

## Theoretical Exam - Short Questions

1. What would be the mean temperature on the Earth's surface if we ignore the greenhouse effect, assume that the Earth is a perfect black body and take into account its non-vanishing albedo? Assume that the Earth's orbit around the Sun is circular.

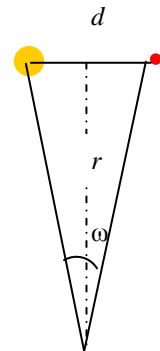
**Answer :**  $T_{\oplus} = T_{\odot} (1 - \alpha)^{1/4} \sqrt{\frac{R_{\odot}}{2r_{\oplus}}} = 246 \text{ K}$  or  $T_{\oplus} = -27^{\circ} \text{ C}$  **(10 Points)**

(Note:  $r_{\oplus}$  = average Earth-Sun distance, the albedo,  $a = 0.39$  is given in the *additional material*).

2. Let us assume that we observe a hot Jupiter planet orbiting around a star at an average distance  $d = 5 \text{ AU}$ . It has been found that the distance of this system from us is  $r = 250 \text{ pc}$ . What is the minimum diameter,  $D$ , that a telescope should have to be able to resolve the two objects (star and planet)? We assume that the observation is done in the optical part of the electromagnetic spectrum ( $\lambda \sim 500 \text{ nm}$ ), outside the Earth's atmosphere and that the telescope optics are perfect.

**Answer:** From the figure we have:

$$\omega(\text{rad}) \approx \frac{d}{r} = \frac{5 \text{ AU}}{250 \text{ pc}} = \frac{5 \times 1.5 \times 10^{11} \text{ m}}{250 \times 3.09 \times 10^{16} \text{ m}} = 9.70 \times 10^{-8} \text{ rad} \quad \text{(5 Points)}$$



Let us assume that  $D$  is the minimum diameter of our space telescope.

Its angular resolution is

$$\Delta\theta = 1.22 \times \frac{\lambda}{D} \rightarrow 9.70 \times 10^{-8} \text{ rad} = 1.22 \times \frac{500 \times 10^{-9} \text{ m}}{D} \rightarrow D = 6 \text{ m}$$

**(5 Points)**

3. It is estimated that the Sun will have spent a total of about  $t_1 = 10$  billion years on the main sequence before evolving away from it. Estimate the corresponding amount of time,  $t_2$ , if the Sun were 5 times more massive.

**Answer:** For the average luminosity of a main sequence star we have:  $L \propto M^4$  (where  $M$  the initial mass of the star). We assume that the total energy  $E$  that the star produces is proportional to its mass  $E \propto M$ . Therefore the amount of time that the star spends on the main sequence is

$$\text{approximately } t_{MS} \approx \frac{E}{L} \propto \frac{M}{M^4} \approx M^{-3}.$$

