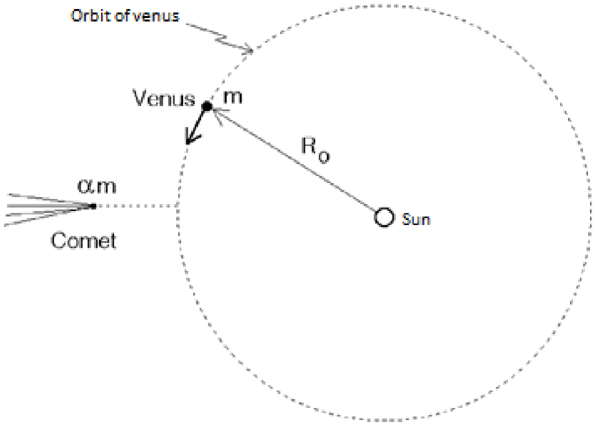
SOLUTION

TQ14 [35 points]



14.1 Let M be the mass of the Sun, then

$$\frac{GMm}{R_0^2} = \frac{mv_0^2}{R_0}$$

$$v_0^2 = \frac{GM}{R_0}$$ [1 point]

14.2 The mechanical energy of Venus (before collision) is:

$$E_i = -\frac{GMm}{R_0} + \frac{1}{2}mv_0^2 = -\frac{GMm}{2R_0}$$ [1 point]

14.3 Since the comet moves radially towards the Sun, it has no angular momentum (with respect to the origin in the Sun). Then, the angular momentum of Venus is the same as that of "Venus-2"

$$L = R_0 m v_0$$ [2 point]

This allows us to find the angular component of the velocity of Venus, v_θ, just after the collision. The angular momentum just after the collision is

$$L = R_0 m v_0 = R_0(m + \alpha m)v_\theta$$ [1 points]

Since the angular momentum is conserved it follows that

$$v_\theta = \frac{v_0}{1+\alpha}$$

The comet has zero energy, so

$$K = -U = +\frac{GM\alpha m}{R_0} = \frac{1}{2}\alpha m v_c^2$$ [2 points]

where v_c is the velocity of the comet just before the collision. It follows that

$$v_c^2 = \frac{2GM}{R_0} = 2v_0^2$$ [1 point]

The collision between the comet and Venus is inelastic, thus the total energy is not conserved. Also notice that the mass of "Venus-2" is $(1 + \alpha)m$. [2 point]

We now apply the conservation of linear momentum. Along the radial direction we get

$$\alpha m v_c = (m + \alpha m)v_r$$ [1 point]

where v_r is the velocity of "Venus-2" just after the collision. It follows that

$$v_r = \frac{\alpha}{1+\alpha}v_c = \frac{\sqrt{2}\,\alpha}{1+\alpha}v_0$$ [1 point]

14.4 Venus-2 mechanical energy (we evaluate it just after the collision) is:

$$E_f = U + K = -\frac{GMm(1+\alpha)}{R_0} + \frac{1}{2}m(1 + \alpha)\left(v_\theta^2 + v_r^2\right)$$ [4 points]

$$= -(1 + \alpha)\frac{GMm}{R_0} + \frac{1}{2}(1 + \alpha)\left[2\left(\frac{\alpha}{1+\alpha}\right)^2 + \frac{1}{(1+\alpha)^2}\right]\frac{GMm}{R_0}$$

$$= -\frac{GMm}{2R_0}\left(\frac{1+4\alpha}{1+\alpha}\right) = E_i\left(\frac{1+4\alpha}{1+\alpha}\right)$$ [1 point]

14.5 ¨Venus-2¨ orbit is no longer a circle. Since the energy is negative it must, therefore, be elliptical. It follows that [1 points]

$$E_f = -G\frac{M(1+\alpha)m}{2a_f}$$ [2 points]

where a_f is the semi-major axis of the orbit of Venus-2. Now, solving for a_f:

$$-G\frac{M(1+\alpha)m}{2a_f} = E_i\left(\frac{1+4\alpha}{1+\alpha}\right) = -G\frac{Mm}{2R_0}\left(\frac{1+4\alpha}{1+\alpha}\right)$$

$$\frac{(1+\alpha)}{a_f} = \frac{1}{R_0}\left(\frac{1+4\alpha}{1+\alpha}\right)$$

$$a_f = R_0\frac{(1+\alpha)^2}{1+4\alpha}$$ [2 points]

