8th International Olympiad on Astronomy and Astrophysics
Suceava - Gura Humorului - August 2014

Content

Indications

1. The problems were elaborated concerning two aspects:
    a. To cover merely all the subjects from the syllabus;
    b. The average time for solving the items is about 15 minutes per a short problem;
2. In your folder you will find out the following:
    c. Answer sheets
    d. Draft sheets
    e. The envelope with the subjects in English and the translated version of them in your mother tongue;
3. The solutions of the problems will be written down only on the answer sheets you receive on your desk. **PLEASE WRITE ONLY ON THE PRINTED SIDE OF THE PAPER SHEET. DON'T USE THE REVERSE SIDE**. The evaluator will not take into account what is written on the reverse of the answer sheet.
4. The draft sheets is for your own use to try calculation, write some numbers etc. BEWARE: These papers are not taken into account in evaluation, at the end of the test they will be collected separately . Everything you consider as part of the solutions have to be written on the answer sheets.
5. Each problem have to be started on a new distinct answer sheet.
6. On each answer sheet please fill in the designated boxes as follows:
    a. In PROBLEM NO. box write down only the number of the problem: i.e. 1 – 15 for short problems, 16 – 19 for long problems. Each sheet containing the solutions of a certain problem, should have in the box the number of the problem;
    b. In Student ID – fill in your ID you will find on your envelope, consisted of 3 leters and 2 digits.
    c. In page no. box you will fill in the number of page, starting from 1. We advise you to fill this boxes after you finish the test
7. We don't understand your language, but the mathematic language is universal, so use as more relationships as you think that your solution will be better understand by the evaluator. If you want to explain in words we kindly ask you to use short English propositions.
8. Use the pen you find out on the desk. It is advisable to use a pencil for the sketches.
9. At the end of the test:
    a. Don't forget to put in order your papers;
    b. Put the answer sheets in the folder 1. Please verify that all the pages contain your ID, correct numbering of the problems and all pages are in the right order and numbered. This is an advantage of ease of understanding your solutions.
    c. Verify with the assistant the correct number of answer sheets used fill in this number on the cover of the folder and sign.
    d. Put the draft papers in the designated folder, Put the test papers back in the envelope.
    e. Go to swim

GOOD LUCK !

Problem 1. Lagrange Points

The *Lagrange* points are the five positions in an orbital configuration, where a small object is stationary relative to two big bodies, only gravitationally interacting with them. For example, an artificial satellite relative to Earth and Moon, or relative to Earth and Sun. In the **Figure 1** are sketched two possible orbits of Earth relative to Sun and of a small satellite relative to the Sun. Find out which of the two points L_3^1 and L_3^2 could be the real Lagrange point relative to the system Earth – Sun, and calculate its position relative to Sun. You know the following data: the Earth - Sun distance $d_{ES} = 15 \cdot 10^7$ km and the Earth – Sun mass ratio $M_E / M_S = 1/332946$

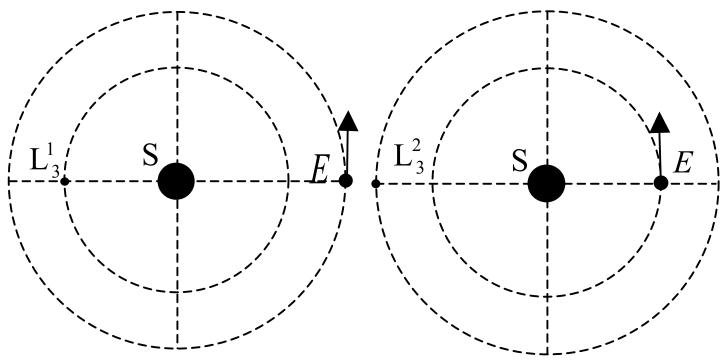


Figure 1

Problem 1. Marking scheme Lagrange Point

1.	Correct derivation of forces' equilibrium	6 points
2.	Correct identification of the Lagrange point	2 points
3.	Correct calculation of the position of Lagrange point	2 points
4.	Deduction for incorrect value	1 point

According to the notations in fig.1.1 and fig. 2.1

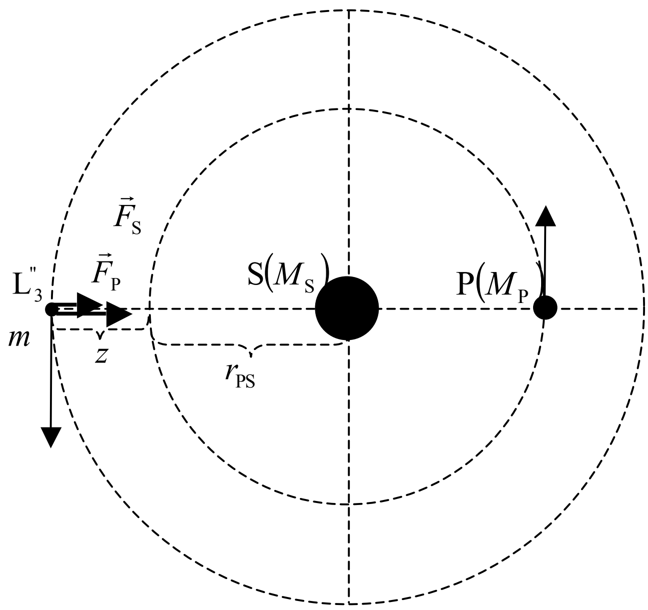


Figure 1a

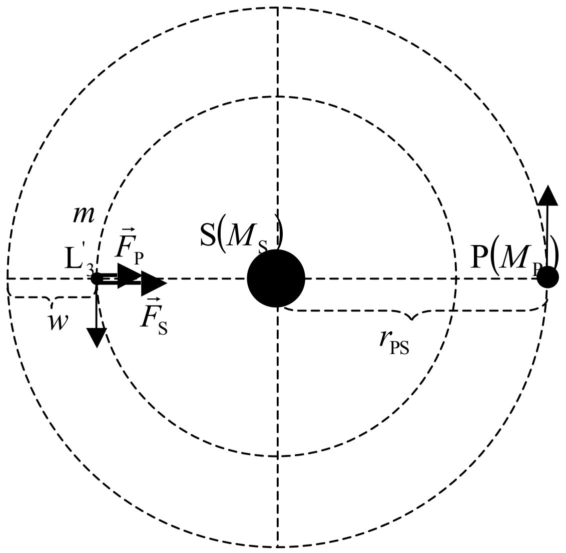


Figure 1 b

$$\vec{F}_S + \vec{F}_P = \vec{F}_{cp} = m\vec{a}_{cp}$$

$$F_S + F_P = ma_{cp} = m\omega^2(r_{PS} \pm w);$$ 2 points

The sign "+" for position $L_3^{'}$ and „– "for $L_3^{''}$

$$K\frac{mM_S}{(r_{PS} \pm w)^2} + K\frac{mM_P}{(2r_{PS} \pm w)^2} = m\omega^2(r_{PS} \pm w);$$

Using the assumption that $w << r_{PS}$

$$\left(1 \pm \frac{w}{r_{PS}}\right)^{-2} \approx 1 \mp 2\frac{w}{r_{PS}} \quad \left(1 \pm \frac{w}{2r_{PS}}\right)^{-2} \approx 1 \mp \frac{w}{r_{PS}};$$

2points p

The rotation speed

$$\omega^2 = \frac{KM_S}{r_{PS}^3}$$ 2p

The final relation

$$w = \mp \frac{M_P r_{PS}}{(12M_S + M_P)}$$

The value has to be positive, thus the $L_3^{''}$ is the position of one Lagrange point 2p

$w \approx 37{,}54\,Km$ 2p

Problem 2. Sun gravitational catastrophe!

In a gravitational catastrophe, the mass of the Sun mass decrease instantly to half of its actual value. If you consider that the actual Earth orbit is elliptical, its orbital period is $T_0 = 1\,year$ and the eccentricity of the Earth orbit is $e_0 = 0{,}0167$.

Find the period of the Earth`s orbital motion, after the gravitational catastrophe, if it occurs on: a) 3rd of July b) 3rd of January.

Problem 2. Marking scheme Sun gravitational catastrophe!

- Correct analyze of the initial conditions when the catastrophe occurs **(A)** **5 points**
- Correct calculations **(B)** **5 points**
  - Correct use of laws of conservation 2 points
  - Finding out that in the first case the orbit will be elliptic, relations (1)

and (2) 2 points
- Correct conduct of calculations 1 point

Detailed solution

(A) The orbit of Earth is elliptical, so the shape of the orbit after the solar catastrophe will depend on the moment when the decrees of the mass of the Sun will occur.

Initial analysis of the problem

a) In 3rd July the Earth is at the aphelion. The speed of the Earth is smaller than the speed of Earth on a circular orbit with radius $r_{0,max} = a_0(1+e_0)$.

b) In 3rd January the Earth is at perihelion. The speed of the Earth is bigger than the speed of Earth on a circular orbit with radius $r_{0,max} = a_0(1-e_0)$.

Conclusion (A) the period should be calculated only for situation a). The expected trajectory in this case is an elliptic one. The possibility that Earth hit the Sun is available too.

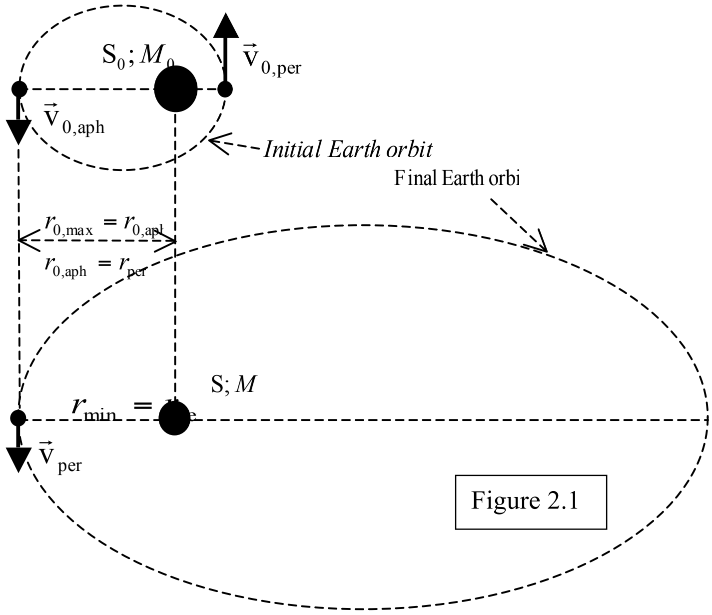


Figure 2.1

(B) Calculations:

In 3rd July the distance from Sun is maximum: fig. 2.1,

$r_{0,max} = a_0(1+e_0)$.

Before the catastrophe:

$v_{0,aph}$ - the speed of Earth on aphelion,

a_0 - big Earth's elliptical orbit semi axis

v_0 - the speed of Earth if its orbit is circular with radius $r_0 = a_0$

According to Keppler's second law and the law of energy conservation (see figure 2.1) the following relations can be written :

$v_{0,per}\, r_{0,per} = v_{0,aph}\, r_{0,aph}$;

$$\frac{v_{0,per}^2}{2} - K\frac{M_0}{r_{0,per}} = \frac{v_{0,aph}^2}{2} - K\frac{M_0}{r_{0,aph}}$$

$r_{0,min} = r_{0,per} = a_0(1-e_0)$

$r_{0,max} = r_{0,aph} = a_0(1+e_0)$

$KM_0 = v_0^2 r_0 = v_0^2 a_0$

$$v_0 = \sqrt{K\frac{M_0}{r_0}} = \sqrt{K\frac{M_0}{a_0}}$$

$$v_{0,per} = v_0\sqrt{\frac{1+e_0}{1-e_0}} > v_0$$

$v_{0,per} > v_0$ **(1)**

$$v_{0,aph} = v_0\sqrt{\frac{1-e_0}{1+e_0}}$$

$v_{0,aph} < v_0$ **(2)**

Conclusion – According to the relations (1) and (2) the new orbit of the Earth could be an elliptic one.

For the new elliptical Earth orbit:

$$r_{per} = r_{0,aph};$$
$$r_{min} = r_{per} = a(1-e);$$
$$a_0(1+e_0) = a(1-e);\ a = a_0\frac{1+e_0}{1-e};$$
$$v_{per} = v_{0,aph};$$
$$v_{per} = v\sqrt{\frac{1+e}{1-e}},$$

Where v este is the Earth's speed on a circular orbit with the radius $r = a$, when the mass of the Sun becomes $M = M_0/2;$

$$v\sqrt{\frac{1+e}{1-e}} = v_0\sqrt{\frac{1-e_0}{1+e_0}};$$
$$v = \sqrt{K\frac{M}{r}} = \sqrt{K\frac{M_0}{2a}} = \sqrt{K\frac{M_0}{a_0}}\sqrt{\frac{a_0}{2a}} = v_0\sqrt{\frac{a_0}{2a}};$$
$$e = 1-2e_0;\ a = a_0\frac{1+e_0}{2e_0}.$$

Conclusion

$$T_0 = \frac{2\pi r_0}{v_0} = \frac{2\pi a_0}{v_0}$$
$$T = \frac{2\pi r}{v} = \frac{2\pi a}{v};$$
$$\frac{T}{T_0} = \frac{a}{a_0}\frac{v_0}{v} = \frac{1+e_0}{2e_0}\sqrt{\frac{2a}{a_0}} = \frac{1+e_0}{2e_0}\sqrt{2}\sqrt{\frac{1+e_0}{2e_0}};$$
$$T = T_0\sqrt{2}\left(\frac{1+e_0}{2e_0}\right)^{3/2} \approx 230\text{ years}$$

b) In 3rd of January the Earth is at perihelion. In that moment the Erath speed is larger than the speed necessary for an Earth's circular orbit. Thus the trajectory of the Earth after the catastrophe will be an open trajectory, i.e. an hyperbolic or parabolic orbit.

Conclusion it is not necessary to calculate the period of revolution or could be issued as infinite

Problem 3. Cosmic radiation

During studies concerning cosmic radiation, a neutral unstable particle – the π^0 meson was identified. The rest-mass of meson π^0 is much larger than the rest-mass of the electron. The studies reveal that during its flight, the meson π^0 disintegrates into 2 photons.

Find an expression for the initial velocity of the meson π^0, if after its disintegration, one of the photons has the maximum possible energy $E_{\max}$ and, consequently, the other photon has the minimum possible energy $E_{\min}$. You may use as known **c -** the speed of light.

Problem 3. Marking scheme Cosmic radiation

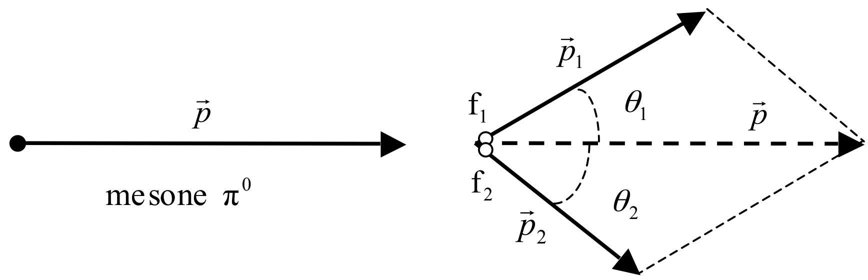
- Correct use of general laws of conservation **(A)**	**5 points**
- Correct applying of the laws of conservation for the conditions stated in the problem **(B)**	**4 points**
- Correct conduct of calculations and final solution (C)	**1 point**

Detailed solution

(A)

In the disintegration process the laws of energy conservation and the law of the conservation of momentum are both obeyed.

In the general case the law of conservation of the momentum is represented in the down below figure.



the total initial energy of the π^0 meson is

$$E^2 = p^2c^2 + m_0^2c^4$$

And its kinetic energy is

$$E_c = E - m_0c^2$$

The expressions of the 2 conservation laws written after the disintegration are:

$$\vec{p} = \vec{p}_1 + \vec{p}_2;$$

$$E + m_0c^2 = E_1 + E_2,$$

The energy of the photon 1 can be calculated using the notations in the figure

$$E = E_c + m_0c^2;$$

$$p_1 \sin\theta_1 = p_2 \sin\theta_2;$$

$$\frac{E_1}{c}\sin\theta_1 = \frac{E_2}{c}\sin\theta_2;$$

$$E^2 = p^2c^2 + m_0^2c^4;$$

$$p^2c^2 = E^2 - m_0^2c^4;$$

$$E = E_1 + E_2;$$

$$E_2 = E - E_1 = \left(E_c + m_0c^2\right) - E_1;$$

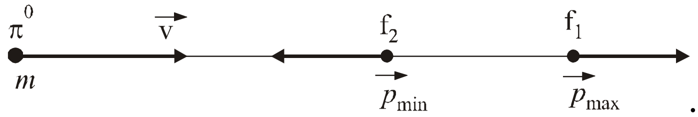
$$E_1 = \frac{m_0{}^2c^4}{2}\frac{1}{E_c + m_0c^2 - \cos\theta_1\sqrt{E_c\left(E_c + m_0c^2\right)}}.$$

Similar the second photon energy is:

$$E_2 = \frac{m_0{}^2c^4}{2}\frac{1}{E_c + m_0c^2 - \cos\theta_2\sqrt{E_c\left(E_c + m_0c^2\right)}}$$

(B)

If one of the photon has the maximum possible energy E_{max} and consequently the other photon has the minimum possible energy E_{min} the law of momentum conservation is sketched:



Thus the relations become very simple:

$$mv = \frac{m_0\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}$$

$$E_{max} = \frac{m_0c^2}{2\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}\left(1 + \frac{\mathbf{v}}{c}\right);$$

$$E_{min} = \frac{m_0c^2}{2\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}\left(1 - \frac{\mathbf{v}}{c}\right);$$

$$v = c\frac{E_{max} - E_{min}}{E_{max} + E_{min}}. \qquad \textbf{(C)}$$

Problem 4. Mass function of a visual binary stellar system

For a visual binary stellar system consisted of the stars σ_1 and σ_2, the following relation represents the mass function of the system:

$$f\left(M_1; M_2\right) = \frac{M_2^3 \sin^3 i}{\left(M_1 + M_2\right)^2},$$

where M_1 is the mass of star σ_1, M_2 is the mass of star σ_2 and i is the angle between the plane of the stars' orbits and a plane perpendicular on the direction of observation.

The recorded spectrum of radiations emitted by the star σ_1, during several months, reveals a sinusoidal variation of radiation wavelength, with the period T = 7 days and a shift factor $z = (\Delta\lambda)/\lambda = 0{,}001$.

a. Prove that the mass function of the system is:

$$f\left(M_1; M_2\right) = \frac{M_2^3 \sin^3 i}{\left(M_1 + M_2\right)^2} = \frac{T}{2\pi K}\left(\mathrm{v}_1 \cdot \sin i\right)^3,$$

Where: $v_1 \cdot \sin i$ is the maximum speed of star σ_1 relatively to the observer; $K -$ the gravitational constant , i is the angle between the plane of the orbits and the plane normal to the observation direction.
Assumptions: The orbits of the stars are circular,

b. Derive an expression for the mass function of the system. The following values are known: $c = 3 \times 10^8$ m/s; $K = 6{,}67 \times 10^{-11}$ Nm^{2}kg^{-2}.

Problem 5. The Astronaut saved by … ice from a can!

An astronaut, with mass $M = 100$ kg, get out of the space ship for a repairing mission. He has to repair a satellite standing still relatively to shuttle, at about $d = 90$ m distance away from the shuttle. After he finished his job he realizes that the systems designated to assure his come-back to shuttle were broken. He also observed that he has air only for 3 minutes. He also noticed that he possessed a hermetically closed cylindrical can (base section $S = 30$ cm^2) firmly attached to its glove, with $m = 200$ g of ice inside. The ice did not completely fill the can.

Determine if the astronaut is able to arrive safely to the shuttle, before his air reserve is empty. Briefly explain your calculations. Note that he cannot throw away anything of its equipment, or touch the satellite.

You may use the following data: $T = 272$ K / the temperature of the ice in the can, $p_s = 550$ Pa - the pressure of the saturated water vapors at the temperature $T = 272$ K; $R = 8300$ J/(kmol·K) - the constant of perfect gas; $\mu = 18$ kg/kmol - the molar mass of the water.

Problem 5. Marking scheme The Astronaut saved by … ice from a can!