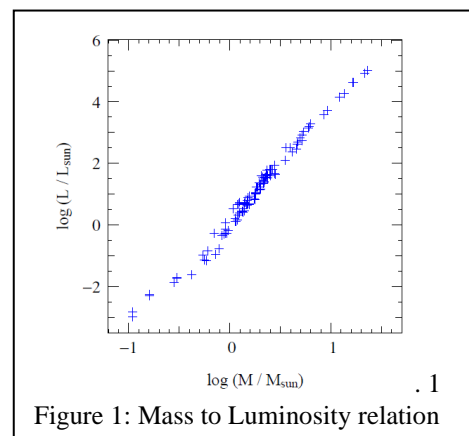
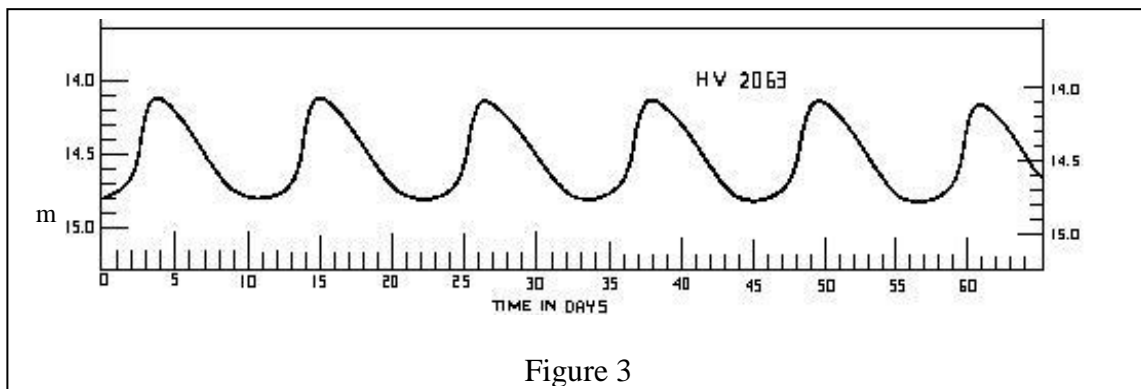
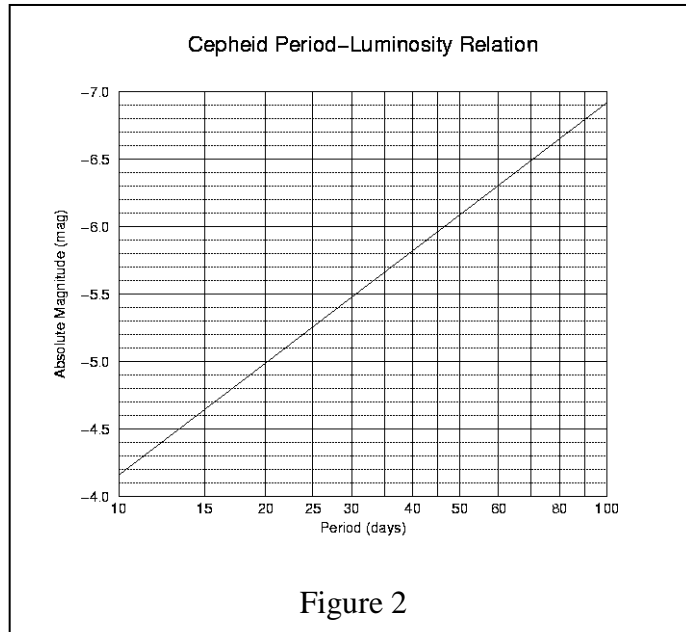


Figure 1: Mass to Luminosity relation

**(5 Points)**

Therefore,  $\frac{t_1}{t_2} \approx \frac{1^{-3}}{5^{-3}} = 5^3 \rightarrow t_2 = \frac{1}{125} \times 10^{10} \text{ yr}$  or  $t_2 = 8 \times 10^7 \text{ yr}$ . (5 Points)

4. Figure 2 shows the relation between absolute magnitude and period for classical cepheids. Figure 3 shows the light curve (apparent magnitude versus time in days) of a classical cepheid in a local group galaxy. (a) Using these two figures estimate the distance of the cepheid from us. (b) Revise your estimate assuming that the interstellar extinction towards the cepheid is  $A = 0.25 \text{ mag}$ .



**Answer :** (a) From Figure 2, the period of the cepheid is  $P \sim 11$  days and its average apparent magnitude is  $\sim (14.8 + 14.1) / 2 \text{ mag}$ , i.e.  $m = 14.45 \text{ mag}$ . (2 points)

[A careful student will notice that the graph is not *upside-down* symmetrical so he/she choose some value closer to the bottom; that is  $m = 14.5 \text{ mag}$ . (1 point)

From Figure 1, we derive that for a period of 11 days the expected absolute magnitude of the cepheid is  $M \approx -4.2$ . (1 point)



[A careful student will notice that the graph is logarithmic so he/she choose a value closer to 4.3]

**(1 point)**

Using the formula  $m - M = -5 + 5 \log r$ , where  $r$  is the distance of the cepheid, we get

$$\log r = (14.45 + 4.2 + 5) / 5 = 4.73, \text{ thence } r = 10^{4.73} \approx 57500 \text{ pc or } r = \mathbf{57.5 \text{ kpc}} \quad \mathbf{(3 \text{ points})}$$

