SOLUTION

TQ15 [55 points]

PART A:

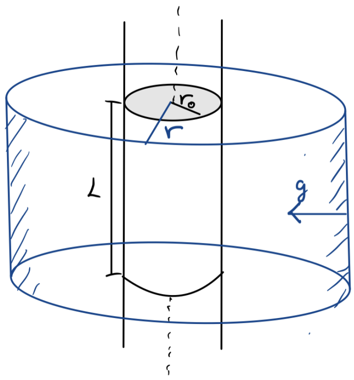
A.1 In cylindrical coordinates, the gravitational field is given by

$$\vec{g} = \vec{g}(r) = g(r)\,\hat{r} = g\,\hat{r}, \qquad \text{[1 point]}$$

where, in view of the axial symmetry, g can be computed from the Gauss law for the gravitational field:

$$\vec{g} \bullet \vec{A} = -\,4\pi G M_{in}, \qquad \text{[1 point]}$$

where M_{in} is the mass enclosed by the surface A, as shown in the figure below:



Taking the surface to be a cylinder of length L and radius r, from the above equation:

$$2\pi r L g = -\,4\pi G M_{in}$$

$$g = -\frac{2GM_{in}}{rL} \qquad \text{[1 point]}$$

There are two cases (regions):

- For $r > r_0$, one has $M_{in} = \mu L$ and so [0.5 points]

$$g = -\frac{2G\mu}{r} \qquad \text{[1 point]}$$

- For $r < r_0$, one has $M = \frac{r^2}{r_0^2}\mu L$ and so [0.5 points]

$$g = -\frac{2G\mu}{r_0}\left(\frac{r}{r_0}\right) \qquad \text{[1 point]}$$

A.2 Writing $\vec{g}(r_0)$, in terms of G, μ and r_0:

$$\vec{g}(r_0) = -\frac{2G\mu}{r_0}\hat{r}$$

$$g_0 \equiv |\vec{g}(r_0)| = \frac{2G\mu}{r_0}$$ [1 point]

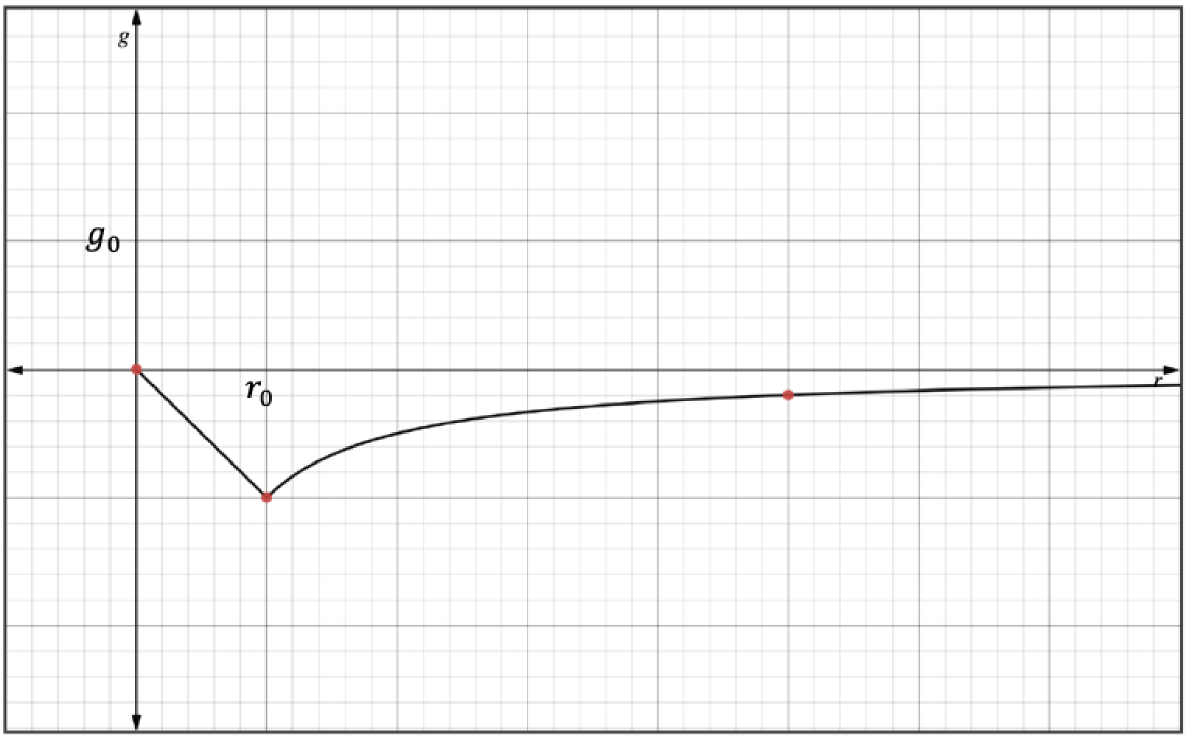
A.3

The following points should be included in the sketch:

- $(0, 0)$ [0.5 points]
- $(1, -1)$ [0.5 points]

Additionally,

- the sketch is linear between $(0, 0)$ and $(1, -1)$ [0.8 points]
- the sketch is concave for $r > r_0$, $\left(\sim \frac{1}{r}\right)$ [0.8 points]
- the sketch is drawn below x-axis. [0.4 points]



A.4 The centripetal acceleration equals the gravitational acceleration, thus

$$\frac{v^2}{R} = \frac{2G\mu}{R} \quad \text{[1 point]}$$

$$\frac{2\pi R}{\tau} = \sqrt{2G\mu} \quad \text{[1 point]}$$

$$R = \frac{\sqrt{2G\mu}}{2\pi}\tau$$

$$A = \frac{\sqrt{2G\mu}}{2\pi} \quad \text{[1 point]}$$

$$\alpha = 1 \quad \text{[1 point]}$$

A.5 The potential energy of the particle is given by

$$U(r) = -\int_b^r \vec{F}(r)\cdot d\vec{r}$$

$$= m\int_b^r g(r)dr \quad \text{[1 point]}$$

$$= 2Gm\mu\int_b^r \frac{1}{r}dr \quad \text{[1 point]}$$

$$= 2Gm\mu \ln\left(\frac{r}{b}\right) \quad \text{[1 point]}$$

Where b is a constant that sets the 0 of U.

Note 1: Usually, $U = 0$ is set at $b = \infty$. For a cosmic string, this is not possible. Instead, we can choose $U = 0$, for example, at the string's surface $b = r_0$ *(this choice is not relevant!)*. Thus

$$U = 2Gm\mu \ln\left(\frac{r}{r_0}\right)$$

Note 2: For completeness, notice that the potential inside the string, $r < r_0$ is

$$U = 2Gm\mu\left(\frac{r}{r_0}\right)^2 - U_0,$$

where $U_0 = 2Gm\mu\left(\frac{r}{r_0}\right)^2 - 2\,Gm\mu\,\ln\left(\frac{r}{r_0}\right)$ is a constant (again, not relevant!) that can be chosen such that U is continuous at the surface of the string. However, for the solution of the problem, it is not necessary to show this.

A.6 The total energy of the particle is conserved, thus:

$$\frac{1}{2}mv^2 + 2Gm\mu\ \ln\left(\frac{r}{b}\right) = 2Gm\mu\ln\left(\frac{R_{max}}{b}\right)$$ [2 point]

Solving for R_{max}:

$$R_{max} = Re^{\frac{v^2}{4G\mu}}$$ [2 point]

Notice that the answer does not depend on b.

A.7 NO. [1 point]

From the previous result,

$$R_{max} = Re^{\frac{v^2}{4G\mu}},$$

we see that it is **not possible** to escape the gravitational field since for any speed, there is always a maximum distance $R_{max} < \infty$

PART B:

B.1 The energy density is given by

$$\rho = aT^4$$ [1 point]

Equivalently:

$$\rho = \frac{4\sigma}{c}T^4 = \frac{\pi^2 k_B^4}{15\,\hbar^3 c^3}T^4$$ [1 point]

Note 3: This result arrives from the integration of the spectral energy density given by the Planck law over all frequencies. However, it is not required to show this integral.

B.2

$$r_0 = \frac{\hbar^{n_1} c^{n_2}}{k_B T}$$

- r_0 has dimensions of length: $[l]$
- $\hbar$ has dimensions of energy × time: $[e\ t]$
- c has dimensions of speed: $[l/t]$
- $k_B T$ has dimensions of energy: $[e]$

Then:

$$[l] = \frac{[e\,t]^{n_1}\left[\frac{l}{t}\right]^{n_2}}{[e]}$$

Since the LHS and the RHS should have the same dimensions, we get:

$$[l]: 1 = n_2$$
$$[e]: 0 = n_1 - 1$$
$$[t]: n_1 = n_2$$ [2 points]

The solution is

$$n_1 = 1$$ [1 point]
$$n_2 = 1$$ [1 point]