- **A.** For the use with an adequate justify of one of the relationships (4) — 3 points
- **B. Reasoning** The student describe correctly the processes before and after the can is opened. — 4points
- **C.** Calculations according to the reasoning, and/or as support for reasoning — 2 points
- **D.** Correct result — 1 point

Detailed solution

Theoretical considerations:

The incident particle flux on a wall (i.e. a certain direction on a surface) is:

$$\Phi = m_0 \cdot \Omega = \frac{1}{6} \cdot m_0 \cdot n \cdot S \cdot \bar{v} \qquad (3)$$

$$m_0 \cdot n = m_0 \cdot \frac{N}{a^3} = \frac{m_0 N}{a^3} = \frac{m}{V} = \rho,$$

where: m_0 is the mass of one molecule; $m -$ mass of the gas in the cube ; $V -$ volume of the cube ; $\rho -$ the density of the gas ;

$$\rho = \frac{\mu p}{RT},$$

where p – the pressure of the gas in the cube; The relation (3) – the mass flux relation becomes:

A $$\Phi = \frac{1}{6}\cdot\rho\cdot S\cdot\overline{v} = \frac{1}{6}\cdot\frac{\mu p}{RT}\cdot S\cdot\sqrt{\frac{3RT}{\mu}} = \frac{1}{6}\cdot p\cdot S\cdot\sqrt{\frac{3\mu}{RT}}. \qquad (4)$$

$$\Phi = \frac{1}{6}\cdot\rho\cdot S\cdot\overline{v} = \frac{1}{6}\cdot\frac{\mu p}{RT}\cdot S\cdot\sqrt{\frac{3RT}{\mu}} = \frac{1}{6}\cdot p\cdot S\cdot\sqrt{\frac{3\mu}{RT}}. \qquad (5)$$ 3points

B. **Reasoning**

Because the cylindrical can is not full of ice, in the empty part of it there are saturated vapors, i.e the mass flux of the molecule which sublimate is equal with mass flux of gass which transform into ice. Thus the pressure in the can is the saturated vapor pressure p_s and it has the corresponding maximum density ρ_s See figure 6.2

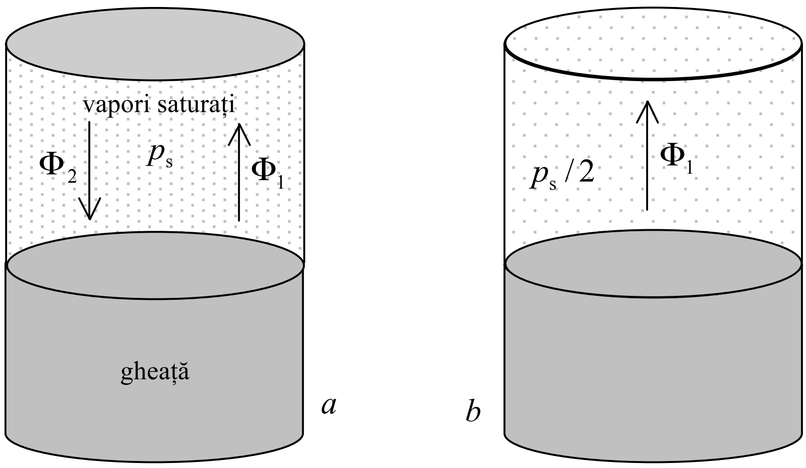


Fig. 6.2

$$\Phi_1 = \Phi_{sublimation} = \frac{1}{6}\cdot\rho_s\cdot S\cdot\overline{v} = \frac{1}{6}\cdot p_s\cdot S\cdot\sqrt{\frac{3\mu}{RT}};$$
$$\Phi_2 = \Phi_{solidification} = \frac{1}{6}\cdot\rho_s\cdot S\cdot\overline{v} = \frac{1}{6}\cdot p_s\cdot S\cdot\sqrt{\frac{3\mu}{RT}};$$
$$\Phi_1 = \Phi_2.$$

After the can was opened, there be no molecules which sublimate thus the mass flux of the molecules which gather the ice become null. So the pressure becomes $(p_s/2)$.

Thus the force acting on the astronaut will be

C. **Calculations according to the reasoning**, and/or as support for reasoning 2 points

$$F = \frac{p_s}{2}\cdot S,$$

Opening the can the astronaut will be accelearated with:

$$a = \frac{F}{M} = \frac{p_s\cdot S}{2M} = \frac{550\ \mathrm{Nm^{-2}}\cdot 30\cdot 10^{-4}\ \mathrm{m^2}}{2\cdot 10^2\ \mathrm{kg}} = 0{,}00825\ \mathrm{ms^{-2}}.$$

The total time of the acceleration movement will be the total time of ice sublimation:

$$\tau = \frac{m}{\Phi_1} = \frac{m}{\frac{1}{6}\cdot p_s \cdot S \cdot \sqrt{\frac{3\mu}{RT}}} = \frac{6m}{p_s \cdot S}\cdot\sqrt{\frac{RT}{3\mu}} \approx 150\,\mathrm{s}.$$

D. Correct result

The travel distance in this time will be :

$$L = \frac{a\tau^2}{2} = \frac{0{,}00825\,\mathrm{ms^{-2}}\cdot 225\cdot 10^2\,\mathrm{s^2}}{2} \approx 93\,\mathrm{m},$$

The astronaut could arrive safely in at the shuttle if he didn't lose to much time by solving the problem.

Problem 6. The life –time of a star from the main sequence

The plot of the function $\log(L/L_S) = f(\log(M/M_S))$ for data collected from a large number of stars is represented in figure 3. The symbols represents: L and M the luminosity respectively the mass and of a star and L_S and respectively M_S the luminosity and the respectively the mass of the Sun.

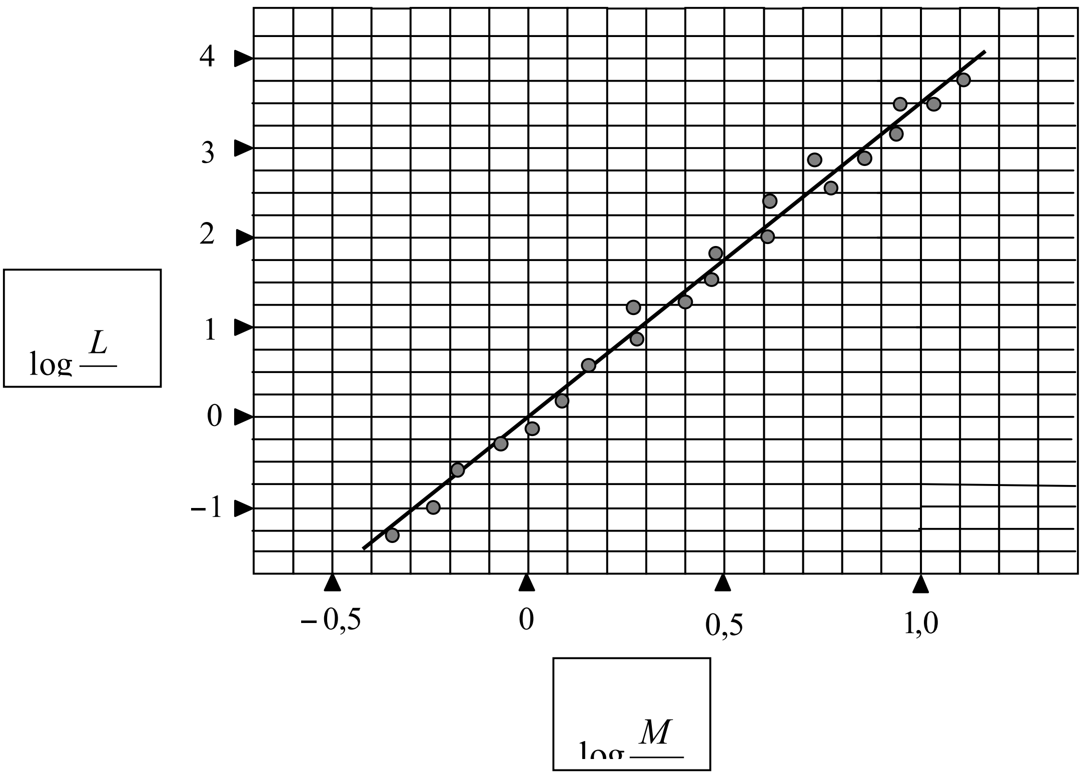


Figure 6

Find an expression for the life- time for each star in the Main Sequence from Hertzprung – Rossell diagram if the time spent by Sun in the same Main Sequence is τ_S. Consider the following assumptions: for each star the percentage of its mass which changed into energy is η, the percent of the mass of Sun which changes into energy is η_S, the mass of each star is $M = nM_S$ and the luminosity of each star remains constant, during its entire life time.

Problem 6. Marking scheme The life –time of a star from the main sequence

- **A.** The analysis of the graph — 6 points
    - Obtaining the formula (1) from the linearity of the graph
    - Correct use of the luminosity formula (2) for finding out the final formula — 4 points

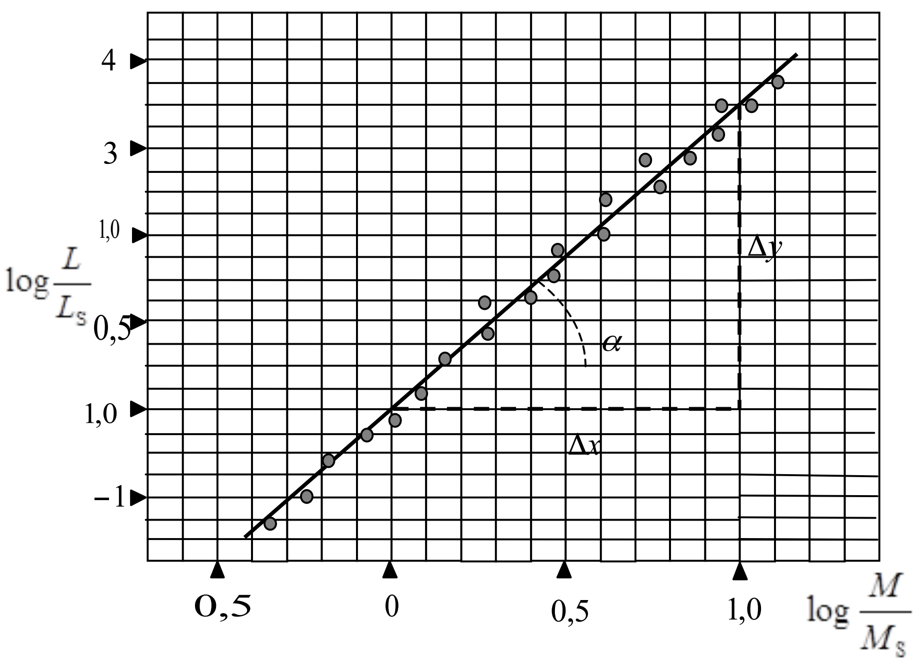
Detailed solution

A. The analysis of the graph :

The graph is linear:

$y = ax + b = ax;$

From the graph it can be obtain the following data:



$$\log \frac{L}{L_S} = a \cdot \log \frac{M}{M_S};$$

$$a = \tan\alpha = \frac{\Delta y}{\Delta x} = \frac{3{,}5}{1} = 3{,}5;$$

$L \sim M^{3,5}$. **(1)**

The total energy of the star is:

$E = Mc^2,$

So the emitted energy due to the mass variation of the star is:

$\Delta E = c^2 \Delta M,$

According to the text

$\Delta M = \eta M;$

$\Delta E = c^2 \eta M.$

By using the definition of the luminosity :

$\frac{\Delta E}{\Delta t} = L;$ **(2)**

$\Delta t = \tau;$

$\frac{c^2 \eta M}{\tau} = L;$

$\tau = \dfrac{c^2 \eta M}{L}$, **(2)**

Which represents the life-time of the star.

By using the results from the graph analysis

$L = \dfrac{L_S}{M_S^{3,5}} \cdot M^{3,5}$,

Thus :

$\tau = \dfrac{c^2 \eta M_S^{3,5}}{L_S} \cdot M^{-2,5}$.

If use the same calculations for the Sun it can be obtain

$E_S = M_S c^2$;

$\tau_S = \dfrac{c^2 \eta_S M_S}{L_S}$,

Which is the life-time of the Sun

$\tau = \dfrac{\eta}{\eta_S} \cdot \tau_S \cdot M_S^{2,5} \cdot M^{-2,5}$;

$\dfrac{\tau}{\tau_S} = \dfrac{\eta}{\eta_S} \cdot \left(\dfrac{M}{M_S}\right)^{-2,5}$; $M = nM_S$;

$\tau = \dfrac{\eta}{\eta_S} (n)^{-2,5} \tau_S$.

Problem 7. The effective temperature on the surface of a star

A star emits radiation with wavelength values in a narrow range $\Delta\lambda << \lambda$, i.e. the wavelength have values between λ and λ and $\lambda + \Delta\lambda$. According to Planck's relationship (for an absolute black body), the following relation define, the energy emitted by star in the unit of time, through the unit of area of its surface, per length-unit of the wavelength range:

$$r = \frac{2\pi hc^2}{\lambda^5 \left(e^{hc/k\lambda c} - 1\right)}.$$

The spectral intensities of two radiations with wavelengths λ_1 and respectively λ_2, both in the range $\Delta\lambda$ measured on Earth are $I_1(\lambda_1)$ and, respectively $I_2(\lambda_2)$.

a. Establish the equation which, in a general case, allows determining the effective temperature on the surface of the star using only spectral measurements.
b. Find out the approximate value of the effective temperature on the star surface if $hc >> \lambda kT$.
c. Find out the relation between wavelength λ_1 and λ_2, if $I_1(\lambda_1) = 2I_2(\lambda_2)$, when $hc << \lambda kT$.

You know: h – Planck's constant; k – Boltzmann's constant; c – speed of light in vacuum.

Problem 7. Marking scheme The effective temperature on the surface of a star

a. Identifying the expression of spectral intensity and correct use of the given relation for obtaining the relation **(1)** 4 points

b. correct use of the assumption $hc >> \lambda kT$ and find out relation (2) 3 points

c. correct use of the assumption $hc << \lambda kT$. and find out relation (3) 3 points

Detailed solution

a. We start from the definition of r:

$$r = \frac{\Delta E}{\Delta t \cdot S_{stea} \cdot \Delta\lambda} = \frac{\Delta E}{\Delta t \cdot 4\pi R^2 \cdot \Delta\lambda}$$ where R is the radius of the star

$$r = \frac{2\pi hc^2}{\lambda^5\left(e^{hc/k\lambda T} - 1\right)}$$

Considering **d** as the distance from the star to the Earth, the definition- relation of the spectral intensity can be written as follows:

$$I(\lambda) = \frac{\frac{\Delta E}{\Delta t \cdot \Delta\lambda}}{4\pi d^2}$$

$$I(\lambda) = \frac{2\pi hc^2 R^2}{d^2\lambda^5\left(e^{hc/\lambda kT} - 1\right)};$$

Particularly for each wavelength:

$$I_1(\lambda_1) = \frac{2\pi hc^2 R^2}{d^2\lambda_1^5\left(e^{hc/\lambda_1 kT} - 1\right)};\quad I_2(\lambda_2) = \frac{2\pi hc^2 R^2}{d^2\lambda_2^5\left(e^{hc/\lambda_2 kT} - 1\right)};$$

The ratio of the 2 above relations

$$\frac{I_1(\lambda_1)}{I_2(\lambda_2)} = \left(\frac{\lambda_2}{\lambda_1}\right)^5 \cdot \frac{e^{hc/\lambda_2 kT} - 1}{e^{hc/\lambda_1 kT} - 1} \qquad (1)$$

Represents an equation which allow to find out the temperature of star's surface **T by using spectral measurements**

b. If we consider that

$hc >> \lambda kT$, then:

$e^{hc/\lambda_1 kT} - 1 \approx e^{hc/\lambda_1 kT}$ and $e^{hc/\lambda_2 kT} - 1 \approx e^{hc/\lambda_2 kT}$

The relation (1) becomes

$$\frac{I_1(\lambda_1)}{I_2(\lambda_2)} = \left(\frac{\lambda_2}{\lambda_1}\right)^5 \cdot \frac{e^{hc/\lambda_2 kT}}{e^{hc/\lambda_1 kT}} = \left(\frac{\lambda_2}{\lambda_1}\right)^5 \cdot e^{hc/\lambda_2 kT - hc/\lambda_1 kT}$$

$$\frac{I_1(\lambda_1)}{I_2(\lambda_2)}\cdot\left(\frac{\lambda_1}{\lambda_2}\right)^5 = e^{hc/\lambda_2 kT - hc/\lambda_1 kT}$$

$$T = \frac{hc(\lambda_1 - \lambda_2)}{k\lambda_1\lambda_2 \cdot \ln\left[\frac{I_1(\lambda_1)}{I_2(\lambda_2)}\cdot\left(\frac{\lambda_1}{\lambda_2}\right)^5\right]}$$

c. If $hc << \lambda kT$, then:

$$\frac{hc}{k\lambda_1 T} << 1;\ e^{hc/\lambda_1 kT} - 1 \approx 1 + \frac{hc}{k\lambda_1 T} - 1 = \frac{hc}{k\lambda_1 T}$$

$$\frac{hc}{k\lambda_2 T} << 1;\ e^{hc/\lambda_2 kT} - 1 \approx 1 + \frac{hc}{k\lambda_2 T} - 1 = \frac{hc}{k\lambda_2 T}$$

The relation (1) becomes:

$$\frac{I_1(\lambda_1)}{I_2(\lambda_2)} = \left(\frac{\lambda_2}{\lambda_1}\right)^5 \cdot \frac{\frac{hc}{k\lambda_2 T}}{\frac{hc}{k\lambda_2 T}} = \left(\frac{\lambda_2}{\lambda_1}\right)^4$$

$$I_1 = 2I_2;\ \left(\frac{\lambda_2}{\lambda_1}\right)^4 = 2;\ \lambda_2 = \lambda_1 \cdot \sqrt[4]{2} \approx 1{,}2 \cdot \lambda_1.$$

Problem 8. Gradient temperatures

The spectra of two stars with different temperatures T_1 and respectively T_2 were compared. In the spectrum of each star, two very close spectral lines corresponding to the wavelength with values λ_1 and respectively λ_2 were found. For each line of this spectral lines, the difference between the corresponding visual apparent magnitudes of the stars are $\Delta m_{\lambda_1} = m_{1,\lambda_1} - m_{2,\lambda_1}$ and $\Delta m_{\lambda_2} = m_{1,\lambda_2} - m_{2,\lambda_2}$. m_{1,λ_1} is the apparent magnitude of the star 1 for the wavelength λ_1, m_{1,λ_2} is the apparent magnitude of the star 1 for the wavelength λ_2, m_{2,λ_1} is the apparent magnitude of the star 2 for the wavelength λ_1, m_{2,λ_2} is the apparent magnitude of the star 2 for the wavelength λ_2 .

Determine the temperature T_1 of one of the two stars, if the temperature T_2 of the other star is already known, by using the Plank expression of black body radiation:

$$r(\lambda)=\frac{2\pi hc^2}{\lambda^5}\left(e^{hc/k\lambda c}-1\right)^{-1},$$

where: h – Planck's constant; k – Boltzmann's constant; c – speed of light in vacuum. You will consider that $hc >> k\lambda T$.