(b) Assuming  $A = 0.25$ , then  $\log r = (14.45 + 4.2 + 5 + 0.25) / 5 = 4.78$ , thence  $r = 10^{4.78} \sim 53000 \text{ pc or}$

$$r = \mathbf{64.5 \text{ kpc.}} \quad \mathbf{(2 \text{ points})}$$

5. The optical spectrum of a galaxy, whose distance had been measured to be 41.67 Mpc, showed the Balmer H $\alpha$  line ( $\lambda_0 = 656.3 \text{ nm}$ ) redshifted to  $\lambda = 662.9 \text{ nm}$ . (a) Use this distance to calculate a value of the Hubble constant,  $H_0$ . (b) Using your results, estimate the Hubble time of the Universe.

$$\text{Answer: (a) } z = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{662.9 - 656.3}{656.3} \approx 0.01 \quad \mathbf{(3 \text{ Point})}$$

This is small enough that we can use the classical equation for the expansion of the Universe.

$$H_0 = \frac{cz}{r} = \frac{3 \times 10^5 \text{ km s}^{-1} \times 10^{-2}}{41.67 \text{ Mpc}} \quad \text{or } H_0 = 72.4 \frac{\text{km s}^{-1}}{\text{Mpc}} \quad \mathbf{(4 \text{ Points})}$$

$$\text{(b) } t_H \approx \frac{1}{H_0} \quad \text{or } t_H = \mathbf{13.5 \text{ Gyr}} \quad \mathbf{(3 \text{ Point})}$$

6. A star has an effective temperature  $T_{\text{eff}} = 8700 \text{ K}$ , absolute magnitude  $M = 1.6$  and apparent magnitude  $m = 7.2$ . Find (a) the star's distance,  $r$ , (b) its luminosity,  $L$ , and (c) its radius,  $R$ . (Ignore extinction).

$$\text{Answer: (a) Its distance is calculated from equation: } m - M = 5 \log(r) - 5, \text{ or } 7.2 - 1.6 + 5 = 5 \log(r) \rightarrow \log(r) = 2.12 \text{ and } r = \mathbf{132 \text{ pc}} \quad \mathbf{(3 \text{ Points})}$$

$$\text{(b) Its Luminosity is calculated from equation: } M_{\odot} - M = 2.5 \log\left(\frac{L}{L_{\odot}}\right) \text{ or}$$

$$4.8 - 1.6 = 2.5 \log\left(\frac{L}{L_{\odot}}\right) \text{ or } \log\left(\frac{L}{L_{\odot}}\right) = 1.28 \rightarrow \left(\frac{L}{L_{\odot}}\right) = 19.05 \text{ or } L = 19.15 \times L_{\odot} = 19.15 \times 3.9 \times 10^{33} \text{ and } L = \mathbf{7.4 \times 10^{34} \text{ erg sec}^{-1}} \quad \mathbf{(4 \text{ Points})}$$

(c) Its radius can be easily calculated from equation:  $L = 4\pi\sigma R^2 T_{\text{eff}}^4$ , from which

$$R = \frac{1}{T_{\text{eff}}^2} \sqrt{\frac{L}{4\pi\sigma}} \text{ from which we get: } R = \mathbf{1.35 \times 10^{11} \text{ cm}} \quad \mathbf{(3 \text{ Points})}$$

7. A star has visual apparent magnitude  $m_v = 12.2$  mag, parallax  $\pi = 0''.001$  and effective temperature  $T_{eff} = 4000$  K. Its bolometric correction is  $B.C. = -0.6$  mag. (a) Find its luminosity as a function of the solar luminosity. (b) What type of star is it? (i) a *red giant*? (ii) a *blue giant*? or (iii) a *red dwarf*? Please write (i), (ii) or (iii) in your answer sheet.

**Answer :** (a) First its bolometric magnitude is calculated from equation:  $M_V - m_v = 5 - 5 \log(r)$  or equivalent:  $M_V - m_v = 5 + 5 \log \pi \rightarrow M_V = 12.2 + 5 + 5 \log(0''.001) = 12.2 + 5 - 15 = 2.2$  mag. Its barycentric correction is:  $B.C. = M_{bol} - M_V$  and  $M_{bol} = B.C. + M_V$  or  $M_{bol} = -0.6 + 2.2$  or  $M_{bol} = 1.6$  mag . **(4 Points)**