14.6 Using Kepler's third law, we have that

$$\frac{T_f}{T_0} = \left(\frac{a_f}{R_0}\right)^{3/2}$$ [2 points]

$$\frac{T_f}{T_0} = \left(\frac{(1+\alpha)^2}{1+4\alpha}\right)^{3/2} = \frac{(1+\alpha)^3}{(1+4\alpha)^{3/2}}$$ [1 points]

For $\alpha < 2$, the year is shorter, for $\alpha > 2$, the year is longer, while for $\alpha = 2$ the duration of the year does not change.

14.7 For the 'Venus-2' to collide with the Sun, the perihelion radius of the post-collision orbit should be $R_\odot$. [1 point]

$$r_p = R_\odot = 6.955\times10^8 m = \frac{6.955\times10^8}{0.723\times1.496\times10^{11}}R_0 = 0.00643R_0$$

Let $x = \frac{R_\odot}{R_0} = 0.00643$.

Now, we apply conservation of angular momentum right after the collision and at the moment "Venus-2" is at the perihelion:

$$L = mv_0R_0 = (1+\alpha)mv_pR_\odot$$

where v_p is the speed of Venus-2 at the perihelion. Solving for v_p

$$v_p = \frac{v_0}{x(1+\alpha_c)}$$

Applying conservation of energy between the two same instants we obtain:

$$E_f = -\frac{GM(1+\alpha_c)m}{R_\odot} + \frac{1}{2}m(1+\alpha_c)v_p^2$$

$$-\left(\frac{1+4\alpha_c}{1+\alpha_c}\right)\frac{GMm}{2R_0}=-\frac{GM\left(1+\alpha_c\right)m}{x\,R_0}+\frac{1}{2}m\left(1+\alpha_c\right)\left(\frac{v_0}{x\left(1+\alpha_c\right)}\right)^2$$

$$-\left(\frac{1+4\alpha_c}{1+\alpha_c}\right)\frac{GMm}{2R_0}=-\frac{\left(1+\alpha_c\right)}{x}\frac{GMm}{R_0}+\frac{1}{2}\left(1+\alpha_c\right)\left(\frac{1}{x(1+\alpha_c)}\right)^2\frac{GMm}{R_0}$$

$$-\frac{1}{2}\left(\frac{1+4\alpha_c}{1+\alpha_c}\right)=-\frac{\left(1+\alpha_c\right)}{x}+\frac{1}{2}\left(1+\alpha_c\right)\left(\frac{1}{x\left(1+\alpha_c\right)}\right)^2$$

$$\left(1+4\alpha_c\right)=\frac{2\left(1+\alpha_c\right)^2}{x}-\frac{1}{x^2}$$ [3 points]

$$2x\alpha_c^{\ 2}+4x(1-x)\alpha_c-(1-x)^2=0$$

$$\alpha_c=(1-x)\left(\sqrt{1+\frac{1}{2x}}-1\right)$$

$$\alpha_c=7.824$$ [1 point]

14.8 For post-collision orbit,

$$v_\theta=\frac{1}{1+\alpha_c}v_0$$

$$v_r=\frac{\sqrt{2}\alpha_c}{1+\alpha_c}v_0$$

$$v_f=\sqrt{v_r^2+v_\theta^2}$$ [1 point]

$$v_f=\frac{\sqrt{2\alpha_c^2+1}}{1+\alpha_c}v_0$$ [3 point]

Thus,

$$\delta v=v_0-v_f$$

$$=\left(1-\frac{\sqrt{2\alpha_c^2+1}}{1+\alpha_c}\right)v_0$$

$$=-0.259\,v_0$$

$$\frac{\delta v}{v_0} = 25.9\%$$ [0.5 points]

$$\tan \delta\theta = \left(\frac{v_r}{v_\theta}\right)$$

$$\tan \delta\theta = \left(\sqrt{2}\alpha_c\right)$$

$$\delta\theta = 1.480 \text{ rad}$$ [0.5 points]

$$\delta\theta = 84.84°$$