Problem 8. Marking scheme . Gradient temperatures

- Finding out the relations (1) and (2) by the correct using of the approximation $hc >> k\lambda T$. — 4 points
- Correct using of the Pogson's formula — 3 points
- For correct conduct of calculations and obtaining the final formula (5) — 3points

Detailed solution

By using the Plank expression the spectral intensities for the two wavelengths in the doublet emitted by the star number 1 are:

$$r_1(\lambda_1)=\frac{2\pi hc^2}{\lambda_1^5\left(e^{hc/\lambda_1 kT_1}-1\right)};\ r_1(\lambda_2)=\frac{2\pi hc^2}{\lambda_2^5\left(e^{hc/\lambda_2 kT_1}-1\right)};$$

And by considering $hc >> k\lambda T$;

$$r_1(\lambda_1)=\frac{2\pi hc^2}{\lambda_1^5}e^{-hc/\lambda_1 kT_1};\ r_1(\lambda_2)=\frac{2\pi hc^2}{\lambda_2^5}e^{-hc/\lambda_2 kT_1}. \qquad (1)$$

Respectively, the spectral intensities for the two wavelengths in the doublet emitted by the star number 2 are:

$$r_2(\lambda_1)=\frac{2\pi hc^2}{\lambda_1^5\left(e^{hc/\lambda_1 kT_2}-1\right)};\ r_2(\lambda_2)=\frac{2\pi hc^2}{\lambda_2^5\left(e^{hc/\lambda_2 kT_2}-1\right)};$$

And by considering $hc >> k\lambda T$;

$$r_2(\lambda_1)=\frac{2\pi hc^2}{\lambda_1^5}e^{-hc/\lambda_1 kT_2};\ r_2(\lambda_2)=\frac{2\pi hc^2}{\lambda_2^5}e^{-hc/\lambda_2 kT2}. \qquad (2)$$

Using the Pogson formula for star 1 and 2 and for λ_1, result:

$$\log\frac{r_1(\lambda_1)}{r_2(\lambda_1)}=-0,4\left(m_{1,\lambda_1}-m_{2,\lambda_1}\right)=-0,4\cdot\Delta m_{\lambda_1};$$

$$\log e^{hc/\lambda_1 k(1/T_2-1/T_1)}=-0,4\cdot\Delta m_{\lambda_1} \qquad (3)$$

Similar the Pogson formula for λ_2,:

$$\log\frac{r_1(\lambda_2)}{r_2(\lambda_2)}=-0,4\left(m_{1,\lambda_2}-m_{2,\lambda_2}\right)=-0,4\cdot\Delta m_{\lambda_2}$$

$$\log e^{hc/\lambda_2 k(1/T_2-1/T_1)}=-0,4\cdot\Delta m_{\lambda_2} \qquad (4)$$

Using the relations (3) and (4)

$$\log e^{hc/\lambda_2 k(1/T_2-1/T_1)}-\log e^{hc/\lambda_1 k(1/T_2-1/T_1)}=-0,4\cdot\Delta m_{\lambda_2}+0,4\cdot\Delta m_{\lambda_1};$$

$$\log e^{hc/\lambda_2 k(1/T_2-1/T_1)} + \log e^{-hc/\lambda_1 k(1/T_2-1/T_1)} = -0{,}4\cdot\Delta m_{\lambda_2} + 0{,}4\cdot\Delta m_{\lambda_1};$$

$$\left[\frac{hc}{k}\left(\frac{1}{T_2}-\frac{1}{T_1}\right)\cdot\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)\right]\cdot\log e = 0{,}4\left(\Delta m_{\lambda_1}-\Delta m_{\lambda_2}\right);$$

Because $\log e \approx 0{,}43;\ \frac{0{,}4}{\log e}\approx 0{,}93;$

$$\frac{1}{T_2}-\frac{1}{T_1} = -0{,}93\cdot\frac{\left(\Delta m_{\lambda_1}-\Delta m_{\lambda_2}\right)}{\frac{hc}{k}\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)} = -0{,}93\cdot\frac{k\lambda_1\lambda_2\left(\Delta m_{\lambda_1}-\Delta m_{\lambda_2}\right)}{hc\left(\lambda_1-\lambda_2\right)};$$

$$T_1 = T_2\cdot\frac{hc\left(\lambda_1-\lambda_2\right)}{hc\left(\lambda_1-\lambda_2\right)+0{,}93\cdot k\lambda_1\lambda_2 T_2\left(\Delta m_{\lambda_1}-\Delta m_{\lambda_2}\right)}. \quad \textbf{(5)}$$

Problem 9. Pressure of light

One particle of star dust is in static equilibrium at a certain distance from Sun. Assuming that the particle is spherical and its density is ρ, calculate the diameter of the particle.

The following assumption may be useful for solving the problem:

The pressure of electromagnetic radiation is equal with the volume density of the electromagnetic radiations

Problem 9. Marking scheme . Pressure of light

- Correct use of the formula (1) for the pressure of light — 3 points
- Correct identify of the equilibrium condition — 3 points
- Correct solution and reasoning — 4 points

The pressure of the emitted radiation is

$$p_{\text{rad}} = w = \frac{\phi_{planet,D}}{c}$$

$$p_{rad} = \frac{\sigma T^4_{planet} R^2_{planet}}{cD^2} \quad (1)$$

As seen in the below image, the pressure due to the solar radiation is effectively acting on an equivalent plane disc with the diameter *d* of the spherical star dust particle

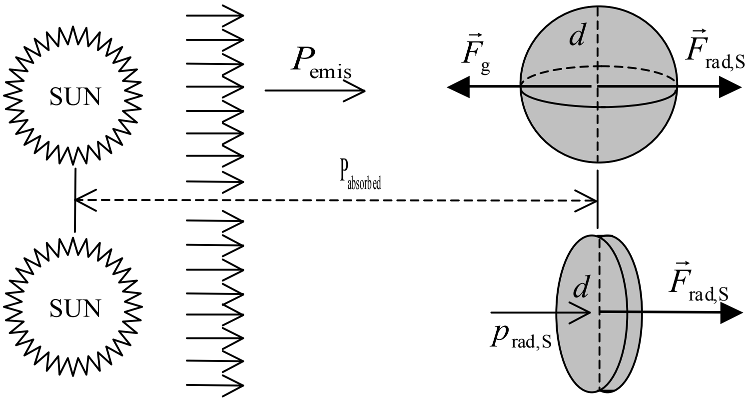


Fig.12 b

Thus the force acting by the Sun radiation on the star-dust particle is:

$$F_{rad,S} = p_{rad,S} \cdot \pi r^2 = p_{rad,S} \cdot \frac{\pi d^2}{4} = \frac{\sigma T_S^4 R_S^2}{cL_S^2} \cdot \frac{\pi d^2}{4},$$

The equilibrium condition is

$$F_{rad,S} = F_g,$$

Where F_g is the gravitational attraction force between Sun and the star-dust particle.

$$\frac{\sigma T_S^4 R_S^2}{cD_S^2} \cdot \frac{\pi d^2}{4} = K\frac{mM_S}{D_S^2};$$

$$m = \rho V = \rho \frac{4\pi r^3}{3} = \rho \frac{4\pi}{3}\frac{d^3}{8} = \rho \frac{\pi d^3}{6};$$

$$\frac{\sigma T_S^4 R_S^2}{cD_S^2} \cdot \frac{\pi d^2}{4} = K\rho \frac{\pi d^3}{6}\frac{M_S}{D_S^2};$$

$$d = \frac{3}{2} \cdot \frac{\sigma}{\rho K} \cdot \frac{T_S^4 R_S^2}{M_S}.$$

Problem 10. The density of the star

In a very simple model, a star is assumed to be a sphere of gas in a state of equilibrium in its on gravitational field. The stellar gas is consisted of plasma, i.e. hydrogen and helium atoms, completely ionized. Find an expression for the value of the mass of the star if you know: r – radius of the star; T – the temperature of the star; n – the relative proportion of hydrogen in the mass of the star; μ_H – molar mass of the hydrogen; μ_{He} – molar mass of the helium; R – universal gas constant; K – gravitation constant. You may use the formula of the pressure of radiation inside the star $p_{rad} = \frac{1}{3}aT^4$, where a is a known constant. The rotation of the star is negligible.

Problem 10. Marking scheme The density of the star

- Correct reasoning for finding the expression of the inner stellar gas (2) 2 points
- Correct expression for the equilibrium condition 2 puncte
- The correct conduct of calculations, and obtain the final correct result 2 puncte
- Correct solution and reasoning 4 points

Detailed solution

The hydrostatic equilibrium inside the star means that in each point of the inside of the star the gravitational forces are compensated by the hydrostatic pressure forces. That means that the mater of the star remains localized in a region of space.

The total pressure of the stellar gas has two components: the pressure due to the movement of the stellar –gas particles (p_{gaz}) and the pressure due to the emitted radiation by the stellar-gas particles (p_{rad}), thus:

$$p_{\text{total}} = p_{\text{gaz}} + p_{\text{rad}};$$

$$p_{\text{gaz}} = p_{\text{H}} + p_{\text{He}};$$

$$p_{\text{rad}} = \frac{1}{3}aT;$$

$$p_{\text{total}} = \rho\frac{n\mu_{\text{He}} + (1-n)\mu_{\text{H}}}{\mu_{\text{H}}\mu_{\text{He}}}RT + \frac{1}{3}aT. \textbf{(1)}$$

In order to calculate the gravitational pressure of the stellar /gas let's consider a narrow cylinder with section area ΔS ,along the radius of the star . See the figure bellow. If the total gravitational force acting on this cylinder is $\vec{F}_g$ than the gravitational pressure exert by the gas –column is $p_{\text{grav}} = F_g / \Delta S$

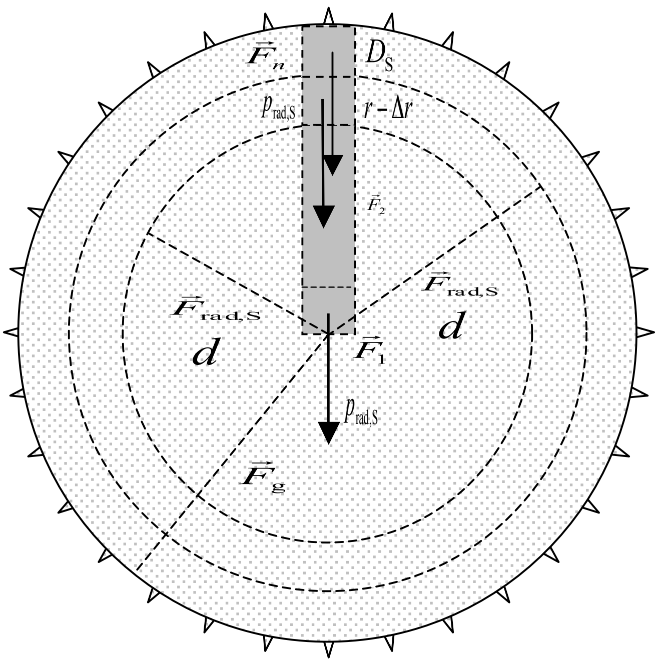


Fig. 15 a

In order to calculate $\vec{F}_g$ let divide the cylinder in n identically small cylinders each of it with height Δr and mass Δm Considering an homogenous star :

$$F_1 = K\frac{\Delta m \cdot M_1}{\left(r - \frac{\Delta r}{2}\right)^2} = K\frac{\Delta m \cdot M_1}{r^2\left(1 - \frac{\Delta r}{2r}\right)^2} = K\frac{\Delta m \cdot M_1}{r^2}\left(1 - \frac{\Delta r}{2r}\right)^{-2};$$

$$\frac{M}{r^3} = \frac{M_1}{(r - \Delta r)^3};\ M_1 = M \cdot \frac{(r-\Delta r)^3}{r^3} = M \cdot \left(1 - \frac{\Delta r}{r}\right)^3;$$

$$F_1 \approx K\frac{\Delta m \cdot M}{r^2}\left(1 - 2\frac{\Delta r}{r}\right);$$

$$F_2 \approx K\frac{\Delta m \cdot M}{r^2}\left(1 - 3\frac{\Delta r}{r}\right);$$

………………………………………………..

$$F_n \approx K\frac{\Delta m \cdot M}{r^2}\left(1 - (n+1)\frac{\Delta r}{r}\right);$$

$$F_g = F_1 + F_2 + \ldots\ldots + F_n;$$

$$F_g = K\frac{m \cdot M}{nr^2}\left[n - \frac{1}{n}\frac{(1+n)n}{2}\right];\ F_g = K\frac{m \cdot M}{r^2}\left[1 - \frac{(1+n)}{2n}\right];$$

$$F_g = K\frac{m \cdot M}{r^2}\frac{n-1}{2n};\ \frac{n-1}{2n} \approx \frac{1}{2};$$

$$F_g = K\frac{m \cdot M}{2r^2}. (2)$$

Thus the gravitational pressure is

$$p_g = \frac{F_g}{\Delta S} = \frac{1}{\Delta S}\frac{\rho \cdot r \cdot \Delta S \cdot \rho \cdot \frac{4\pi r^3}{3}}{2r^2}$$

$$p_g = \frac{2}{3}\pi K r^2 \rho^2$$

Using the relations (1) and (2) in the pressures equilibrium relationship

$$p_{\text{total}} = p_{\text{gravitational}}$$

Results :

$$\frac{2}{3}\pi K r^2 \rho^2 = \rho\frac{n\mu_{He} + (1-n)\mu_H}{\mu_H \mu_{He}}RT + \frac{1}{3}aT;$$

$$\frac{2}{3}\pi K r^2 \cdot \rho^2 - \frac{n\mu_{He} + (1-n)\mu_H}{\mu_H \mu_{He}}RT \cdot \rho - \frac{1}{3}aT = 0;$$

This is an second degree equation in ρ

$$A\rho^2 - B\rho - C = 0$$

Where the coefficients are

$$A = \frac{2}{3}\pi K r^2$$

$$B = \frac{n\mu_{\mathrm{He}} + (1-n)\mu_{\mathrm{H}}}{\mu_{\mathrm{H}}\mu_{\mathrm{He}}} RT$$

$$C = \frac{1}{3} aT$$

The positive solution is the valid one i.e.

$$\rho = \frac{B + \sqrt{B^2 + 4AC}}{2A} \quad \rho = \frac{\frac{n\mu_{\mathrm{He}} + (1-n)\mu_{\mathrm{H}}}{\mu_{\mathrm{H}}\mu_{\mathrm{He}}} RT + \sqrt{\left(\frac{n\mu_{\mathrm{He}} + (1-n)\mu_{\mathrm{H}}}{\mu_{\mathrm{H}}\mu_{\mathrm{He}}} RT\right)^2 + \frac{8}{9}\pi K r^2 aT}}{\frac{4}{3}\pi K r^2}$$

Problem 11. Space – ship orbiting the Sun

A spherical space –ship orbits the Sun on a circular orbit, and spin around one of its axes. The temperature on the exterior surface of the ship is T_N. Find out the apparent magnitude of the Sun and the angular diameter of the Sun as seen by the astronaut on board of the space – ship. The following values are known:, T_S - the effective temperature of the Sun; R_S - the radius of the Sun; d_0 - the Earth –Sun distance; m_0 - apparent magnitude of Sun measured from Earth; R_N - the radius of the space –ship.

Problem 11. Marking scheme Space – ship orbiting the Sun

- Correct use of the formulas (1) for the apparent brightness — 3 points
- Correct use of the formula (2) by using Pogson formula — 3 points
- Correct solution and reasoning — 4 points

Detailed solution

According to the Stefan – Boltzmann law, the luminosity of the Sun is:

$$L_{sun} = Q_{sun} \cdot 4\pi R_{Sun}^2 = \sigma T_{Sun}^4 \cdot 4\pi R_{Sun}^2, \quad (1)$$

At distance d from the Sun , where the space ship is the energy which passes the unit of surface in an unit of time is:

$$\phi_{Sun,d} = \frac{L_S}{4\pi d^2} = \frac{\sigma T_S^4 \cdot 4\pi R_S^2}{4\pi d^2}. \quad (2)$$

The space ship receive through its entire surface, in the unit of time, the energy:

$$P_{received} = \frac{\sigma T_{Sun}^4 \cdot 4\pi R_{Sun}^2}{4\pi d^2} \cdot \pi R_{ship}^2.$$

Corresponding to its temperature, T_N, according the Stefan - Boltzmann law, the emitted energy by starship through its hole surface in the unit of time :

$$P_{emis,N} = \sigma T_N^4 \cdot 4\pi R_N^2.$$

When the temperature stabilized at thermic equilibrium :

$$P_{received,\mathrm{N}} = P_{\mathrm{emis,N}}$$

$$\frac{\sigma T_{\mathrm{S}}^4 \cdot 4\pi R_{\mathrm{S}}^2}{4\pi d^2} \cdot \pi R_{\mathrm{N}}^2 = \sigma T_{\mathrm{N}}^4 \cdot 4\pi R_{\mathrm{N}}^2$$

the distance of orbiting the Sun of the space ship is:

$$d = \frac{T_{\mathrm{S}}^2 R_{\mathrm{S}}}{2T_{\mathrm{N}}^2},$$

The angular diameter of the Sun as seen from the space ship :

$$\alpha / 2 = \frac{R_{\mathrm{S}}}{d}$$

$$\alpha = \frac{2R_{\mathrm{S}}}{d} = 4\left(\frac{T_{\mathrm{N}}}{T_{\mathrm{S}}}\right)^2$$

According the Pogson formula written for Sun seen from Earth and space ship the following relation occurs:

$$\lg \frac{E_{\mathrm{S,Nava}}}{E_{\mathrm{S,P}}} = -0{,}4(m - m_0) \qquad (2)$$

$$2 \cdot \lg\left(\frac{d_0}{d}\right) = -0{,}4(m - m_0)$$

The apparent magnitude of the Sun as seen from the space ship

$$m = m_0 - 5^{\mathrm{m}} \cdot \lg \frac{2d_0 T_{\mathrm{N}}^2}{R_{\mathrm{S}} T_{\mathrm{S}}^2}$$

Problem 12. The Vega star in the mirror

Inside a photo camera a plane mirror is placed along the optical axis of the lens of the objective (as seen in figure 13). The length of the mirror is half of the focal distance of the lens of the objective. The photo camera is oriented as on the photographic plate situated in the focal plane of the photo camera are captured two images with different illuminations of the Vega star. Find out the difference between the apparent photographical magnitudes of the two images of the Vega stars.

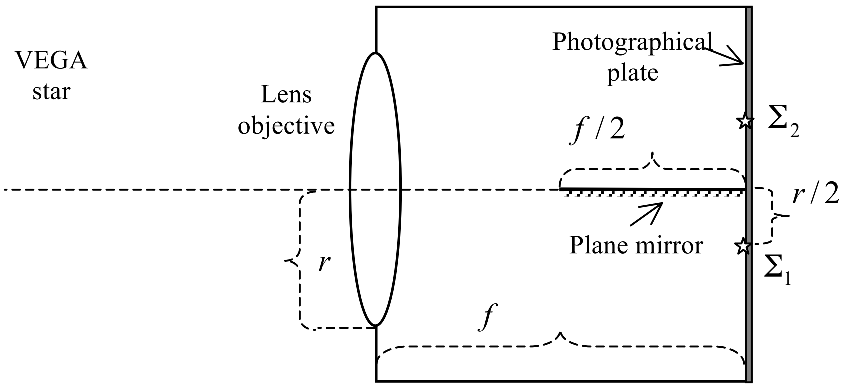


Figure 12

Problem 12. Marking scheme The Vega star in the mirror

The light beam arriving from **Vega Star** can be considered paraxial, due to the distance from it to the observer on Earth. The explanation for the existence of two distinct images of the star is that the optical axis of the objective is not parallel with the light beam from the star.

The images on the camera plate are symmetrical placed relative to the principal optical axis.

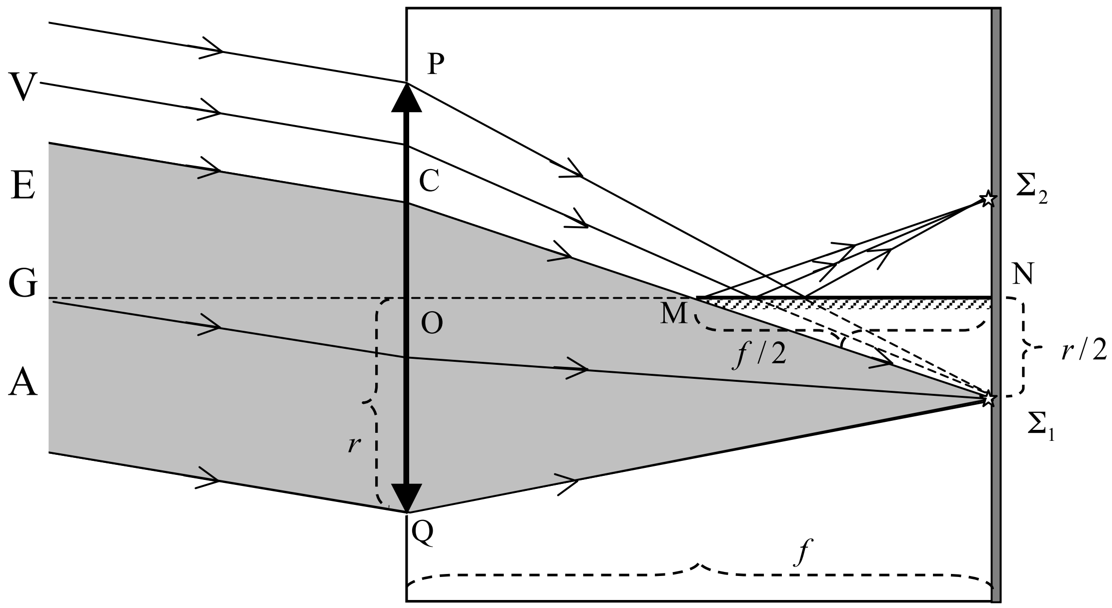


Fig. 12

Each of the point images of the Vega Star Σ_1 and Σ_2 didn't concentrate the same light fluxes. In the down below figure it can be seen the sections of the lens which correspond to each image. The sector APBC is passed by the light which concentrates in the image Σ_2 and the light passing the sector ACBQ concentrates into the point image Σ_1 See the picture in figure 13 .

