{96}{5}\pi^{8/3}\left(\frac{GM_{chirp}}{c^3}\right)^{5/3}\right]$$

$$M_{chirp} = \left[\left(\frac{5}{256}\right)^{3/5}\frac{c^3}{G}\pi^{-8/5}\right](\text{slope})^{3/5}$$ **2.0**

$$\text{slope} = 2.10 \times 10^{-6}$$ **2.0**

$$\therefore M_{chirp} = 2.39 \times 10^{30}\,\text{kg} = 1.20 M_\odot$$ **2.0**

with 95% confidence interval

$$\Delta\text{slope} = 9 \times 10^{-8}$$

$$\frac{\Delta M_{chirp}}{M_{chirp}} = \frac{3}{5}\frac{\Delta\text{slope}}{\text{slope}} = 0.04$$

$$\Delta M_{chirp} = 0.03 M_\odot$$

We realise that what is measured by the ground-based GW detectors is actually the detector-frame masses, which are related to the source frame masses by

$$m_{\text{detector}} = (1+z)m$$

where z is the redshift of the binary.

(2.3) It is known that host galaxy NGC 4993 has a red shift $z = 0.009783$, find the source frame chirp mass. **(2pt)**

$$M_{chirp} = \frac{M_{chirp_{\text{detactor}}}}{1+z} = \frac{1.20M_\odot}{1.009783} = 1.19M_\odot$$ **2.0**

(2.4) Find distance to the NGC 4993. **(3pt)**

According to the Hubble law,

$$D = \frac{zc}{H} = \frac{0.009783 \times 3 \times 10^5}{70} = 41.9\,\text{Mpc}$$ **2.0**

(2.5)The mass ratio $q = m_1/m_2$, is much harder to measure. Advanced waveform analysis shows that q for this system was in the range of 0.73 to 1.0. Calculate range of values for the masses m_1 (primary) and m_2 (secondary). **(6pt)**

$$M_{chirp} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

$$\therefore M_{chirp} = m_2 \sqrt[5]{\frac{q^3}{1+q}}$$ **2.0**

This is an increasing function of q

$$\therefore m_{2,min} = M_{chirp}2^{1/5} = 1.37M_\odot$$

$$m_{2,max} = 1.60M_\odot$$ **1.0**

$$\text{Also, } M_{chirp} = \frac{m_1}{q}\sqrt[5]{\frac{q^3}{1+q}} = \frac{m_1}{\sqrt[5]{q^2(1+q)}}$$ **2.0**

$$\therefore m_{1,max} = M_{chirp}2^{1/5} = 1.37M_\odot$$

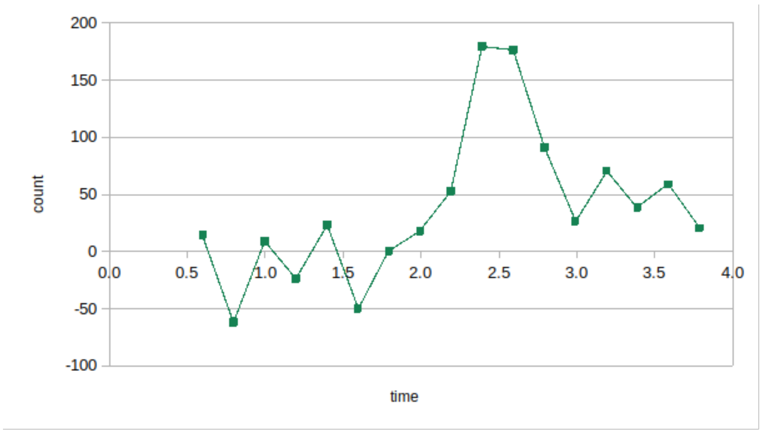
$$m_{2,min} = 1.17M_\odot$$ **1.0**

3 Speed of the gravitational wave

Gamma-ray burst GRB 170817A was observed by the Fermi Gamma-ray Burst Monitor to be almost simultaneous with gravitational wave event. The same neutron star merger was identified as the source of the signal. In the table below, the first column is time since the arrival of the gravitational wave peak, which happened on 2017 August 17 at 12:41:04 UTC. This peak symbolized the start of the merger of the neutron stars. The second column gives detector counts as measured in Fermi GBM. Background is already subtracted from the signal.

$t(s)$	event count (/s)
0.5981	15
0.7975	-62
0.9968	9
1.1962	-24
1.3956	24
1.5949	-50
1.7943	0
1.9937	18
2.1930	53
2.3924	179
2.5918	176
2.7911	91
2.9905	26
3.1899	71
3.3892	38
3.5886	59
3.7880	21

(3.1) Plot event count over time on a millimeter paper. **(6pt)**



(3.2) Estimate the delay Δt between start of the merger and start of the Gamma-ray Burst. **(1pt)**

It appears that the event is starting from $\Delta t = 1.8\,\text{s}$. Some students may also write $\Delta t = 2.2\,\text{s}$ as that is when the signal is truly above fluctuations seen before the event. We will accept both the answers.

1.0

From this measurement it is possible to determine what is called the fractional speed difference during the trip.

$$\frac{\Delta v}{v_{EM}} = \frac{v_{GW} - v_{EM}}{v_{EM}}$$

(3.3) Express this quantity in terms of Δt and distance to the source D. **(1pt)**

By simple substitution we can calculate that

$$\frac{\Delta v}{v_{EM}} = \frac{\Delta t v_{GW}}{D} \approx \frac{\Delta t c}{D}$$

1.0

(3.4) If we conservatively assume that the peak of the gravitational wave signal and the first GRB photons were emitted simultaneously, thus attributing the entire lag to faster travel by the gravitational wave signal, this time difference provides an upper bound on Δv. Calculate this upper bound. **(1pt)**

$$\frac{\Delta v}{v_{EM}} = 4.2 \times 10^{-16} \text{ to } 5.1 \times 10^{-16}$$

1.0

(3.5) To obtain a lower bound on Δv, one can assume that the two signals were emitted at times differing by more than Δt with the faster EM signal making up some of the difference. Take maximum time delay as 10s and find lower bound. **(1pt)**

$$\Delta t' = \Delta t - 10 = -8.2\,\text{s to } 7.8\,\text{s}$$

From this time delay

$$\frac{\Delta v}{v_{EM}} = -1.9 \times 10^{-15} \text{ to } 1.8 \times 10^{-15}$$

1.0