B.3 The energy of a piece of the string of length L is

$$E = \rho\pi r_0^2 L = Mc^2$$ [1 point]

where $M = \mu L$ is the mass of the piece. Solving for μ:

$$\mu = \frac{\rho\pi r_0^2}{c^2}$$ [1 point]

B.4 The weak field condition is

$$\frac{2G\mu}{c^2} \ll 1$$
$$2G\frac{\rho\pi r_0^2}{c^2} \ll 1$$ [1 point]
$$2G\frac{\pi}{c^2}\left(aT^4\right)\left(\frac{\hbar c}{k_B T}\right)^2 \ll 1$$ [1 point]
$$\frac{2Ga\hbar^2\pi}{c^2 k_B^2}\, T^2 \ll 1$$ [2 point]

Where we used: $\mu = \frac{\rho\pi r_0^2}{c^2}$, $\rho = aT^4$, $r_0 = \frac{\hbar c}{k_B T}$

$$\frac{2Ga\hbar^2\pi}{c^2 k_B^2}\, T^2 \ll 1$$
$$\frac{2G\hbar^2\pi}{c^2 k_B^2}\left(\frac{\pi^2 k_B^4}{15\,\hbar^3 c^3}\right) T^2 \ll 1$$
$$\frac{2\pi^3}{15}\left(\frac{Gk_B^2}{\hbar c^5}\right) T^2 \ll 1$$
$$\frac{2\pi^3}{15}\,\frac{T^2}{T_{Pl}^2} \ll 1$$ [2 point]

where $T_{Pl} = 1.416784\times10^{32} K$ (known as the Planck Temperature)

Note 4: the numerical factor $\frac{2\pi^3}{15}\sim 4.13$*, (a 5% error tolerance is accepted)*
Note 5: From **A.4** we get that

$$2G\mu = v^2$$

where v is the speed with which a particle would orbit a string. Thus the weak field is rewritten as

$$\frac{v^2}{c^2} \ll 1$$

which is equivalent to a non-relativistic condition.

B.5 The weak field condition is

$$4.13\left(\frac{T}{T_{Pl}}\right)^2 \ll 1$$

equivalently

$$\sim 2\frac{T}{T_{Pl}} \ll 1$$ [1 point]

i. $\frac{2T_{EW}}{T_{Pl}} \sim 2\times\frac{10^{15}}{10^{32}} \approx 1.4\times10^{-17} \ll 1$ [1 point]

ii. $\frac{2T_{GUT}}{T_{Pl}} \sim 2\times\frac{10^{29}}{10^{32}} \approx 1.4\times10^{-3} \ll 1$ [1 point]

Note 6: Using the following expression $4.13\left(\frac{T}{T_{Pl}}\right)^2 \ll 1$
One gets

i. $4.13\left(\frac{T_{EW}}{T_{Pl}}\right)^2 \sim 2.1\times10^{-34} \ll 1$

ii. $4.13\left(\frac{T_{GUT}}{T_{Pl}}\right)^2 \sim 2.1\times10^{-6} \ll 1$

And are also acceptable answers *(a 5% error tolerance is accepted)*

B.6

i. Yes [0.5 point]
ii. Yes [0.5 point]

PART C:

C.1 From the figure, the asymptotic condition for the observer to see a second image is that the light ray travelling from O directly to S should bend and travel along SE. This is the maximum angle of bending.