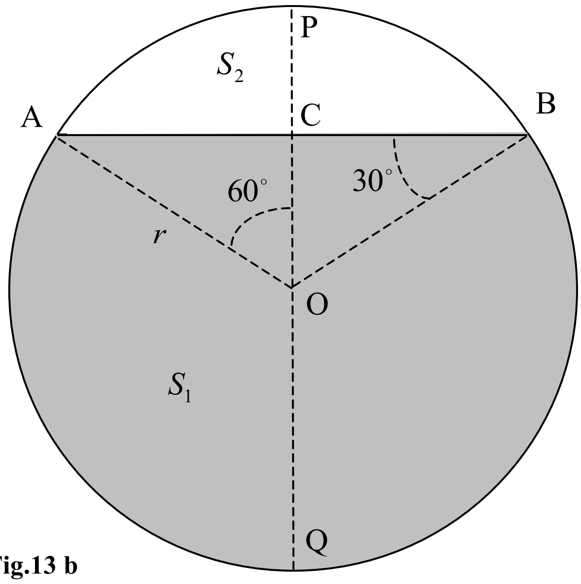


Fig.13 b

The ratio between the light fluxes concentrated into the two image points will directly depend on the ratio of the two sectors areas.

From the geometry of the figure 2 results :

$$\mathrm{MN} = \mathrm{OM};\ \mathrm{N}\Sigma_1 = \mathrm{OC} = \frac{r}{2};$$

$$\angle(\mathrm{CBO}) = 30^\circ;\ \angle(\mathrm{BOC}) = 60^\circ;\ \angle(\mathrm{AOB}) = 120^\circ;$$

$$\frac{S_1}{S_2} = \frac{8\pi + 3\sqrt{3}}{4\pi - 3\sqrt{3}} \approx 4.$$

Using the Pogson formula :

$$\log\frac{E_1}{E_2} = \log\frac{\dfrac{\sigma T_{\mathrm{V}}^4 \cdot 4\pi R_{\mathrm{V}}^2}{4\pi d_{\mathrm{PV}}^2}\cdot S_1}{\dfrac{\sigma T_{\mathrm{V}}^4 \cdot 4\pi R_{\mathrm{V}}^2}{4\pi d_{\mathrm{PV}}^2}\cdot S_2} = -0{,}4(m_1 - m_2);$$

$$\log\frac{S_1}{S_2} = -0{,}4(m_1 - m_2);$$

$$m_2 - m_1 = 1{,}5^{\mathrm{m}}.$$

Problem 13. Stars with Romanian names

Two Romanian astronomers Ovidiu Tercu and Alex Dumitriu from The Astronomical Observatory of the Museum Complex of Natural Sciences in Galati Romania, recently discovered – in September 2013- two variable stars. They used for that a telescope with the main mirror diameter of 40 cm and a SBIG STL-6303e CCD camera.

With the accord of the **AAVSO** (**A**merican **A**ssociation of **V**ariable **S**tars **O**bservers), the two stars have now Romanian names: Galati V1 and respectively Galati V2. The two stars are circumpolar, located in Cassiopeia and respectively in Andromeda constellation. The two stars are visible above the horizon, form the territory of Romania, all over the year. The galactic coordinates of the two stars are: $\text{Galati V1}\left(G_1 = 114.371^\circ; g_1 = -11.35^\circ\right)$ and $\text{Galati V2}\left(G_2 = 113.266^\circ; g_2 = -16.177^\circ\right)$.

Another star, discovered by the Romanian astronomer Nicolas Sanduleak, has also a Romanian name – Sanduleak -69° 202; it explodes as the supernova SN 1987. This star was localized in the Dorado constellation from the Large Magellan Cloud, by the coordinates:

$$\alpha = 5^{h}35^{min}28{,}03^{s};\ \delta = -69^\circ 16' 11{,}79'';\ G = 279{,}7^\circ;\ g = -31{,}9^\circ.$$

Estimate the angular distance between the stars Galati V1 and Galati V2

Problem 13. Marking scheme Stars with Romanian names

1. Correct geometrical calculations 5 points
2. Correct calculations 5 points

In the figure bellow the two stars σ_1 and σ_2, are located using the galactic coordintates $\left(G_1; g_1\right)$ and respectively $\left(G_2; g_2\right)$ on the geocentric celestial sphere. The spherical triangles $\sigma_1 A \sigma_2$ $\left(G_1; g_1\right)$ and respectively may be considered rectangular plane triangles because the angles $\Delta G = G_2 - G_1$ $\left(G_1; g_1\right)$ and respectively $\Delta g = g_2 - g_1$ are very small

Thus:

$$\sigma_1\sigma_2 = \sqrt{\left(\sigma_1 A\right)^2 + \left(\sigma_2 A\right)^2},$$

or:

$$\sigma_1\sigma_2 = \sqrt{\left(\sigma_1 B\right)^2 + \left(\sigma_2 B\right)^2},$$

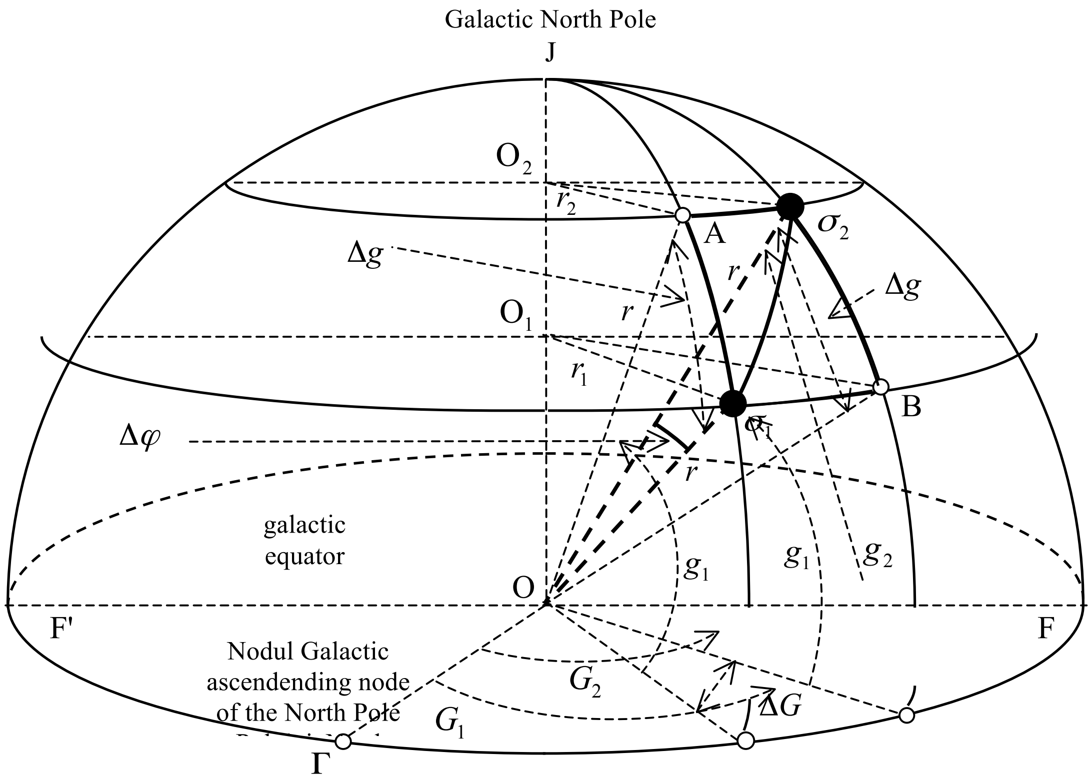


Fig.

$$\sigma_1 \mathrm{A} = r \cdot \Delta g;$$

$$\sigma_2 \mathrm{A} = r_2 \cdot \Delta G = r \cdot \cos g_2 \cdot \Delta G;$$

$$\sigma_1 \sigma_2 = r \cdot \Delta\varphi,$$

Where $\Delta\varphi$ is the angular distance between two stars

$$r \cdot \Delta\varphi = \sqrt{(r \cdot \Delta g)^2 + (r \cdot \cos g_2 \cdot \Delta G)^2};$$

$$\Delta\varphi = \sqrt{(\Delta g)^2 + (\cos g_2 \cdot \Delta G)^2};$$

$$\sigma_1 \mathrm{B} = r_1 \cdot \Delta G = r \cdot \cos g_1 \cdot \Delta G;$$

$$\Delta\varphi = \sqrt{(\cos g_1 \cdot \Delta G)^2 + (\Delta g)^2};$$

$$(G_1 = 114.371^\circ; g_1 = -11.35^\circ); (G_2 = 113.266^\circ; g_2 = -16.177^\circ);$$

$$\Delta G = G_2 - G_1 = -1{,}105^\circ; \Delta g = g_2 - g_1 = -4{,}827^\circ;$$

$$\cos g_1 = 0{,}98; \cos g_2 = 0{,}96;$$

$$\Delta\varphi = \sqrt{(-4{,}827^\circ)^2 + (0{,}96)^2 \cdot (-1{,}105^\circ)^2} \approx 4{,}942^\circ;$$

$$\Delta\varphi = \sqrt{(0{,}98)^2 \cdot (-1{,}105^\circ)^2 + (-4{,}827^\circ)^2} \approx 4{,}946^\circ,$$

The angular distance between Galați V1 and Galați V2.

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Problem 14. Apparent magnitude of the Moon

You know that the absolute magnitude of the Moon is $M_L = 0{,}25^m$. Calculate the values of the apparent magnitudes of the Moon corresponding to the following Moon –phases : full-moon and the first quarter. You know: the Moon – Earth distance - $d_{LP} = 385000\,\text{km}$, the Earth – Sun distance - $d_{PS} = 1\,\text{AU}$, the Moon –Sun distance, $d_{LS} = 1\,\text{AU}$

Problem 14. Marking scheme Apparent magnitude of the Moon

1.	General analysis of the problem	6 points
2.	The analysis of the 2 particular situations	4 points

The apparent magnitude of a planet from the Solar System depends on the phase angle $M = M(\Psi)$.
The apparent magnitude of the body is given by the relation:

$$m = M + 2{,}5 \cdot \log \frac{d_{C,S}^2 \cdot d_{C,O}^2}{d_0^4 \cdot p(\Psi)},$$

unde: $d_{B,S}$ – the distance between the body and the Sun; $d_{B,O}$ – distance between the body and observer; $d_0 = 1\,\text{AU}$; Ψ – the phase angle ; $p(\Psi)$ – the phase function :

$$p(\Psi) = \frac{2}{3} \cdot \left[\left(1 - \frac{\Psi}{\pi}\right)\cos\Psi + \frac{1}{\pi}\sin\Psi\right],$$

Ψ as seen in the figure bellow is given by the cosine law.

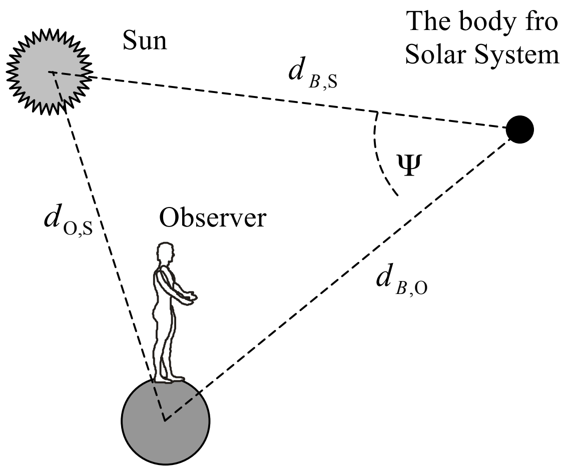
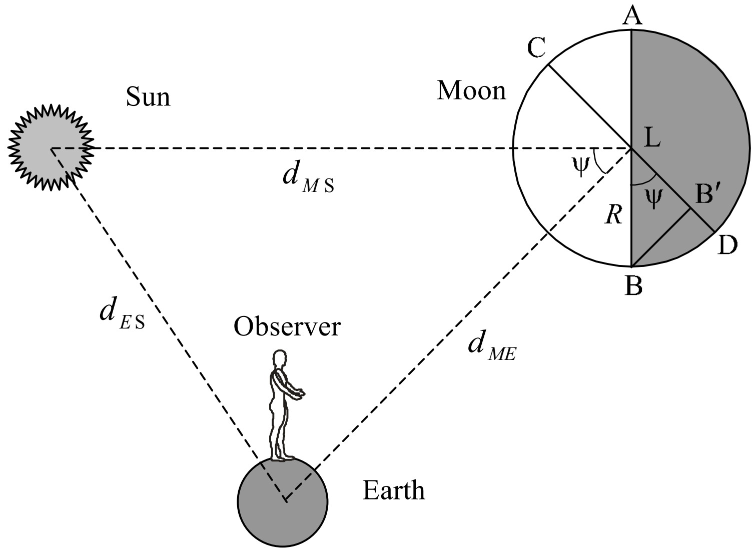


Fig.

$$\cos\Psi = \frac{d_{BO}^2 + d_{BS}^2 - d_{OS}^2}{2d_{BO} \cdot d_{BS}}.$$

In particularaly for the Moon



$$\cos\Psi = \frac{d_{ME}^2 + d_{MS}^2 - d_{ES}^2}{2d_{ME}\cdot d_{MS}};$$

$$p(\Psi) = \frac{2}{3}\cdot\left[\left(1-\frac{\Psi}{\pi}\right)\cos\Psi + \frac{1}{\pi}\sin\Psi\right];$$

$$m_M = M_M + 2{,}5\cdot\log\frac{d_{M,S}^2\cdot d_{ME}^2}{d_0^4\cdot p(\Psi)}.$$

Particular cases:

1) Full moon

$$\Psi = 0;$$
$$\cos\Psi = 1;\ \sin\Psi = 0;$$
$$p(\Psi) = \frac{2}{3};$$
$$d_{MS} = 1\,\text{AU};\ d_{ME} = 385000\,\text{km} \approx 0{,}00256\,\text{AU} = 256\cdot 10^{-5}\,\text{SU};\ d_0 = 1\,\text{S}U;$$

$$m_M = M_M - 12{,}5^{\text{m}} = 0{,}25^{\text{m}} - 12{,}5^{\text{m}} = -12{,}25^{\text{m}}.$$

2) First Quarter

$$\Psi = 90^\circ;$$
$$\cos\Psi = 0;\ \sin\Psi = 1;$$
$$p(\Psi) = \frac{2}{3\pi} \approx 0{,}2;$$

$$\frac{d_{M,S}^2 \cdot d_{ME}^2}{d_0^4 \cdot p(\Psi)} = \frac{65536 \cdot 10^{-10}}{0,2} = 491520 \cdot 10^{-10};$$

$$m_M = M_M - 10,75^{\text{m}} = 0,25^{\text{m}} - 10,75^{\text{m}} = -10,5^{\text{m}}.$$

Problem 15. Absolute magnitude of a cepheide

The cepheides are cariable stars, whom luminosities and luminosities varies due to volume oscillations. The period of the oscilations of a cepheide star is:

$$P = 2\pi R\sqrt{\frac{R}{KM}},$$

where: R – the radius of the cepheide; M – the mass of the cepheid (constant during oscillation);

$$R = R(t); \ P = P(t).$$

Demonstrate that the absolute magnitude of the cepheide M_{cef}, depend on the period of cepheide's pulsation P according the following relation:

$$M_{\text{cef}} = -2,5^{\text{m}} \cdot \log k - \left(\frac{10}{3}\right)^{\text{m}} \cdot \log P,$$

where k *is* constant; $P = P(t); \ M_{\text{cef}} = M_{\text{cef}}(t).$

Problem 15. Marking scheme Apparent magnitude of the Moon

$$P = 2\pi R\sqrt{\frac{R}{KM}},$$

rezults

$$P^2 = \frac{4\pi^2 R^3}{KM}; \ R = \sqrt[3]{\frac{KMP^2}{4\pi^2}} = \left(\frac{KM}{4\pi^2}\right)^{1/3} \cdot P^{2/3};$$

$$R^2 = \left(\frac{KM}{4\pi^2}\right)^{2/3} \cdot P^{4/3}.$$

The absolute brightness is:

$$L_{\text{cef}} = \sigma T_{\text{cef}}^4 \cdot 4\pi R^2,$$

And the apparent brightness :

$$E_{\text{cef}} = \frac{L_{\text{cef}}}{4\pi d_{\text{P,cef}}^2} = \frac{\sigma T_{\text{cef}}^4 \cdot 4\pi R^2}{4\pi d_{\text{P,cef}}^2},$$

$d_{\text{P,cef}}$ is the distance between the observer on Erath and the cepheide

$$E_{\text{cef}} = \frac{\sigma T_{\text{cef}}^4 \cdot 4\pi \cdot \left(\frac{KM}{4\pi^2}\right)^{2/3} \cdot P^{4/3}}{4\pi d_{\text{P,cef}}^2}.$$

Similarly for Sun

$$E_S = \frac{L_S}{4\pi d_{PS}^2} = \frac{\sigma T_S^4 \cdot 4\pi R_S^2}{4\pi d_{PS}^2}.$$

By using the Pogson formula:

$$\log\frac{E_{cef}}{E_S} = -0{,}4(m_{cef} - m_S);$$

$$M_{cef} = M_S - 5^m \log\frac{|d_{P,cef}|}{|d_{PS}|} - 2{,}5 \cdot \log\frac{E_{cef}}{E_S};$$

$$\frac{T_{cef}^4 \cdot \left(\frac{KM}{4\pi^2}\right)^{3/2} \cdot d_{PS}^2}{T_S^4 \cdot R_S^2 \cdot d_{P,cef}^2} = k_1 = \text{constant};$$

$$M_{cef} = M_S - 5^m \log\frac{|d_{P,cef}|}{|d_{PS}|} - 2{,}5 \cdot \log k_1 - \frac{10}{3} \cdot \log P;$$

$$M_S - 5^m \log\frac{|d_{P,cef}|}{|d_{PS}|} - 2{,}5 \cdot \log k_1 = -2{,}5 \cdot \log k;$$

$$k = \text{constant};$$

$$M_{cef} = -2{,}5 \cdot \log k - \frac{10}{3} \cdot \log P.$$

8th International Olympiad on Astronomy and Astrophysics
Suceava - Gura Humorului - August 2014

Content

Indications

1. The problems were elaborated concerning two aspects:
   a. *To cover merely all the subjects from the syllabus;*
   b. *The average time for solving the items is about 15 minutes per a short problem;*
2. In your folder you will find out the following:
   a. *Answer sheets*
   b. *Draft sheets*
   c. *The envelope with the subjects in English and the translated version of them in your mother tongue;*
3. The solutions of the problems will be written down only on the answer sheets you receive on your desk. **PLEASE WRITE ONLY ON THE PRINTED SIDE OF THE PAPER SHEET. DON'T USE THE REVERSE SIDE**. The evaluator will not take into account what is written on the reverse of the answer sheet.
4. The draft sheets is for your own use to try calculation, write some numbers etc. BEWARE: These papers are not taken into account in evaluation, at the end of the test they will be collected separately . Everything you consider as part of the solutions have to be written on the answer sheets.
5. Each problem have to be started on a new distinct answer sheet.
6. On each answer sheet please fill in the designated boxes as follows:
   a. *In **PROBLEM NO**. box write down only the number of the problem: i.e. **1** to **15** for short problems, **16** to **19** for long problems. Each sheet containing the solutions of a certain problem, should have in the box the number of the problem;*
   b. *In **Student ID** – fill in your ID you will find on your envelope, consisted of 3 leters and 2 digits.*
   c. *In page no. box you will fill in the number of page, starting from 1. We advise you to fill this boxes after you finish the test*
7. We don't understand your language, but the mathematic language is universal, so use as more relationships as you think that your solution will be better understand by the evaluator. If you want to explain in words we kindly ask you to use short English propositions.
8. Use the pen you find out on the desk. It is advisable to use a pencil for the sketches.
9. At the end of the test:
   a. *Don't forget to put in order your papers;*
   b. *Put the answer sheets in the folder 1. Please verify that all the pages contain your ID, correct numbering of the problems and all pages are in the right order and numbered. This is an advantage of ease of understanding your solutions.*
   c. *Verify with the assistant the correct number of answer sheets used fill in this number on the cover of the folder and sign.*
   d. *Put the draft papers in the designated folder, Put the test papers back in the envelope.*
   e. *Go to swim*

GOOD LUCK !