Then its Luminosity is calculated from:

$$M_{\odot} - M_{bol} = 2.5 \log\left(\frac{L}{L_{\odot}}\right), \text{ or } 4.72 - 1.6 = 2.5 \log\left(\frac{L}{L_{\odot}}\right) \text{ or } \log\left(\frac{L}{L_{\odot}}\right) = 1.25 \text{ and } L = \mathbf{17.70 L_{\odot}}$$

**(2 Point)**

(b) Type of star: A star with  $M_{bol} = 1.6$  mag,  $L = 17.7 L_{\odot}$  and  $T_{eff} = 4000$  K is much brighter and much cooler than the Sun (see *Table of constants*). Therefore it is **(i)** a red giant star. **(4 Points)**

8. A binary system of stars consists of star (a) and star (b) with brightness ratio 2. The binary system is difficult to resolve and is observed from the Earth as one star of 5<sup>th</sup> magnitude. Find the apparent magnitude of each of the two stars ( $m_a, m_b$ ).

**Answer :** The apparent magnitude of star (a) is  $m_a$ , of star (b) is  $m_b$  and that of the system as a whole is  $m_{a+b}$ . The corresponding apparent brightnesses are  $l_a, l_b$  and  $l_{a+b} = l_a + l_b$ .

For star (a) :

$$m_{a+b} - m_a = -2.5 \log\left(\frac{l_a + l_b}{l_a}\right) \text{ and because } \frac{l_b}{l_a} = \frac{1}{2}, \text{ we get } m_a = m_{a+b} + 2.5 \log(1 + \frac{1}{2}) \text{ or}$$

$$m_a = 5 + 2.5 \log(3/2) \text{ and finally } \mathbf{m_a = 5.44 \text{ mag}} . \text{ Similarly for star (b):} \quad \mathbf{(5 \text{ point})}$$

$$m_{a+b} - m_b = -2.5 \log\left(\frac{l_a + l_b}{l_b}\right) \text{ and because } \frac{l_a}{l_b} = 2, \text{ we get } m_b = m_{a+b} + 2.5 \log(3) \text{ or}$$

$$m_b = 5 + 2.5 \log(3) \text{ and finally } \mathbf{m_b = 6.19 \text{ mag}} . \quad \mathbf{(5 \text{ Point})}$$

9. Find the equatorial coordinates (*hour angle* and *declination*) of a star at Madrid, geographic latitude  $\phi = 40^\circ$ , when the star has zenith angle  $z = 30^\circ$  and azimuth  $A = 50^\circ$  (azimuth as measured from the South)

**Answer :** From the position triangle  $\Pi Z_v \Sigma$  (Figure 3) of the star,  $\Sigma$ , we get, by using the *cosine law* for a spherical triangle:

$$\cos(90 - \delta) = \cos(90 - a) \times \cos(90 - \phi) + \sin(90 - a) \times \sin(90 - \phi) \times \cos(180 - A) \quad \mathbf{(2 \text{ Point})}$$

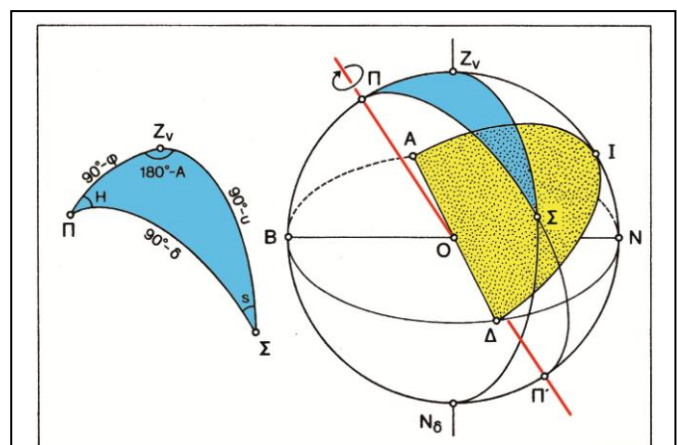


Figure 3. The position triangle



where  $\delta$  is the star's *declination*,  $a$  its *altitude* ( $a = 90^\circ - z$ ),  $\varphi$  the *geographical latitude* of the observer,  $H$  the stars *hour angle* and  $A$  the star's *azimuth*.