As all angles are small, we can safely use

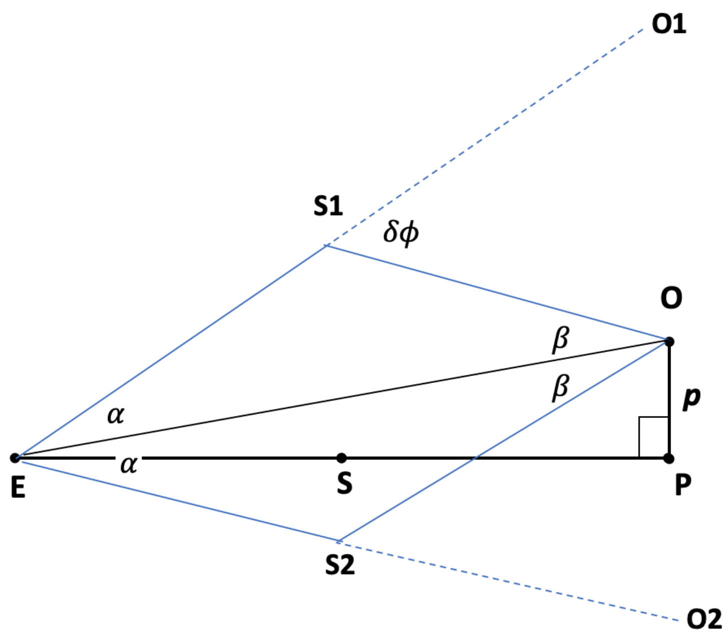
$$\sin x \approx \tan x \approx x$$

Now,

$$\frac{p}{SP} < \delta\phi \qquad \text{[2 points]}$$
$$< \frac{4\pi G\mu}{c^2}$$
$$< 2\pi\left(\frac{2G\mu}{c^2}\right)$$
$$< 2\pi\left(\frac{2\pi^3}{15}\right)\left(\frac{T^2}{T_{Pl}^2}\right) \qquad \text{[2 points]}$$

And $SP \approx \left(D_{OE} - D_{ES}\right)$

$$p < \frac{4\pi^4}{15T_{Pl}^2}T^2\left(D_{OE} - D_{ES}\right) \qquad \text{[2 points]}$$



C.2 In the figure, the blue lines represent the bending of two light rays corresponding to the images O1 and O2. Note that $D_{ES} \approx D_{ES1} \approx D_{ES2}$ and $D_{OS} \approx D_{OS1} \approx D_{OS2}$. Thus the angle $S1EO \approx S2EO \equiv \alpha$ and $S1OE \approx S2OE \equiv \beta$.2α is the angular separation we are looking for.

[2 points]

Further, notice

$$\alpha + \beta = \delta\phi$$ [1 point]

Using law of sines in the triangle **EOS1**

$$\frac{\alpha}{D_{S1O}} = \frac{\beta}{D_{ES1}}$$ [1 point]

we get

$$2\alpha = 2\delta\phi\left(\frac{D_{OE}-D_{ES}}{D_{OE}}\right)$$ [2 points]

C.3 If $D_{OE} = 2D_{ES}$,

$$2\alpha = \delta\phi = 2\pi\left(\frac{2\pi^3}{15}\right)\left(\frac{T^2_{GUT}}{T^2_{Pl}}\right)\approx 1.29\times 10^{-5}$$ [2 points]

and

$$\delta\phi = 1.22\frac{\lambda}{D}$$

Thus, using $\lambda \in [3\times 10^{-7} m, 8\times 10^{-7} m]$

$$D = 1.22\frac{\lambda}{\delta\phi} \in \left[3.75\times 10^{-2} m, 7.51\times 10^{-2} m\right]$$ [2 points]

Any answer within this interval is valid.

A remarkable result of this model is light deflection by a cosmic string, which leads to the possibility of detection through gravitational lensing. For instance, cosmic string trings moving across the line of sight with respect to an Earth-based observer will cause line-like discontinuities as shown in the figure below (taken from [1]).

Gravitational lensing by cosmic strings 1737

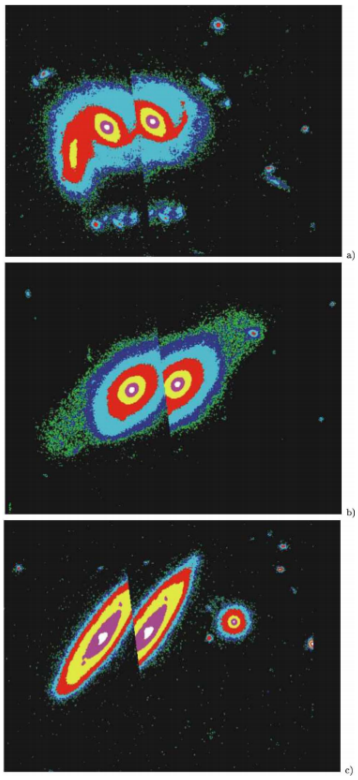


Figure 6. (a) This picture represents the lensed image of a spiral galaxy. The direction of a string is almost perpendicular to galaxy plane. Duplicated details are clearly visible. (b) In this case, the string is inclined with respect to the galaxy plane. As a result sharp edge appears. (c) Also in this case the direction of string is inclined with respect to galactic plane.

[1] M. V. Sazhin, O. S. Khovanskaya, M. Capaccioli, G. Longo, M. Paolillo, G. Covone, N. A. Grogin, E. J. Schreier, Gravitational lensing by cosmic strings: what we learn from the CSL-1 case, *Monthly Notices of the Royal Astronomical Society*, Volume 376, Issue 4, March 2007, Pages 1731–1739, https://doi.org/10.1111/j.1365-2966.2007.11543.x