16. Long problem 1. Eagles on the Caraiman Cross !

In the Bucegi mountains, part of the Carpathian mountains, after the end of the First World War an iron cross was built by the former King of Romania called Ferdinand the I-st and his wife Queen Maria. The cross is an unique monument in Europe. The monument is an impressive iron cross called „The Heros' Cross" which in 2013 entered in the Guiness Book as the cross build on the highest altitude mountain peek.

The cross was built on the plane plateau situated on the top of the peek called Caraiman, at the altitude $H = 2300\,m$ from sea level. Its height, including the base-support is $h = 39{,}3\,m$. The horizontal arms of the cross are oriented on the N-S direction. The latitude of the Cross is $\varphi = 45^{\circ}$.

A. In the evening of 21st of March 2014, the summer equinox day, two eagles stop from their flight, first near the monument, and the second, on the top of the Cross as seen in figure 1. The two eagles are on the same

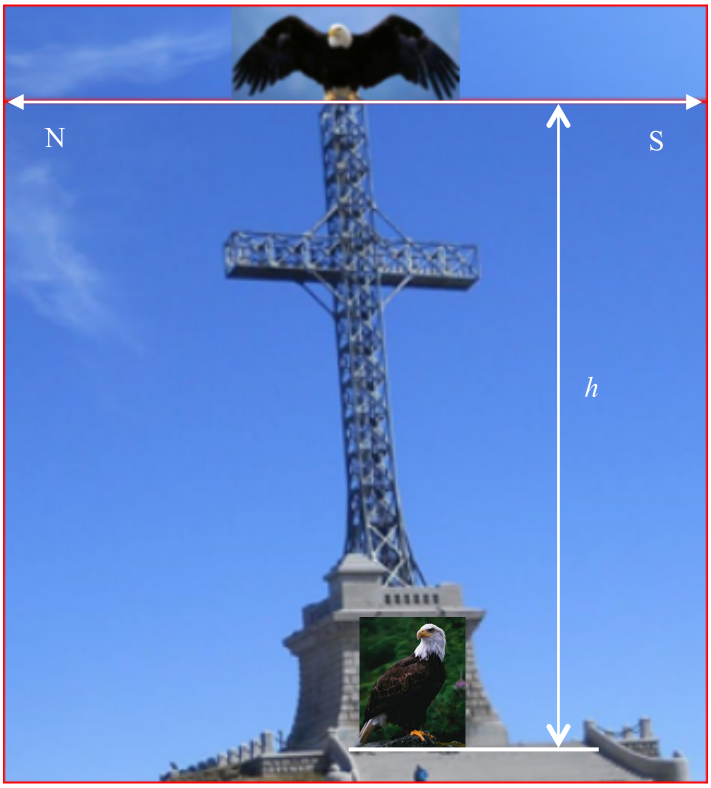


Figure 1

vertical direction. The sky was very clear, so the eagles could see the horizon and observe the Sun set. Each eagle start to fly right in the moment that each of it observes that the Sun completely disappears.

In the same time, an astronomer located at the sea-level, at the base of the Bucegi Mountains. Assume that he is on the same vertical with the two eagles.

Assuming negligible the atmospheric refraction, solve the following questions:

1) *Calculate* the duration of the sunset, measured by the astronomer.

2) *Calculate* the durations of sunset measured by each of the two eagles and indicate which of the eagles leaves first the Cross. What is the time interval between the leaving moments of the two eagles.

The following information is necessary:

The duration of the sunset measurement starts when the solar disc is tangent to the horizon line and stops when the solar disc completely disappears.

The Earth's rotation period is $T_E = 24\,\text{h}$, the radius of the Sun $R_S = 6{,}96 \cdot 10^5\ \text{km}$, Earth – Sun distance $d_{ES} = 15 \cdot 10^7\ \text{km}$, the local latitude of the Heroes Cross $\varphi = 45^\circ$.

B) At a certain moment of the next day, 22nd March 2014, the two eagles come back to the Heroes Cross. One of the eagles lands on the top of the vertical pillar of the Cross and the other one land on the horizontal plateau, just in the end point of the shadow of the vertical pillar of the Cross.

1) Calculate the distance between the two eagles, if this distance has the minimum possible value.

2) Calculate the length of the horizontal arms of the Cross l_b, if the shadow on the plateau of one of the arm of the cross has the length $u_b = 7\ \text{m}$

C) At midnight, the astronomer visit the cross and, from the top of it, he identifies a bright star at the limit of the circumpolarity. He named this star „Eagles Star". Knowing that due to the atmospheric refraction the horizon lowering is $\xi = 34'$, calculate:

1) The "Eagles star" declination;

2) The "Eagles star" maximum height above the horizon.

Long problem 1. Marking scheme - Eagles on the Caraiman Cross

1)	**10**
2)	**10**
B1)	10
B2)	10
C1)	5
C2)	5

A. 1 The following notations are used: D_S the diameter of the Sun, d_{ES} Earth-Sun distance, θ angular diameter of the Sun as seen from the Earth:

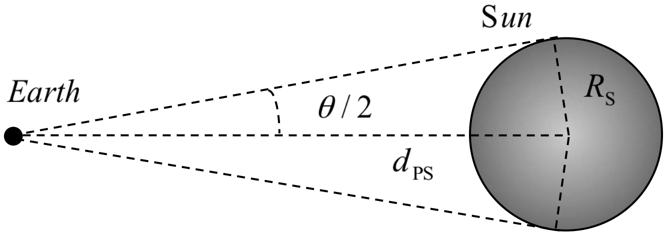


Fig. 1

According the fig. 1 the angular diameter of the Sun can be calculated as follows

$$\sin\frac{\theta}{2} = \frac{R_S}{d_{PS}} \approx \frac{\theta}{2};$$

$$\theta = \frac{2R_S}{d_{PS}} = \frac{D_S}{d_{PS}} = \frac{2\cdot 6{,}96\cdot 10^5\ \text{km}}{15\cdot 10^7\ \text{km}} = 0{,}00928\ \text{rad}.$$

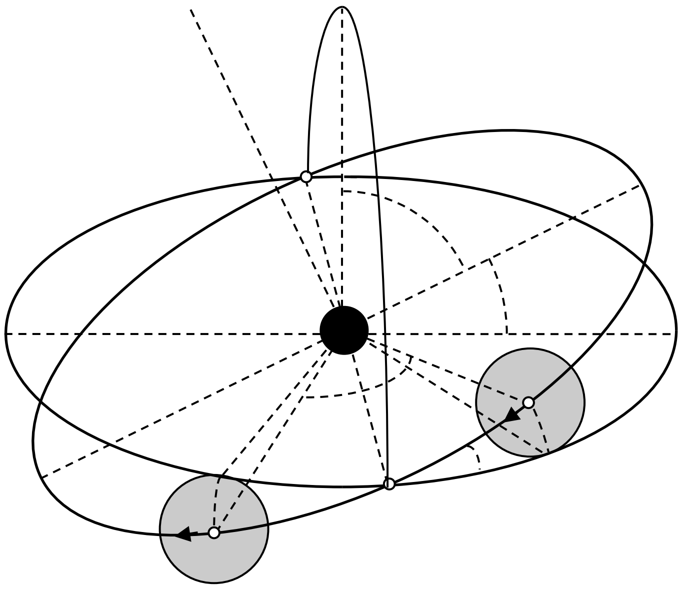
The figure 2 presents the Sun's evolution during sunset as seen by the astronomer. In an equinox day the Sun moves retrograde along the celestial equator. There are marked the following 3 positions of the Sun:

T_{dos} - The solar disc is tangent to the equatorial plane above the standard horizon – the start of the sunset;

S_{dos} - The center of the solar disc on the celestial equator in the moment of the sunset starts;

T_{sos} - The solar disc is tangent to the equatorial plane bellow the standard horizon – the end of the sunset

S_{sos} - The center of the solar disc on the celestial equator in the moment of the sunset ends;



The duration of the sunset is τ. During this time the center of the Sun moves along the equator from S_{dos} to S_{sos}. The vector-radius of the Sun rotates in equatorial plane with angle ϕ and in vertical plane with angle θ. i.e. the angular diameter of the Sun as seen from the Earth.

Considering that the Sun travels the distance $2x$ along the equatorial path with merely constant i.e. during time τ and that the spherical right triangle $S_{dos}T_{dos}V$ can be considered a plane one the following relations can be written:

$$\sin\gamma = \frac{R_S}{x};\ x = \frac{R_S}{\sin\gamma};\ \ 2x = \frac{2R_S}{\sin\gamma} = \frac{D_S}{\sin\gamma};$$

$$\tau = \frac{2x}{v} = \frac{2x}{\omega \cdot d_{PS}} = \frac{\frac{D_S}{\sin\gamma}}{\frac{2\pi}{T_P}\cdot d_{PS}} = \frac{\frac{D_S}{d_{PS}}}{\frac{2\pi}{T_P}\cdot\sin\gamma} = \frac{\theta\cdot T_P}{2\pi\cdot\sin\gamma};$$

$$\sin\gamma = \sin\left(90^\circ - \varphi\right) = \cos\varphi;$$

$$\tau = \frac{\theta\cdot T_P}{2\pi\cdot\cos\varphi};$$

$$\tau = \frac{0{,}00928\,\text{rad}\cdot 24\,\text{h}}{2\cdot 3{,}14\,\text{rad}\cdot\cos\left(45^\circ 21'\right)} = \frac{0{,}22272\cdot 60}{2\cdot 3{,}14\cdot 0{,}707}\,\text{min} \approx 3\,\text{min}.$$

2) If the atmospheric refraction is negligible the eagle on the top of the cross V_1 on figure 3 is on the same latitude (φ), as the astronomer but at the altitude H. Thus from the point of view of the V_1 the horizon line is below the standard horizon line with an angle $\Delta\alpha_1$,

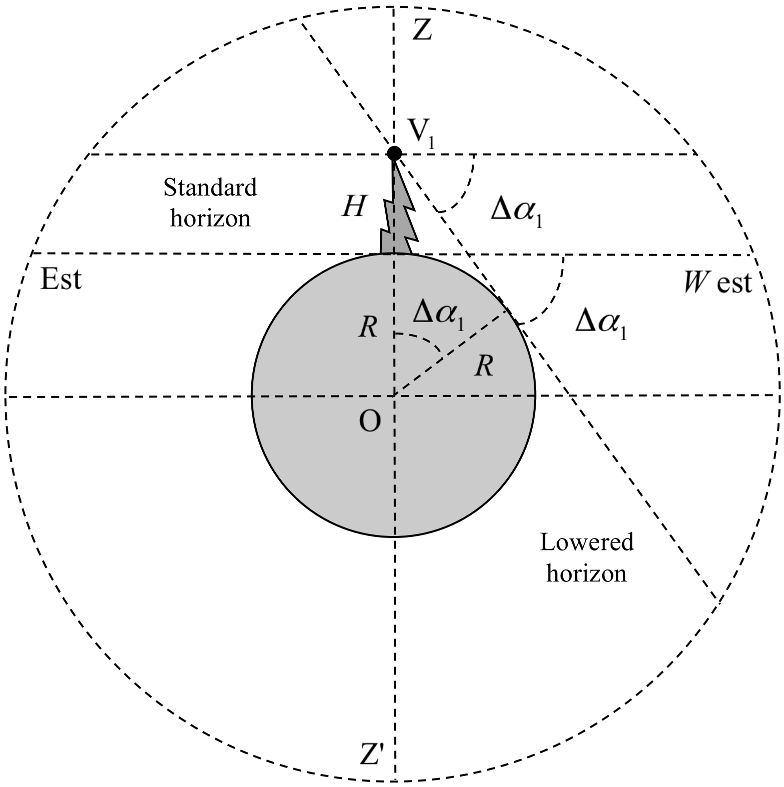


Fig. 3

$$\cos\Delta\alpha_1 = \frac{R}{R+H};$$

$$\sin \Delta\alpha_1 = \frac{\sqrt{(R+H)^2 - R^2}}{R+H} = \frac{\sqrt{2RH + H^2}}{R+H} \approx \frac{\sqrt{2RH}}{R} = \sqrt{\frac{2H}{R}} \approx \Delta\alpha_1;$$

$$\Delta\alpha_1 = \sqrt{\frac{2 \cdot 2{,}3\ \text{km}}{6380\ \text{km}}} \approx 0{,}02685\ \text{rad} \approx 1{,}54^\circ.$$

For the observer V_1 the Sun will go below the lowered horizon after moving down under the standard horizon with angle $\Delta\alpha_1$ and moving along the equator with an angle $\Delta\beta_1$, as seen in fig. 4

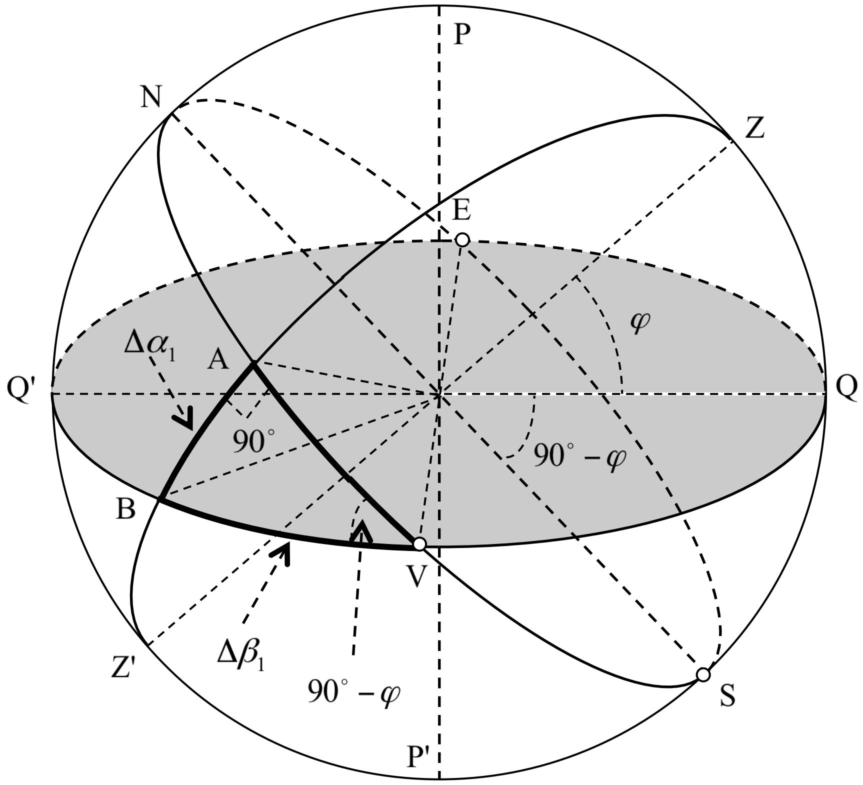


Fig.

In the right spherical triangle ABV by using the sinus formula :

$$\frac{\sin(90^\circ - \varphi)}{\sin \Delta\alpha_1} = \frac{\sin 90^\circ}{\sin \Delta\beta_1};$$

$$\frac{\cos\varphi}{\Delta\alpha_1} = \frac{1}{\Delta\beta_1}; \ \Delta\beta_1 = \frac{\Delta\alpha_1}{\cos\varphi};$$

$$\Delta\beta_1 = \omega \cdot \Delta\tau_1 = \frac{2\pi}{T_P} \cdot \Delta\tau_1;$$

$$\Delta\tau_1 = \frac{\Delta\alpha_1}{\cos\varphi} \cdot \frac{T_P}{2\pi} = \frac{1{,}54^\circ}{\cos(45^\circ 21')} \cdot \frac{24 \cdot 60\ \text{min}}{360^\circ} \approx 8{,}71\ \text{minute},$$

Which represents the delay of the start of the sunset from the point of view of V_1 regardless to the astronomer due to the V_1 observer's altitude.

The altitude effect on the total time of sunset can be calculated by using the fig. 5

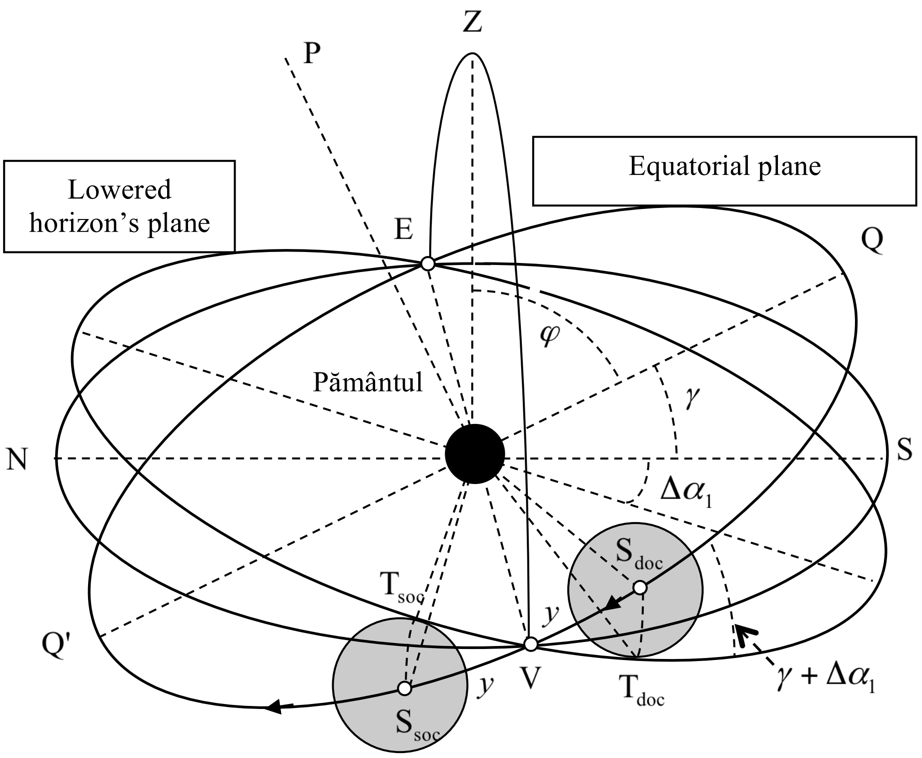


Fig.5

$$\sin(\gamma+\Delta\alpha_1)=\frac{R_S}{y};\ y=\frac{R_S}{\sin(\gamma+\Delta\alpha_1)};\ 2y=\frac{2R_S}{\sin(\gamma+\Delta\alpha_1)}=\frac{D_S}{\sin(\gamma+\Delta\alpha_1)};$$

$$\tau_1=\frac{2y}{\mathrm{v}}=\frac{2y}{\omega\cdot d_{PS}}=\frac{\dfrac{D_S}{\sin(\gamma+\Delta\alpha_1)}}{\dfrac{2\pi}{T_P}\cdot d_{PS}}=\frac{\dfrac{D_S}{d_{PS}}}{\dfrac{2\pi}{T_P}\cdot\sin(\gamma+\Delta\alpha_1)}=\frac{\theta\cdot T_P}{2\pi\cdot\sin(\gamma+\Delta\alpha_1)};$$

$$\gamma=90^\circ-\varphi;$$

$$\sin(\gamma+\Delta\alpha_1)=\sin(90^\circ-\varphi+\Delta\alpha_1)=\sin[90^\circ-(\varphi-\Delta\alpha_1)]=\cos(\varphi-\Delta\alpha_1);$$

$$\tau_1=\frac{\theta\cdot T_P}{2\pi\cdot\cos(\varphi-\Delta\alpha_1)};$$

$$\tau_1 = \frac{0{,}00928\,\text{rad}\cdot 24\ \text{h}}{2\cdot 3{,}14\ \text{rad}\cdot\cos\left(45^\circ - 1{,}54^\circ\right)} = \frac{0{,}22272\cdot 60}{2\cdot 3{,}14\cdot 0{,}725}\,\text{min} \approx 2{,}9350\ \text{min},$$

Which represents the total duration of sunset for V_1 at altitude *H*.