This can be written as:

$$\sin \delta = \cos z \times \sin \varphi - \sin z \times \cos \varphi \times \cos A \text{ or}$$

$$\sin \delta = \cos 30^\circ \times \sin 40^\circ - \sin 30^\circ \times \cos 40^\circ \times \cos 50^\circ \quad (2 \text{ Point})$$

or

$$\sin \delta = 0.866 \times 0.643 - 0.500 \times 0.766 \times 0.643 = 0.311. \quad \delta = 18^\circ 05' \quad (1 \text{ Point})$$

Using the sine law for the spherical triangle, we get:

$$\frac{\sin H}{\sin(90 - a)} = \frac{\sin(180^\circ - A)}{\sin(90^\circ - \delta)} \text{ or } \frac{\sin H}{\sin z} = \frac{\sin A}{\cos \delta} \quad (2 \text{ Point})$$

$$\rightarrow \sin H = \sin 50^\circ \times \frac{\sin 30^\circ}{\cos(18^\circ 07')} = \frac{0.766 \times 0.5}{0.950} \quad (2 \text{ Point})$$

$$\text{or } \sin H = 0.403. \text{ Therefore: } H = 23^\circ 46' \text{ or } \mathbf{H = 1^h 35^m 03^s}. \quad (1 \text{ Point})$$

10. In the centre of our Galaxy, in the intense radio source Sgr A\*, there is a black hole with large mass. A team of astronomers measured the angular distance of a star from Sgr A\* and its orbital period around it. The maximum angular distance was  $0.12''$  (arcsec) and the period was 15 years. Calculate the mass of the black hole in solar masses, assuming a circular orbit.

$$\text{Answer: } F = -\frac{GM_{BH}M_*}{R^2} = -\frac{M_*v^2}{R} \quad (2 \text{ Point})$$

$$\text{But } v = \frac{2\pi R}{P}. \text{ Therefore } \frac{GM_{BH}}{R^2} = 4\pi^2 \frac{R}{P^2} \text{ or } GM_{BH} = 4\pi^2 \frac{R^3}{P^2} \quad (2 \text{ Points})$$

Similarly:

$$GM_\odot = 4\pi^2 \frac{(1AU)^3}{(1yr)^2} \quad \hat{\eta} \quad G = \frac{1}{M_\odot} 4\pi^2 \frac{(1AU)^3}{(1yr)^2} \quad (2 \text{ Point})$$

$$\text{From Kepler's 3}^{rd} \text{ law we get: } \frac{M_{BH}}{M_\odot} = \frac{(R/1AU)^3}{(P/1yr)^2} \quad (1 \text{ Point})$$

Inserting the given data we find the distance of the star from the black hole:

$$R = \frac{0.12}{200,000} (8000)(3 \times 10^{18} \text{ cm}) = 1.4 \times 10^{16} \text{ cm} = 960 \text{ AU} \quad (2 \text{ Point})$$

$$\text{Therefore: } \frac{M_{BH}}{M_\odot} = \frac{(960)^3}{(15)^2} = 4 \times 10^6 \quad \text{form which we calculate the mass of the black hole:}$$

$$M_{BH} = 4 \times 10^6 M_\odot \quad (1 \text{ Point})$$

11. What is the maximum altitude,  $a_M$  (max), at which the Full Moon can be observed from Thessaloniki? The geographical latitude of Thessaloniki is  $\varphi_\oplus = 40^\circ 37'$ . Take into account as many factors as possible.



**Answer:** In order to have Full Moon, the Moon should be diametrically opposite the Sun, i.e. the three bodies, Sun – Earth – Moon should be on a straight line. If the orbital plane of the Moon coincided with the ecliptic, the maximum altitude of the Full Moon would be  $90^\circ - \varphi_\oplus + 23.5^\circ$ .

(2 point).