Similarly for eagle V_2 at the same latitude (φ), but altitude $H+h$ (the top of the cross), the lowering effect on the horizon is measured by angle $\Delta\alpha_2$ thus

$$\cos\Delta\alpha_2 = \frac{R}{R+H+h};$$

$$\sin\Delta\alpha_2 = \frac{\sqrt{(R+H+h)^2 - R^2}}{R+H+h} = \frac{\sqrt{2R(H+h)+(H+h)^2}}{R+H+h} \approx \frac{\sqrt{2R(H+h)}}{R} = \sqrt{\frac{2(H+h)}{R}} \approx \Delta\alpha_2;$$

$$\Delta\alpha_{\cdot 2} = \sqrt{\frac{2\cdot(2{,}3+0{,}0393)\,\text{km}}{6380\ \text{km}}} \approx 0{,}02707\ \text{rad} \approx 1{,}55^\circ;$$

$$\frac{\sin\left(90^\circ - \varphi\right)}{\sin\Delta\alpha_2} = \frac{\sin 90^\circ}{\sin\Delta\beta_2};$$

$$\frac{\cos\varphi}{\Delta\alpha_2} = \frac{1}{\Delta\beta_2};\ \ \Delta\beta_2 = \frac{\Delta\alpha_2}{\cos\varphi};$$

$$\Delta\beta_2 = \omega\cdot\Delta\tau_2 = \frac{2\pi}{T_P}\cdot\Delta\tau_2;$$

$$\Delta\tau_2 = \frac{\Delta\alpha_2}{\cos\varphi}\cdot\frac{T_P}{2\pi} = \frac{1{,}55^\circ}{\cos\left(45^\circ 21'\right)}\cdot\frac{24\cdot 60\ \text{min}}{360^\circ} \approx 8{,}77\ \text{minute},$$

Which represents the delay of the start moment of the sunset for V_2 due to the altitude $H+h$.

Similar the total duration of the sunset for the observer V_2 :

$$\tau_2 = \frac{\theta\cdot T_P}{2\pi\cdot\cos(\varphi - \Delta\alpha_2)};$$

$$\tau_2 = \frac{0{,}00928\,\text{rad}\cdot 24\ \text{h}}{2\cdot 3{,}14\ \text{rad}\cdot\cos\left(45^\circ - 1{,}55^\circ\right)} = \frac{0{,}22272\cdot 60}{2\cdot 3{,}14\cdot 0{,}726}\,\text{min} \approx 2{,}9309\ \text{min},$$

We may note the following:

- the horizon- lowering $\Delta\alpha$ is increased by the increase of the altitude;

$$\left(H < H+h \rightarrow \Delta\alpha_1 < \Delta\alpha_2;\, H\uparrow \rightarrow \Delta\alpha\uparrow\right)$$

- the delay of the moment of sunset start is increased by the increase of the altitude:

$$\left(H < H+h \rightarrow \Delta\tau_1 < \Delta\tau_2;\, H\uparrow \rightarrow \Delta\tau\uparrow\right)$$

- the total duration of sunset is reduced by the increase of the altitude:

$$\left(0 < H < H+h \rightarrow \tau > \tau_1 > \tau_2;\, H\uparrow \rightarrow \tau\downarrow\right)$$

Conclusions:

If we consider t_0 the moment of sunset star for the astronomer

- for V_1 the sunset starts at $t_0 + 8{,}71$ min and ends at $t_0 + 8{,}71\ \text{min} + 2{,}9350\ \text{min} = t_0 + 11{,}6450\ \text{min}$
- for V_2 the sunset starts at $t_0 + 8{,}77$ min and ends at $t_0 + 8{,}77\ \text{min} + 2{,}9309\ \text{min} = t_0 + 11{,}7009\ \text{min}$
- Thus eagle from the plateau leaves first the cross;

- The time between the leaving moments is:

$$\Delta t = t_0 + 11{,}7009\,\text{min} - t_0 - 11{,}6450\,\text{min} = 0{,}0559\,\text{min} = 3{,}354\,\text{s}.$$

b)

As seen in fig. 6 the length of the cross on the plateau will be minimum when the Sun passes the local meridian, i.e. the height of the Sun above the horizon will be maximum:

$(h_{max} = \gamma = 90^\circ - \varphi)$.

Thus the shadow of the horizontal arms of the cross is superposed on the shadow of the vertical pillow.

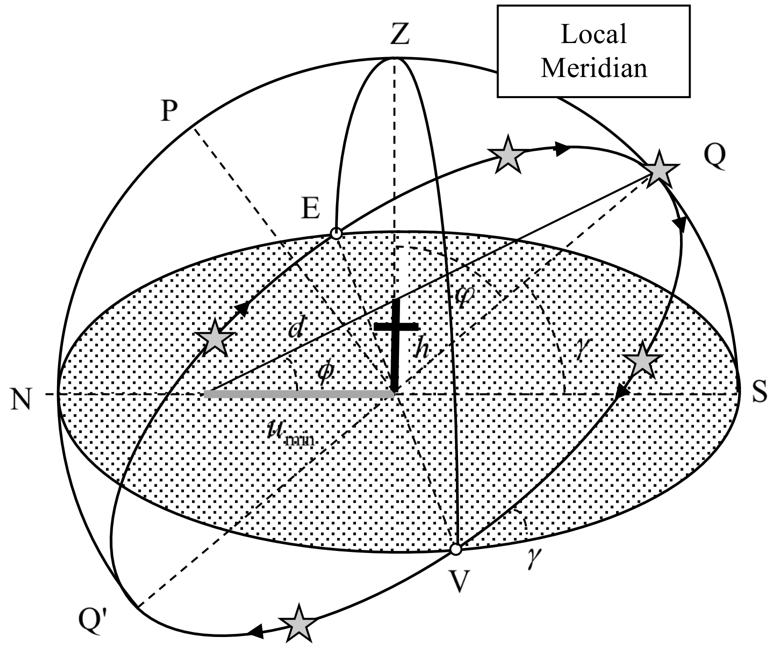


Fig. 6

In this conditions :

$$\sin\phi = \frac{h}{d};\ \phi \approx \gamma = 90^\circ - \varphi;$$

$$d = \frac{h}{\sin\phi} \approx \frac{h}{\sin\gamma} = \frac{h}{\sin(90^\circ - \varphi)} = \frac{h}{\cos\varphi} = \frac{39{,}3\,\text{m}}{\cos 45^\circ} = \frac{39{,}3}{0{,}707}\,\text{m} \approx 55{,}58\,\text{m};$$

The distance between the two eagles is

$$u_{min} = h \cdot \cot\phi \approx h \cdot \cot\varphi = h \cdot \cot(90^\circ - \varphi) = h \cdot \tan\varphi = 39{,}3\,\text{m}.$$

2) In the above mentioned conditions the shadow of the arm oriented toward South is on the vertical pillow of the cross, as seen in fig. 7:

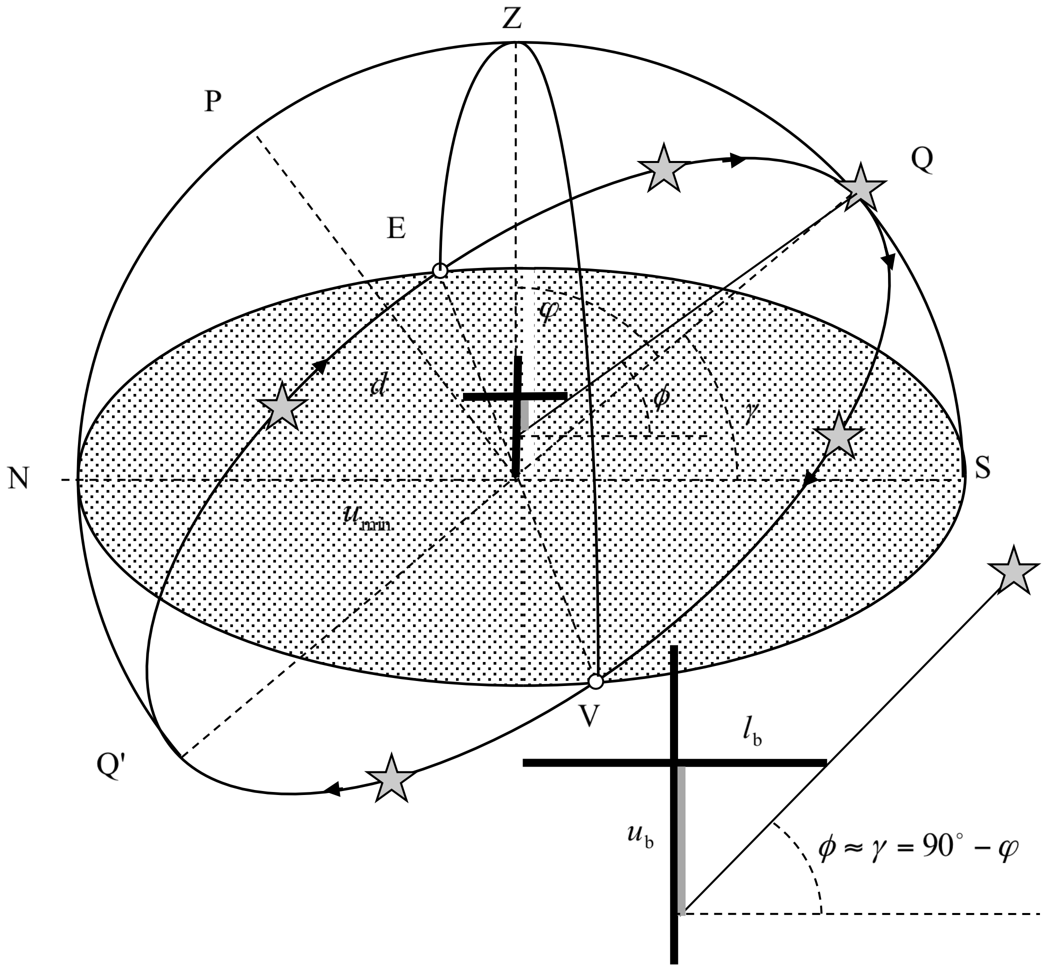


Fig. 7

$$\tan\varphi = \frac{l_b}{u_b};\ l_b = u_b \cdot \tan\varphi = 7\ \text{m} \cdot \tan 45^\circ = 7\ \text{m},$$

Which represents the length of the cross arm.

C)

1) For an observer situated in the center O of the celestial topocentric sphere, at latitude φ, at sea level, al the stars are circumpolar ones see fig. 8. Their diurnal parallels, parallel with the equatorial parallel, are above the real local horizon (N_0S_0). The star σ_0 is at the circumpolar limit because its parallel touches the real local horizon in point N_0 but still remains above it. Thus σ_0 is a marginal circumpolar star. Without taking into account the atmospheric refraction:

From the isosceles triangle $O\sigma_0 N_0$ results the σ_0 declination:

$$\delta_{0,\min} + 90^\circ + (\varphi - \delta_{\min}) = 180^\circ;$$

$$\delta_{0,\min} = 90^\circ - \varphi.$$

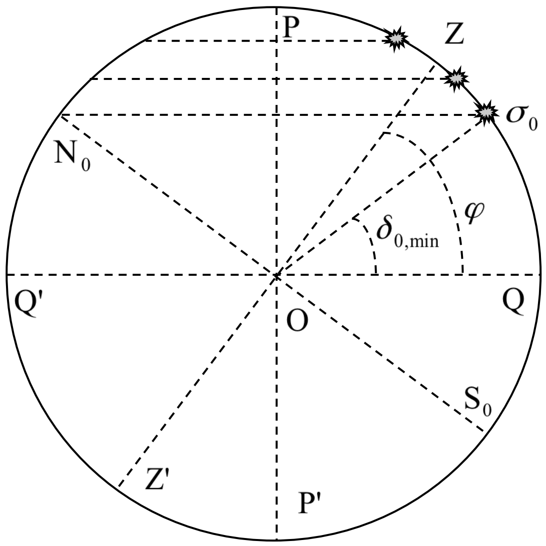


Fig. 8

By taking into account the atmospheric refraction the horizon line changes to line N'S', with angle $\theta' \approx 34'$, as seen in fig. 9. The star σ_0 remains a circumpolar one but above the limit. In this conditions the star σ' gather the limit conditions its declination been $\delta'_{\min} < \delta_{\min}$. In this conditions for an observer situated in the center of the topocentric celestial sphere, at latitude φ and altitude zero, the star σ', with declination $\delta'_{\min} < \delta_{0\min}$ is on the limit of the circumpolarity.

From the isosceles triangle N'Oσ' the declination σ' will be :

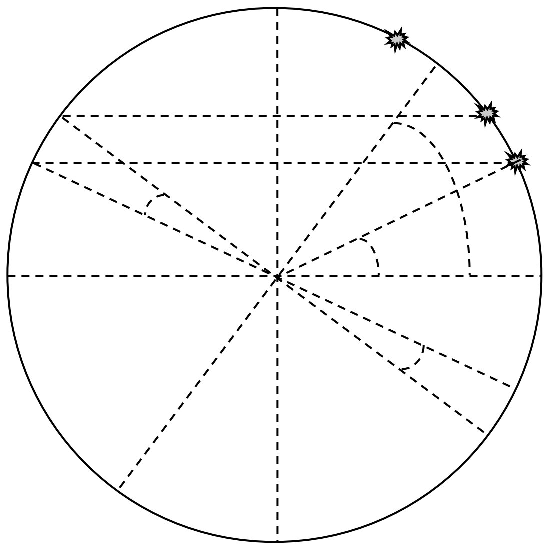


Fig. 9

$$2\delta'_{\min} + (\varphi - \delta'_{\min}) + (90^\circ + \theta') = 180^\circ;$$
$$\delta'_{\min} = 90^\circ - \varphi - \theta'.$$

For the observer at latitude φ, but at height h, taking into account the effect of lowering of the horizon the star σ'' will meet the problem requirements see figure 10. The new horizon is N"S" and declination of σ'' is $\delta''_{min} < \delta_{0\,min}$. Star σ_0 remains a circumpolar one but above the limit.

From the isosceles triangle $N''O\sigma''$ the declination of star σ'' will be:

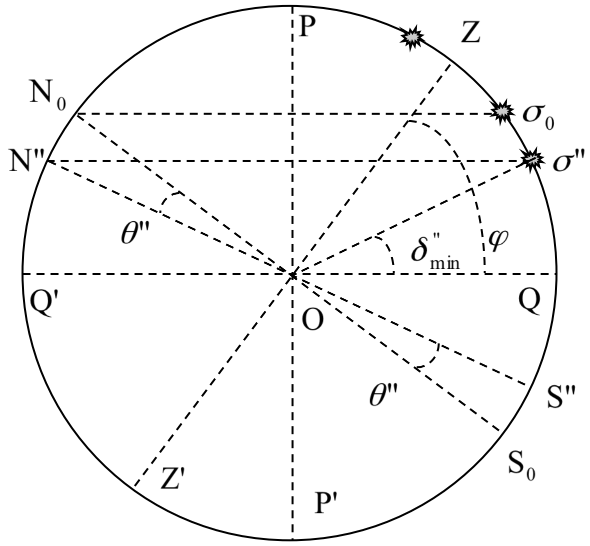


Fig. 10

$$2\delta''_{min} + (\varphi - \delta''_{min}) + (90^\circ + \theta'') = 180^\circ;$$
$$\delta''_{min} = 90^\circ - \varphi - \theta''.$$

By taking into account the refraction effect and the altitude effect, from triangle $NO\sigma$ in figure 11, the declination will be

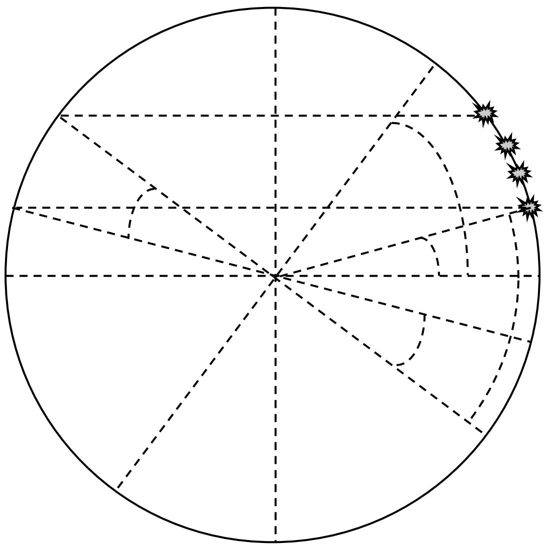


Fig. 11

$$2\delta_{min} + (\varphi - \delta_{min}) + (90^\circ + \theta) = 180^\circ;$$
$$\theta = \theta' + \theta''; \; \theta' = \xi = 34' : \theta'' = \Delta\alpha_2 = 1{,}55^\circ;$$

$$\delta_{\min} = 90^\circ - \varphi - \theta' - \theta'' = 90^\circ - \varphi - \theta' - \Delta\alpha_2;$$
$$\delta_{\min} = 90^\circ - 45^\circ - 0{,}56^\circ - 1{,}55^\circ \approx 42{,}9^\circ.$$

2) The maximum height above the horizon will be

$$h_{\max} = 90^\circ + \delta_{\min} - \varphi = 90^\circ + 42{,}9^\circ - 45^\circ = 87{,}9^\circ.$$

17. Long problem 2. Cosmic Pendulum

A space shuttle (N) orbits the Earth in the equatorial plane on a circular trajectory with radius *r*. From the spaceship. The shuttle has an arm designated to place satellites on the orbit. The arm is a metallic rod (negligible mass) with length $l << r$. The arm is connected to the shuttle with frictionless mobile articulation. A satellite **S** is attached to the arm and let out from the shuttle. At a certain moment the angle between the rod and the shuttle's orbit radius is α, see figure 1. You know the mass of the Earth –**M** , and the gravitational constant **k**.

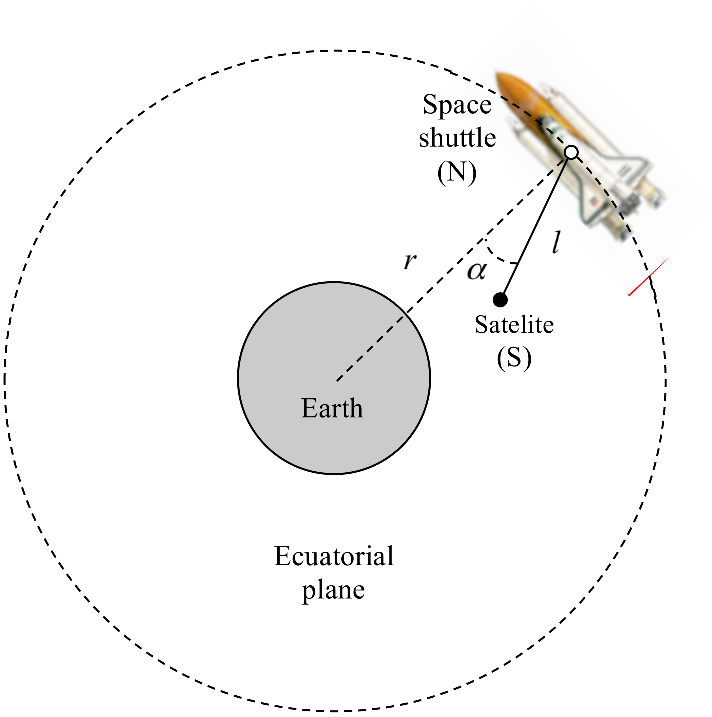


Fig.2

a) Find out the values of angle α for which the configuration of the system shuttle – rod –satellite remains unchanged regardless to Earth (the system is in equilibrium), during orbiting the Earth. For each found value of angle α, specify the type of the system equilibrium i.e. stable or unstable.

You will take in to account the following assumptions: the initial orbit of the shuttle is not affected by the presence of the satellite **S**, all the external friction-type interactions are negligible, the satellite – shuttle gravitational interaction is negligible too. The following data are known: , m_1 - the mass of the shuttle, the mass of the satellite $m_2 << m_1$.

b) In the moment of one stable equilibrium configuration, the rod with the satellite attached is slightly rotated with a very small angle in the orbital plane and then released. Demonstrate that the small oscillations of the satellite **S** relative to the shuttle are harmonically ones. Express the period T_0 of this cosmic pendulum as a function of the orbiting period T of the shuttle around the Earth.

It is known the linear harmonic oscillator equation:

$$\frac{d^2\beta}{dt^2} + \omega_0^2\beta = 0;\ \omega_0 = \frac{2\pi}{T_0},$$

Where : β – the instantaneous angular deviation; T_0 – the period of the linear harmonic oscillator.

c) If we consider that the mass of the satellite S , m_2 is not negligible by comparison with the shuttle's one m_1 in the conditions from point a) the evolution on the orbit of the shuttle would be influenced by the presence of the satellite S rigid attached to the shuttle by the rod. Identify and determine the consequences on the shuttle's movement after one complete rotation around the Earth.

d) Propose a special technical maneuver which can cancels the influence of the non negligible mass satellite S on the shuttles movement.

a)	15
b)	15
c)	10
d)	10

Long problem 2. Marking scheme - Cosmic Pendulum

a) In figure 1 are represented the forces in the system.