Because the orbital plane of the Moon is inclined by  $5.14^\circ$  ( $5^\circ 18'$ ) to the plane of the ecliptic, the maximum angle is larger:  $90^\circ - 40.6^\circ + 23.5^\circ + 5.3^\circ$ , or (P)

$$\alpha_M(\text{max}) = 79.8^\circ \quad (3 \text{ points})$$

Geocentric parallax of Moon for this situation is  $0.33^\circ$ , (3 points)

whereas refraction is only  $0.2'$ . (1 point)

Final answer is therefore:  $79.8^\circ - 0.3^\circ = 79.5^\circ$  (1 point)

**12.** *Sirius A*, with visual magnitude  $m_V = -1.47$  (the brighter star on the sky) and with stellar radius  $R_A = 1.7R_\odot$ , is the primary star of a binary system. The existence of its companion, *Sirius B*, was deduced from astrometry in 1844 by the well known mathematician and astronomer Friedrich Bessel, before it was directly observed. Assuming that both stars were of the same spectral type and that *Sirius B* is fainter by 10 mags ( $\Delta m = 10$ ), calculate the radius of *Sirius B*.

**Answer:** The distance of the two stars from our solar system is the same. Therefore

$$m_B - m_A = 2.5 \log \frac{\frac{L_A}{4\pi r^2}}{\frac{L_B}{4\pi r^2}} = 2.5 \log \frac{L_A}{L_B} \quad (5 \text{ Point})$$

From which we get  $L_A = 10^4 L_B$ . From equation  $L = 4\pi R^2 \sigma T_{\text{eff}}^4$  we get

$$L_A/L_B = (R_A/R_B)^2 (T_A/T_B)^4$$

Assuming that the two stars belong to the same spectral type (and therefore  $T_A = T_B$ ) we get

$$R_B = 0.01R_A = 0.01 \times 1.7 \times 696000 \text{ km or } R_B = 1.2 \cdot 10^4 \text{ km} \quad (5 \text{ Point})$$

**13.** Recently in London, because of a very thick layer of fog, the visual magnitude of the Sun, became equal to the (usual – as observed during cloudless nights) magnitude of the full Moon. Assuming that the reduction of the intensity of light due to the fog is given by an exponential equation, calculate the exponential coefficient,  $\tau$ , which is usually called *optical depth*.

**Answer:** The absorption due to the fog in London is obviously

$$A = -26.8 - (-12.74) = -14.06 \text{ mag.} \quad (2 \text{ Point})$$

$$\text{Rearranging the equation } I_v(r) = I_v(0) \times e^{-\tau}, \text{ we get } \frac{I_v(0)}{I_v(r)} = e^\tau, \quad (5 \text{ Points})$$

or

$$A = \Delta m = 2.5 \times \log(e) \times (-\tau) \text{ or } \tau = (-A)/(2.5 \times \log e) = 14.06/1.08 \text{ \textasciitilde } \tau = 12.9 \quad (3 \text{ Points})$$



14. What is the hour angle,  $H$ , and the zenith angle,  $z$ , of *Vega* ( $d = 38^\circ 47'$ ) in Thessaloniki ( $\lambda_1 = 1^{\text{h}}32^{\text{m}}$ ,  $\phi_1 = 40^\circ 37'$ ), at the moment it culminates at the local meridian of Lisbon ( $\lambda_2 = -0^{\text{h}}36^{\text{m}}$ ,  $\phi_2 = +39^\circ 43'$ )?

**Answer:** By definition at the moment when the star culminates in Lisbon, its hour angle is exactly  $0^\circ$ . Therefore its hour angle in Thessaloniki is  $0^\circ + (\lambda_1 - \lambda_2)$  or  $H = 02^{\text{h}}08^{\text{m}}$ . **(3 Points)**

Using the cosine law equation  $\cos z = \cos(90-\phi) \times \cos(90-d) + \sin(90-\phi) \times \sin(90-d) \times \cos H$ , **(4 Points)**

the zenith distance at Thessaloniki can be calculated to **24° 33'** **(3 Points)**

15. The Doppler shift of three remote galaxies has been measured with the help of Spectral observations:

| Galaxy  | Redshift, $z$ |
|---------|---------------|
| 3C 279  | 0.536         |
| 3C 245  | 1.029         |
| 4C41.17 | 3.8           |

(a) Calculate their apparent recession velocity (1) using the classical approach, (2) using the approximate formula  $v = c \ln(1+z)$ , that is often used by cosmologists and (3) using the special relativistic approach.

(b) For all three formulae, at what percentage of the speed of light do they appear to recede?

(c) Which of (1) classical, (2) special relativity (3) approximate cosmological.