$\vec{F}_1$ – the gravitational attraction force acting on the shuttle due to the Earth;

$\vec{F}_2$ – the gravitational attraction force acting on the satelite due to the Earth;

$\vec{F}$ – the tension force in the suspention rod.

For a value of $\alpha \neq 0$ the movement equations are:
For the shuttle on circular orbit around the Earth:

$$m_1\omega^2 r = K\frac{m_1 M}{r^2} + F\cos\alpha;$$

For the satelite on circular orbit around the Earth:

$$m_2\omega^2\left(r - l\cos\alpha\right) = K\frac{m_2 M}{\left(r - l\cos\alpha\right)^2} - \frac{F}{\cos\alpha}.$$

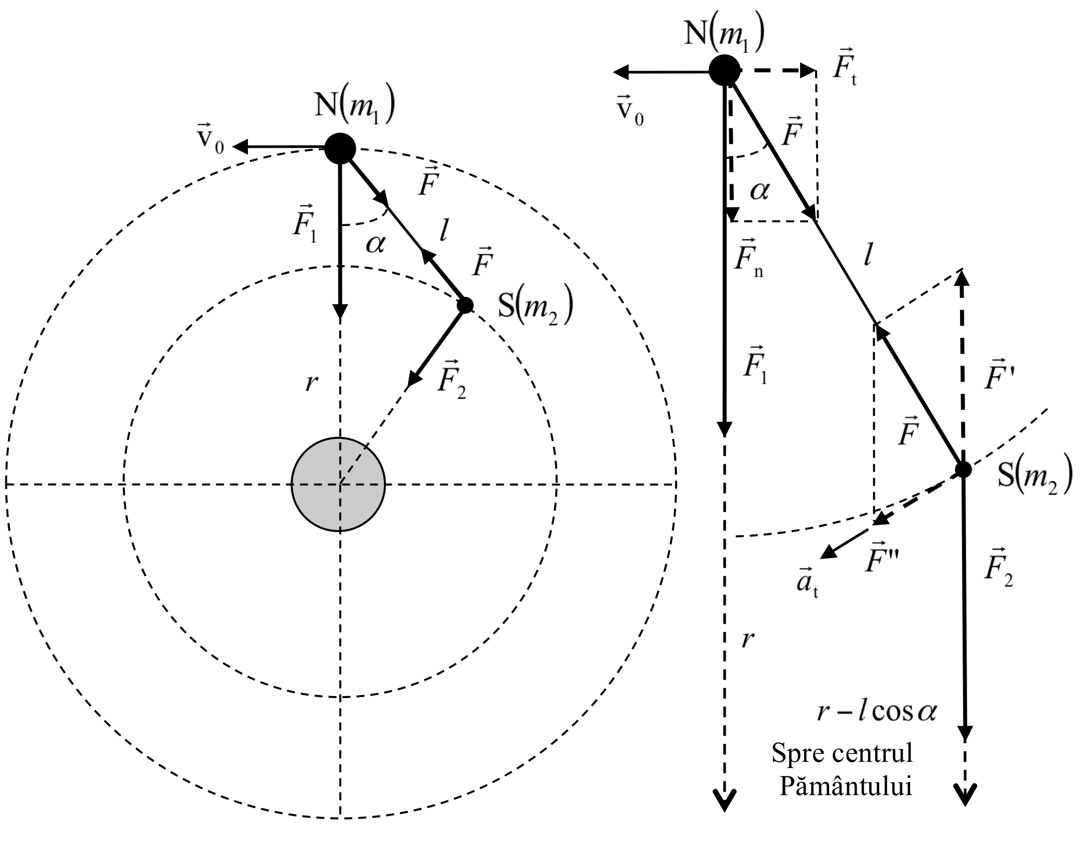


Fig.1

If the gravitational interaction between the shuttle and the satellite is negligible results:

$$F\cos\alpha << K\frac{m_1 M}{r^2};$$

$$m_1\omega^2 r \approx K\frac{m_1 M}{r^2};\ \omega^2 = \frac{KM}{r^3} = \frac{4\pi^2}{T^2};$$

$$T = 2\pi\sqrt{\frac{r^3}{KM}},$$

The revolution time of the shuttle around the Earth:

$$m_2\frac{KM}{r^3}\left(r - l\cos\alpha\right) = K\frac{m_2 M}{\left(r - l\cos\alpha\right)^2} - \frac{F}{\cos\alpha};$$

$$m_2 \frac{KM}{r^3}(r - l\cos\alpha)^3 = Km_2M - \frac{F}{\cos\alpha}(r - l\cos\alpha)^2;$$

$$m_2 \frac{KM}{r^3} r^3 \left(1 - \frac{l}{r}\cos\alpha\right)^3 = Km_2M - \frac{F}{\cos\alpha} r^2 \left(1 - \frac{l}{r}\cos\alpha\right)^2;$$

$$m_2 KM \left(1 - \frac{l}{r}\cos\alpha\right)^3 = Km_2M - \frac{F}{\cos\alpha} r^2 \left(1 - \frac{l}{r}\cos\alpha\right)^2;$$

$$m_2 KM \left(1 - 3\frac{l}{r}\cos\alpha\right) = Km_2M - \frac{F}{\cos\alpha} r^2 \left(1 - 2\frac{l}{r}\cos\alpha\right);$$

$$3Km_2M \frac{l}{r}\cos\alpha \approx \frac{F}{\cos\alpha} r^2;$$

$$F \approx 3Km_2M \frac{l}{r^3}\cos^2\alpha,$$

The tension force in the rod.

The satellite movement regardless to the shuttle is non uniform circular one, described by the equation:

$$m_2 \vec{a}_t = \vec{F}'';$$

$$m_2 a_t = m_2 \varepsilon l = m_2 \frac{d^2\alpha}{dt^2} l = -F'' = -F\tan\alpha = -3Km_2M \frac{l}{r^3}\sin\alpha\cos\alpha;$$

$$\frac{d^2\alpha}{dt^2} + 3\frac{KM}{r^3}\sin\alpha\cos\alpha = 0,$$

The solutions of this equation is $\alpha(t)$.

If during the system evolution the configuration of the system remains the same results:

$$\alpha = \text{constant};\ \frac{d^2\alpha}{dt^2} = 0;$$

$$\sin\alpha \cdot \cos\alpha = 0;$$

$$\sin\alpha = 0;\ \alpha_1 = 0;\ \alpha_2 = \pi;\ \text{(echilibru stabil)};$$

$$\cos\alpha = 0;\ \alpha_3 = \pi/2;\ \alpha_4 = 3\pi/2;\ \text{(echilibru instabil)},$$

As seen in the pictures .

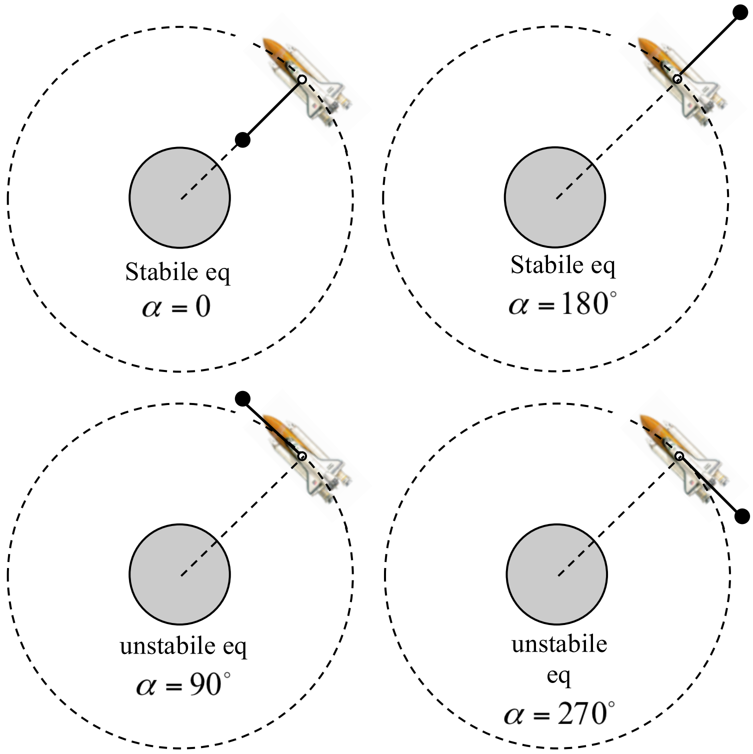


Fig.

b) Coresponding to the position with $\alpha = 0$, the movement equation of the satellite displaced from equilibrium position with a very small angle β:

$$\frac{d^2\beta}{dt^2} + 3\frac{KM}{r^3}\sin\beta \cdot \cos\beta = 0;$$

$$\sin\beta \approx \beta; \cos\beta \approx 1;$$

$$\frac{d^2\beta}{dt^2} + 3\frac{KM}{r^3}\beta = 0,$$

Which represents the linear harmonic oscillator;

$$\frac{d^2\beta}{dt^2} + \omega_0^2\beta = 0;$$

Thus the period of the small oscillations will be

$$\omega_0^2 = 3\frac{KM}{r^3} = \frac{4\pi^2}{T_0^2}; \; T_0 = 2\pi\sqrt{\frac{1}{3}\frac{r^3}{KM}} = \frac{T}{\sqrt{3}},$$

c)

$$m_1\omega^2 r = K\frac{m_1 M}{r^2} + F\cos\alpha;$$

$$m_2\omega^2(r - l\cos\alpha) = K\frac{m_2 M}{(r - l\cos\alpha)^2} - \frac{F}{\cos\alpha};$$

$$\omega^2 = \frac{KM}{r^3} + \frac{F\cos\alpha}{m_1 r};$$

$$m_2\omega^2(r - l\cos\alpha)^3 = Km_2M - \frac{F}{\cos\alpha}(r - l\cos\alpha)^2;$$

$$m_2\omega^2 r^3\left(1 - \frac{l}{r}\cos\alpha\right)^3 = Km_2M - \frac{F}{\cos\alpha}r^2\left(1 - \frac{l}{r}\cos\alpha\right)^2;$$

$$m_2\omega^2 r^3\left(1 - 3\frac{l}{r}\cos\alpha\right) = Km_2M - \frac{F}{\cos\alpha}r^2\left(1 - 2\frac{l}{r}\cos\alpha\right);$$

$$m_2\left(\frac{KM}{r^3} + \frac{F\cos\alpha}{m_1 r}\right)r^3\left(1 - 3\frac{l}{r}\cos\alpha\right) = Km_2M - \frac{F}{\cos\alpha}r^2\left(1 - 2\frac{l}{r}\cos\alpha\right);$$

$$m_2\left(KM + \frac{F\cos\alpha}{m_1}r^2\right)\cdot\left(1 - 3\frac{l}{r}\cos\alpha\right) = Km_2M - \frac{F}{\cos\alpha}r^2\left(1 - 2\frac{l}{r}\cos\alpha\right);$$

$$F = \frac{3Km_2M\frac{l}{r}\cos\alpha}{\left(\frac{m_2}{m_1}\cos\alpha + \frac{1}{\cos\alpha}\right)r^2 - rl\left(\frac{m_2}{m_1}\cos^2\alpha + 1\right)};$$

$$\omega^2 = \frac{KM}{r^3} + \frac{F\cos\alpha}{m_1 r};$$

$$\omega^2 = \frac{KM}{r^3} + \frac{3K\frac{m_2}{m_1}M\frac{l}{r^2}\cos^2\alpha}{\left(\frac{m_2}{m_1}\cos\alpha + \frac{1}{\cos\alpha}\right)r^2 - rl\left(\frac{m_2}{m_1}\cos^2\alpha + 1\right)}.$$

Observation: If $m_2 << m_1$ and $l << r$, the results are already found :

$$\omega^2 = \frac{KM}{r^3} = \omega_0^2;$$

$$F = \frac{3Km_2M\frac{l}{r}\cos\alpha}{\left(\frac{1}{\cos\alpha}\right)r^2} = 3Km_2M\frac{l}{r^3}\cos^2\alpha.$$

Corresponding to the initial circulae orbit with radius r, the total mechanical energy of the system shuttle – Earth is:

$$E_0 = \frac{m_1 v_0^2}{2} - K\frac{m_1M}{r} = -K\frac{m_1M}{2r}.$$

After one complete rotation, the tangential component of the tension in the rod $\vec{F}_t$, acting on the shuttle will determine the change of the radius of the orbit i.e. $(r - \Delta r)$. The total energy of the system will be:

$$E = \frac{m_1 v^2}{2} - K\frac{m_1M}{r - \Delta r} = -K\frac{m_1M}{2(r - \Delta r)}.$$

$$\Delta E = E - E_0 = -K\frac{m_1M}{2(r - \Delta r)} + K\frac{m_1M}{2r} = -K\frac{m_1M}{2}\left(\frac{1}{r - \Delta r} - \frac{1}{r}\right);$$

$$\Delta E = -K\frac{m_1M}{2}\cdot\frac{r - r + \Delta r}{r(r - \Delta r)} = -K\frac{m_1M}{2}\cdot\frac{\Delta r}{r(r - \Delta r)};$$

$$r - \Delta r \approx r;$$

The variation of the mechanical energy after a complete rotation

$$\Delta E \approx -K\frac{m_1 M \cdot \Delta r}{2r^2} < 0,$$

Thus

$$\Delta E = L_t = -2\pi r \cdot F_t;\ F_t = F\sin\alpha;$$

$$-K\frac{m_1 M \cdot \Delta r}{2r^2} = -2\pi r \cdot F_t$$

The variation of the altitude due to the action of the satellite will be:

$$\Delta r = \frac{4\pi r^3 F_t}{K m_1 M} = \frac{4\pi r^3 F\sin\alpha}{K m_1 M},$$

The potentialenergy v

$$\Delta E_p = -K\frac{m_1 M}{r - \Delta r} - \left(-K\frac{m_1 M}{r}\right) = -K m_1 M\left(\frac{1}{r-\Delta r} - \frac{1}{r}\right);$$

$$\Delta E_p = -K m_1 M \cdot \frac{r - r + \Delta r}{r(r - \Delta r)} = -K m_1 M \frac{\Delta r}{r(r-\Delta r)};$$

$$r - \Delta r \approx r;$$

$$\Delta E_p \approx -K\frac{m_1 M \cdot \Delta r}{r^2} < 0,$$

Thus

$$\Delta E_p \approx -K\frac{m_1 M \cdot \Delta r}{2r^2} \cdot 2 = \Delta E \cdot 2 = -2\pi r F_t \cdot 2 = -4\pi r F_t;$$

$$\Delta E_c = \frac{m_1 \mathrm{v}^2}{2} - \frac{m_1 \mathrm{v}_0^2}{2};$$

$$\frac{m_1 \mathrm{v}_0^2}{r} = K\frac{m_1 M}{r^2};\ m_1 \mathrm{v}_0^2 = K\frac{m_1 M}{r};$$

$$\frac{m_1 \mathrm{v}^2}{r - \Delta r} = K\frac{m_1 M}{(r-\Delta r)^2};\ m_1 \mathrm{v}^2 = K\frac{m_1 M}{r - \Delta r};$$

$$\Delta E_c = K\frac{m_1 M}{2(r - \Delta r)} - K\frac{m_1 M}{2r} = K\frac{m_1 M}{2}\left(\frac{1}{r - \Delta r} - \frac{1}{r}\right);$$

$$\Delta E_c = K\frac{m_1 M}{2}\left(\frac{1}{r - \Delta r} - \frac{1}{r}\right) = K\frac{m_1 M}{2}\left(\frac{1}{r - \Delta r} - \frac{1}{r}\right);$$

$$\Delta E_c = K\frac{m_1 M}{2}\left(\frac{1}{r - \Delta r} - \frac{1}{r}\right) = K\frac{m_1 M}{2} \cdot \frac{r - r + \Delta r}{r(r - \Delta r)};$$

$$\Delta E_c = K\frac{m_1 M}{2} \cdot \frac{\Delta r}{r(r - \Delta r)};$$

$$r - \Delta r \approx r;$$

$$\Delta E_c \approx K \frac{m_1 M \cdot \Delta r}{2r^2} > 0,$$

reprezentând variația energiei cinetice a navei, după o rotație completă în jurul Pământului;

$$\Delta E_c > 0;\ v > v_0,$$

rezultat care constituie "paradoxul sateliților", adică, atunci când altitudinea orbitei navei scade, viteza navei crește;

$$\Delta E = \Delta E_c + \Delta E_p;$$

$$\Delta E_c = \Delta E - \Delta E_p = -2\pi r F_t - (-4\pi r F) = 2\pi r F_t > 0;$$

$$\Delta E_c = \frac{m_1 v^2}{2} - \frac{m_1 v_0^2}{2} = \frac{m_1}{2}\left(v^2 - v_0^2\right) = \frac{m_1}{2}(v - v_0)(v + v_0);$$

$$v = v_0 + \Delta v;\ v - v_0 = \Delta v;\ v + v_0 = 2v_0 + \Delta v \approx 2v_0;$$

$$\Delta E_c \approx \frac{m_1}{2} 2v_0 \cdot \Delta v = m_1 v_0 \Delta v = 2\pi r F_t;$$

$$\Delta v = \frac{2\pi r F_t}{m_1 v_0};\ v_0 = \sqrt{K\frac{M}{r}};$$

$$\Delta v = \frac{2\pi r F_t}{m_1} \cdot \sqrt{\frac{r}{KM}}.$$

d) Manevra tehnică propusă este reprezentată în figura alăturată: o forță reactivă, egală în modul și de sens contrar cu tensiunea din tijă:

$$\vec{F}_r = -\vec{F},$$

astfel încât forța rezultantă care acționează asupra navei să rămână forța de atracție gravitațională din partea Pământului:

$$\vec{F}_{\text{Nava}} = \vec{F}_1 + \vec{F} + \vec{F}_r = \vec{F}_1.$$

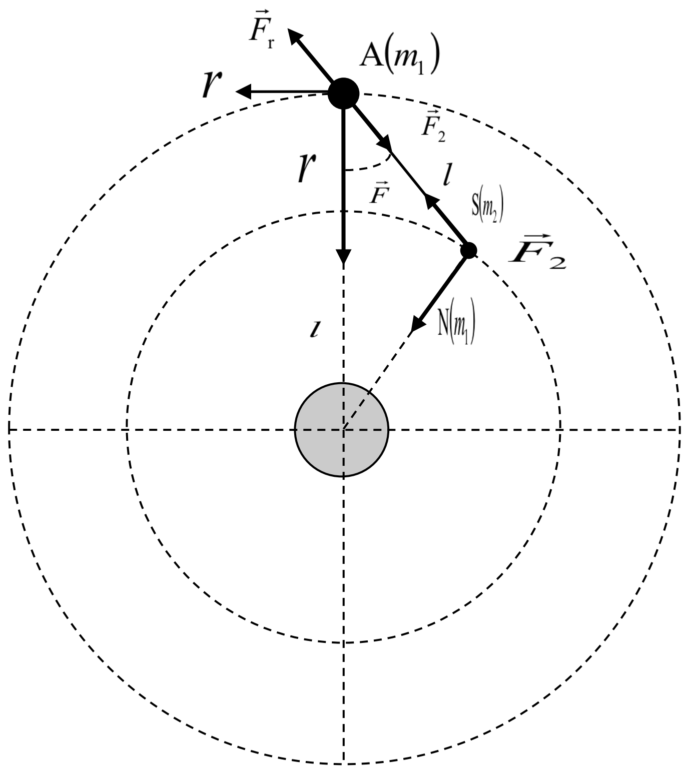


Fig.

18. Long problem 3. From Romania …. to Antipod! … a ballistic messenger

The 8th IOAA organizers plan to send to the **antipode** the official flag using a ballistic projectile. The projectile will be launched from Romania, and the rotation of the Erath will be neglected.

a) Calculate the coordinates of the target-point if the launch-point coordinates are: $\varphi_{\text{Romania}} = 44^\circ$ *North*; $\lambda_{\text{Romania}} = 30^\circ$ Est.

b) Determine the elements of the launching-speed vector, regardless to the center of the Earth, in order that the projectile should hit the target.

c) Calculate the velocity of the projectile when it hits the target.

d) Calculate the minimum velocity of the projectile.

e) Calculate the flying –time of the projectile, from the launch-moment to the impact one. You will use the value of the gravitational acceleration at Erath surface $g_0 = 9{,}81\,\text{ms}^{-2}$;the Earth radius $R = 6370\,\text{km}$.

f) Evaluate the possibility that the projectile to be seen with the naked eye in the moment that it passes at the maximum distance from the Earth. You will use the following values: The Moon albedo $\alpha_M = 0{,}12$; The Moon radius $R_M = 1738\,Km$; the Earth –Moon distance $r_{EL} = 384400$km; the apparent magnitude of the full moon $m_M = -12{,}7^{m}$. You assume that the projectile is perfectly metallic sphere with radius $r_{projectile} = 400\,x\,10^{-3}\ m$ and with perfectly reflective surface.

Long problem 3. Marking scheme - From Romania …. to Antipod! … a ballistic messenger

a)	10
b)	10
c)	10
d)	10
e)	10
f)	10

a) The two places are represented in the figure.

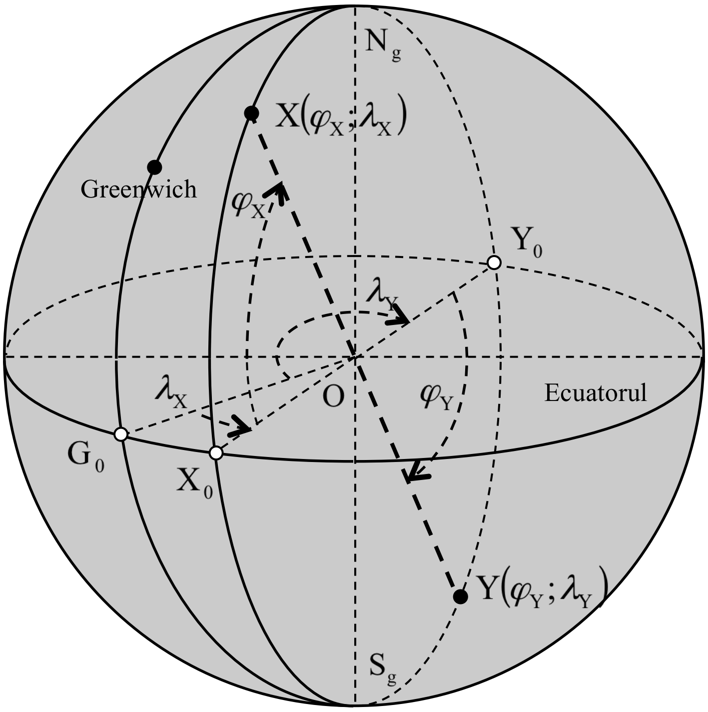


Fig.

$$\varphi_{\mathrm{Y}} = \varphi_{\mathrm{Y,Sud}} = \varphi_{\mathrm{X,Nord}} = \varphi_{\mathrm{X}};$$

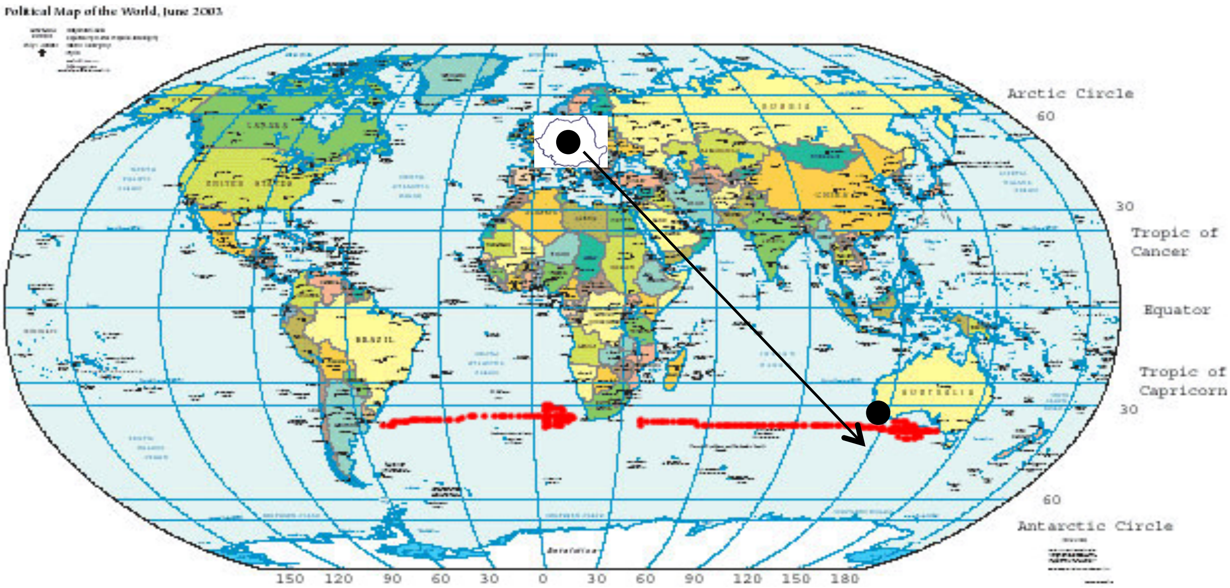
$$\lambda_{\mathrm{Y,Vest}} + \lambda_{\mathrm{X,Eest}} = 180^{\circ};\ \lambda_{\mathrm{Y}} + \lambda_{\mathrm{X}} = 180^{\circ}.$$

$$\varphi_{\mathrm{Romania}} = 43^{\circ}\ \mathrm{Nord};\ \lambda_{\mathrm{Romania}} = 30^{\circ}\ \mathrm{Est},$$

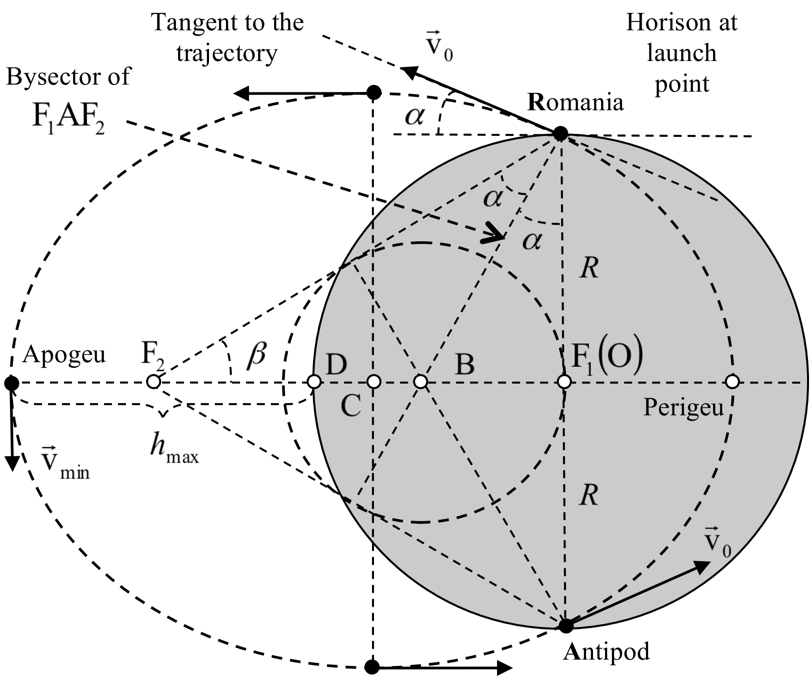
The landing point is

$$\varphi_{\mathrm{Antipod}} = 43^{\circ}\ \mathrm{Sud};\ \lambda_{\mathrm{Antipod}} = 150^{\circ}\ \mathrm{Vest},$$

Somewhere South EEst from Tasmania (South from Australia).



b) The schetch of the trajectory



In order to hit the point the trajectory of the missile has to be an elypse with the Earth center in the center of the Earth. *Se știe că*:

$$F_2B = 2 \cdot F_1B;\ F_1F_2 = 3 \cdot F_1B.$$

Rezultă:

$$\tan 2\alpha = \frac{F_1F_2}{R};\ F_1F_2 = R \cdot \tan 2\alpha;$$

$$\tan \alpha = \frac{F_1B}{R};\ \ F_1B = R \cdot \tan \alpha;$$

$$R \cdot \tan 2\alpha = \ F_1B = R \cdot \tan \alpha;$$

$$\tan 2\alpha = 3 \cdot \tan \alpha;$$

$$\frac{\sin 2\alpha}{\cos 2\alpha} = 3\frac{\sin \alpha}{\cos \alpha};$$

$$\frac{2\sin \alpha \cdot \cos \alpha}{\cos 2\alpha} = 3\frac{\sin \alpha}{\cos \alpha};$$

$$2\cos^2 \alpha = 3\cos 2\alpha;\ 2\cos^2 \alpha = 3(\cos^2 \alpha - \sin^2 \alpha);$$

$$3\sin^2 \alpha = \cos^2 \alpha;\ \tan^2 \alpha = \frac{1}{3};$$

$$\tan \alpha = \frac{\sqrt{3}}{3};\ \alpha = 30^\circ;\ 2\alpha = 60^\circ;\ \beta = 90^\circ - 2\alpha = 30^\circ;\ 2\beta = 60^\circ;$$

$$\Delta(RF_2A) \rightarrow \text{triunghi echilateral};$$

$$RF_2 = AF_2 = RA = 2R;$$

$$RF_2 + RF_1 = 2a = 3R;$$

$$a = \frac{3}{2}R;$$

$$v_0 = \sqrt{KM\left(\frac{2}{r} - \frac{1}{a}\right)};\ r = R;\ g_0 = K\frac{M}{R^2};$$

$$v_0 = \sqrt{\frac{KM}{R^2}\cdot R^2\left(\frac{2}{R} - \frac{2}{3R}\right)} = 2\sqrt{\frac{g_0 R}{3}}.$$

c)

$$v_{Antipod} = v_0.$$

d)

$$F_1F_2 = R\cdot\tan 2\alpha = 2c;\ c = \frac{R}{2}\cdot\tan 2\alpha = \frac{R}{2}\cdot\tan 60^\circ = \frac{\sqrt{3}}{2}R;$$

$$b = \sqrt{a^2 - c^2} = \sqrt{\frac{3}{2}}R;$$

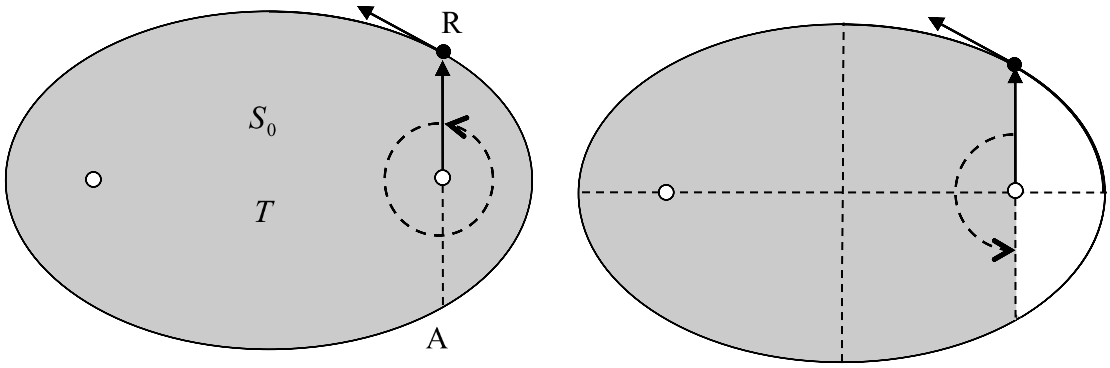
$$2a = 2r_{min} + 2c;\ r_{min} = a - c = \frac{1}{2}\left(3 - \sqrt{3}\right)R;$$

$$r_{max} = 2a - r_{min} = \frac{1}{2}\left(3 + \sqrt{3}\right)R;$$

$$v_{min} = \sqrt{KM\left(\frac{2}{r_{max}} - \frac{1}{a}\right)};\ r_{max} = \frac{1}{2}\left(3 + \sqrt{3}\right)R;$$

$$v_{min} = \sqrt{\frac{KM}{R^2}\cdot R^2\left(\frac{4}{\left(3+\sqrt{3}\right)R} - \frac{2}{3R}\right)} = \sqrt{\frac{2g_0R}{3}\cdot\frac{3-\sqrt{3}}{3+\sqrt{3}}}.$$

e) Accordin to Kepplers laws:



$$\Omega = \frac{dS}{dt} = \text{constant};$$

$$\frac{S_0}{T} = \frac{2\frac{S_0}{4} + 2S_1}{\Delta t};\ \frac{S_0}{T} = \frac{\frac{S_0}{2} + 2S_1}{\Delta t};\ \frac{S_0}{T} = \frac{S_0 + 4S_1}{2\cdot\Delta t};$$

$$S_0 = \pi ab;\ S_1 = \frac{ab}{2}\left[\sqrt{1-\frac{b^2}{a^2}}\cdot\frac{b}{a} + \arcsin\sqrt{1-\frac{b^2}{a^2}}\right];$$

$$\Delta t = \frac{S_0 + 4S_1}{2S_0}\cdot T = \left(\frac{1}{2} + 2\frac{S_1}{S_0}\right)\cdot T;$$

$$T = 2\pi\sqrt{\frac{a^3}{KM}};\ T = \frac{2\pi}{R}\sqrt{\frac{a^3}{g_0}};$$

$$\frac{2S_1}{S_0} = \frac{1}{\pi}\left(\frac{b}{a}\cdot\sqrt{1-\frac{b^2}{a^2}} + \arcsin\sqrt{1-\frac{b^2}{a^2}}\right);$$

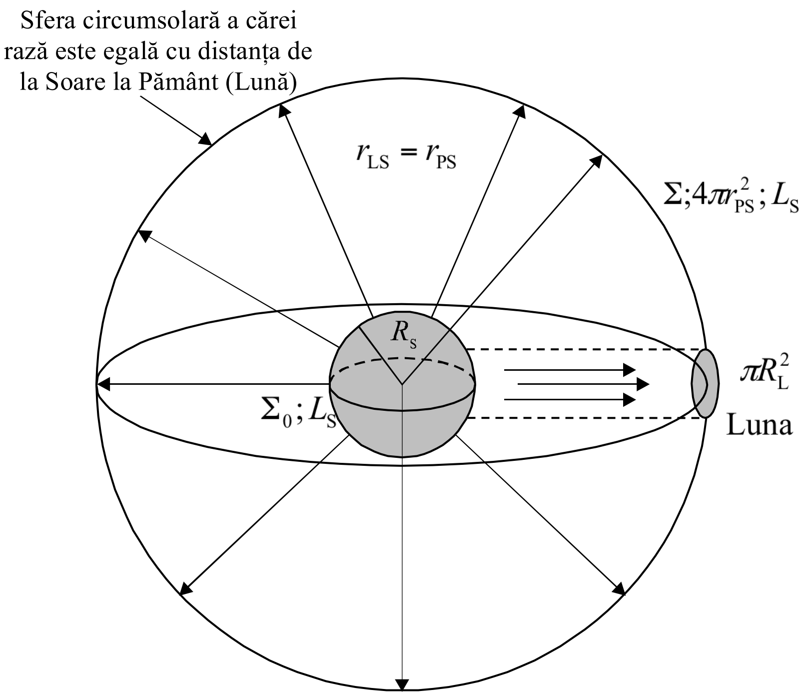
$$\sqrt{1-\frac{b^2}{a^2}} = e;\ \frac{2S_1}{S_0} = \frac{1}{\pi}\left(\frac{b}{a}\cdot e + \arcsin e\right);$$

$$\Delta t = \left(\frac{1}{2} + \frac{eb}{\pi a} + \frac{\arcsin e}{\pi}\right)\cdot T.$$

f) The integral luminosity of Sun:

$$L_\text{S} = \frac{E_\text{emis,Soare}}{t} = 3{,}86\cdot 10^{26}\ \text{W},$$

Dacă For a circumsolar surface Σ with radius r_{PS}, see picture bellow the solar radiation enegy passing through the surface in one second is L_S.



Density of solar flux

$$\phi_{Soare, r_{PS}} = \frac{E_{emis,Soare}}{St} = \frac{\frac{E_{emis,Soare}}{t}}{S} = \frac{L_S}{S} = \frac{L_S}{4\pi r_{PS}^2} = \text{constant}.$$

$$F_{incident,FullMoon} = \phi_{Sun, r_{PS}} \cdot \pi R_L^2.$$

Dacă α_L este albedoul Lunii, rezultă:

$$\alpha_L = \frac{F_{reflectat,FullMoon}}{F_{incident,FullMoon}},$$

unde $F_{reflectat,\text{Luna Plina}}$ – fluxul energetic al radiațiilor reflectate de Luna Plină spre observatorul de pe Pământ;

$$F_{reflectat,FullMoon} = \alpha_L \cdot F_{incident,FullMoon} = \alpha_L \cdot \phi_{Soa,,re,r_{PS}} \cdot \pi R_L^2.$$

În consecință, densitatea fluxului energetic ajuns la observator, după reflexia pe suprafața Lunii, este:

$$\phi_{moon,\ observator} = \frac{F_{reflectat,\ FullMoon}}{2\pi r_{PL}^2} = \alpha_L \cdot \phi_{Soare,r_{PS}} \cdot \frac{\pi R_L^2}{2\pi r_{PL}^2}.$$

Symilarly

$$\phi_{proiectil,\ observator} = \frac{F_{reflectat,\ proiectil}}{4\pi r_{D,proiectil}^2} = \alpha_{proiectil} \cdot \phi_{Soare,r_{PS}} \cdot \frac{\pi R_{proiectil}^2}{4\pi r_{D,proiectil}^2}.$$

În expresia anterioară s-a avut în vedere faptul că densitatea fluxului energetic al proiectilului la observator rezultă din distribuirea prin suprafața sferei cu raza $r_{P,proiectil}$.

Utilizând formula lui Pogson, vom compara magnitudinea aparentă vizuală a Lunii Pline cu magnitudinea aparentă vizuală a proiectilului balistic:

$$\log\frac{\phi_{Luna,\ observator}}{\phi_{proiectil,\ observator}} = -0{,}4\left(m_{Luna\ Plina} - m_{proiectil}\right);$$

$$\log\frac{\phi_{Luna,\ observator}}{\phi_{proiectil,observator}} = \log\frac{\alpha_L \cdot \phi_{Soare,r_{PS}} \cdot \frac{\pi R_L^2}{2\pi r_{PL}^2}}{\alpha_{proiectil} \cdot \phi_{Soare,r_{PS}} \cdot \frac{\pi R_{proiectil}^2}{4\pi r_{D,proiectil}^2}} = \log\frac{\alpha_L \cdot \frac{R_L^2}{r_{PL}^2}}{\alpha_{proiectil} \cdot \frac{R_{proiectil}^2}{2 r_{D,proiectil}^2}};$$

$$\log\frac{\alpha_L \cdot \frac{R_L^2}{r_{PL}^2}}{\alpha_{proiectil} \cdot \frac{R_{proiectil}^2}{2 r_{D,proiectil}^2}} = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$\log\frac{\alpha_L}{\alpha_{proiectil}} \cdot \left(\frac{R_L}{R_{proiectil}}\right)^2 \cdot 2 \cdot \left(\frac{r_{D,proiectil}}{r_{PL}}\right)^2 = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$\alpha_L = 0{,}12;\ \alpha_{proiectil} = 1;$$

$$R_L = 1738\,\text{km};\ R_{proiectil} = 400\,\text{mm};$$

$$r_{D,proiectil} = r_{max,observator-proiectil} = h_{max} = r_{max} - R;\ r_{max} = \frac{1}{2}\left(3+\sqrt{3}\right)R;$$

$$h_{max} = \frac{1}{2}\left(3+\sqrt{3}\right)R - R = \frac{1}{2}\left(1+\sqrt{3}\right)R \approx 8700\ \text{km};$$

$$r_{PL} = r_{observator,Luna} = 384400\,\text{km};\ m_L = -12{,}7^m;$$

$$\log\frac{\alpha_L}{\alpha_{proiectil}} + 2\log\frac{R_L}{R_{proiectil}} + \log 2 + 2\log\frac{r_{D-proiectil}}{r_{PL}} = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$\log(0{,}12) + 2\log\frac{1738000\,\text{m}}{0{,}400\,\text{m}} + \log 2 + 2\log\frac{8700\,\text{km}}{384400\,\text{km}} = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$\log(0{,}12) + 2\log\frac{1738000}{0{,}400} + \log 2 + 2\log\frac{8700}{384400} = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$-0{,}920818754 + 13{,}27597956 + 0{,}301029995 - 3{,}290528253 = -0{,}4\left(m_L - m_{proiectil}\right);$$

$$23{,}4^m = 12{,}7^m + m_{proiectil};$$

$$m_{proiectil} = 10{,}7^m;$$

$$m_{max} \approx 6^m;\ m_{proiectil} > m_{max},$$

The projectile wasn’t seen when it was at its apogee