**Answer:** The recession velocity is calculated by either the classical relation,  $v_c = z \times c$ , or the relativistic relation,  $v_r = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} c$ . The calculations for the three galaxies are summarized in the following Table:

| Galaxy  | $v_c$ (km/s) | $V_a$ (km/s) | $v_r$ (km/s) | $v_r/c \times 100$ |
|---------|--------------|--------------|--------------|--------------------|
| 3C279   | 160800       | 128750       | 121390       | 40%                |
| 3C245   | 308700       | 212260       | 182740       | 61%                |
| 4C41.17 | 1140000      | 470580       | 275040       | 92%                |

(1) Table 1, Columns 2,3,4 **(5 Points)**

(2) Column 5 **(3 Points)**

(3) If the student answers "Classical" **(0 Points)**

If the student answers "Special Relativity" **(1 Point)**

If the student answers "Approximate formula" **(2 Points)**



## Theoretical Exam - Long Questions

### Question 1

In a homogeneous and isotropic universe, the matter (baryonic matter + dark matter) density parameter  $\Omega_m = \frac{\rho_m}{\rho_c} = 32\%$ , where  $\rho_m$  is the matter density and  $\rho_c$  is the critical density of the Universe.

- (1) Calculate the average matter density in our local neighbourhood.
- (2) Calculate the escape velocity of a galaxy 100 Mpc away from us. Assume that the recession velocity of galaxies in Hubble's law equals the corresponding escape velocity at that distance, for the critical density of the Universe that we observe.
- (3) The particular galaxy is orbiting around the centre of our cluster of galaxies on a circular orbit. What is the angular velocity of this galaxy on the sky?
- (4) Will we ever discriminate two such galaxies that are initially at the same line of sight, if they are both moving on circular orbits but at different radii (answer “Yes” or “No”)? [Assume that the Earth is located at the centre of our local cluster.]

**Answer:**

(1) The critical density  $\rho_c = \frac{3H_0^2}{8\pi G}$  **(9 points)**

If the matter density parameter  $\Omega_m = \frac{\rho_m}{\rho_c}$  is 32% , thus  $\rho_m = 0.32 \frac{3H_0^2}{8\pi G}$  . **(3 points)**

From the latest estimate of  $H_0 = 67.8 \frac{km\ s^{-1}}{Mpc}$  we obtain  $\rho_m = 8.6 \times 10^{-27} \text{ kg m}^{-3}$  **(4 points)**

(2) The escape velocity is  $v_{esc} = \sqrt{\frac{2GM}{d}}$  . **(4 points)**

By replacing  $M = \rho_c \frac{4\pi}{3} d^3$  **(2 points)**

we obtain, for the escape velocity within  $d = 100 \text{ Mpc}$

$$v_{esc} = \sqrt{\frac{8\pi}{3} G \times 0.32 \times \frac{3H_0^2}{8\pi G} \times 100 \text{ Mpc}} = 3835 \text{ km s}^{-1} \quad \textbf{(8 points)}$$

(3) If a galaxy is orbiting around the centre of our galaxy, its velocity is  $\frac{1}{\sqrt{2}}$  of its escape velocity. Thus

$$\omega = \frac{v}{d} = \frac{v_{esc} / \sqrt{2}}{d} = \frac{H_0 d \sqrt{\Omega_m / 2}}{d} = \frac{\sqrt{0.32} (67.8 \times 10^3 \text{ m/s}) / (3.09 \times 10^{22} \text{ m})}{\sqrt{2}} = 8.8 \times 10^{-19} \text{ rad/s} \quad \textbf{(12 Points)}$$

This is

$$1.8 \times 10^{-13} \text{ arc sec/s} \quad \text{and it does not depend on the distance } d.$$

(4) Therefore we will never be able to resolve them and the answer is “No”. **(8 Points)**



## Question 2

A spacecraft is orbiting the Near Earth Asteroid (2608) *Seneca* (staying continuously very close to the asteroid), transmitting pulsed data to the *Earth*. Due to the relative motion of the two bodies (the asteroid and the Earth) around the *Sun*, the time it takes for a pulse to arrive at the ground station varies approximately between 2 and 39 minutes. The orbits of the Earth and *Seneca* are coplanar. Assuming that the *Earth* moves around the *Sun* on a circular orbit (with radius  $a_{\text{Earth}} = 1$  AU and period  $T_{\text{Earth}} = 1$  yr) and that the orbit of *Seneca* does not intersect the orbit of the *Earth*, calculate:

- (1) the semi-major axis,  $a_{\text{Sen}}$  the eccentricity,  $e_{\text{Sen}}$  of *Seneca's* orbit around the *Sun*
- (2) the period of *Seneca's* orbit,  $T_{\text{Sen}}$  and the average period between two consecutive oppositions,  $T_{\text{syn}}$  of the *Earth-Seneca* couple
- (3) an approximate value for the mass of the planet Jupiter,  $M_{\text{Jup}}$  (assuming this is the only planet of our Solar system with non-negligible mass compared to the Sun). Assume that the presence of Jupiter does not influence the orbit of *Seneca*.

**Answer :**

- (1) For  $\Delta t_b = 2$  min = 120 sec, the distance travelled by a light pulse is  $R_1 = c \times \Delta t$  or  $R_1 = 0.24$  AU, while for  $\Delta t_a = 39$  min the maximum distance is  $R_2 = c \times \Delta t$  or  $R_2 = 4.67$  AU. **(4 points)**

Since the orbits do not intersect and the  $R_2$  exceeds by far 1 AU, the orbit of *Seneca* is exterior to that of the *Earth*.  $R_1$  corresponds to the minimum relative distance of the two bodies (i.e. at opposition), while  $R_2$  corresponds to the maximum relative distance (i.e. at conjunction).

If  $q = a \times (1 - e)$  is the perihelion and  $Q = a \times (1 + e)$  the aphelion distance of *Seneca*, then  $R_1 = q - 1$  AU (the minimum distance of *Seneca* from the *Sun* minus the semi-major axis of the *Earth's* orbit), while  $R_2 = Q + 1$  AU (the maximum distance of *Seneca* from the *Sun* plus the semi-major axis of the *Earth's* orbit). **(6 points)**

Thus,

$$\begin{aligned} a_{\text{Sen}} (1 - e_{\text{Sen}}) &= 1 + R_1 = 1.24 \text{ AU} \\ a_{\text{Sen}} (1 + e_{\text{Sen}}) &= R_2 - 1 = 3.67 \text{ AU} \end{aligned} \quad \begin{aligned} &\text{from which one finds } a_{\text{Sen}} \sim 2.46 \text{ AU} \\ &\text{and } e_{\text{Sen}} \sim 0.49 \end{aligned} \quad \begin{aligned} &\mathbf{(2 \text{ point})} \\ &\mathbf{(2 \text{ points})} \end{aligned}$$

(figures rounded to 2 dec. digits)

- (2) The period,  $T_{\text{Sen}}$ , of *Seneca's* orbit can be found by using Kepler's 3<sup>rd</sup> law for *Seneca* and the *Earth* (ignoring their small masses, compared to the Sun's):

$$a_{\text{Sen}}^3 / T_{\text{Sen}}^2 = a_{\text{Earth}}^3 / T_{\text{Earth}}^2 = 1 \quad \mathbf{(6 \text{ points})}$$

from which one finds  $T_{\text{Sen}} \sim 3.87$  yr **(2 point)**

(Alternatively one can use Kepler's law for *Seneca* only and use natural units for the mass of the *Sun* and G, i.e. [mass] = 1  $M_{\text{Sun}}$ , [t] = 1 yr and [r] = 1 AU, in which case  $T_{\text{Sen}} = a_{\text{Sen}}^{3/2} = 3.86$  yr) **(8 points)**

Assuming non-retrograde orbit, the synodic period,  $T_{\text{syn}}$  of *Seneca* and *Earth* is given by



$$1/T_{\text{syn}} = 1/T_{\text{Earth}} - 1/T_{\text{Sen}} \text{ (Seneca is superior)} \quad (6 \text{ points})$$

$$\text{which gives } T_{\text{syn}} = 1.35 \text{ yr} \quad (2 \text{ point})$$

(3) From Kepler's law, we can calculate the mass of the *Sun*

$$4\pi^2 a_{\text{Sen}}^3 / T_{\text{Sen}}^2 = G M_{\text{Sun}}, \text{ i.e. } M_{\text{Sun}} \sim 1.988 \times 10^{30} \text{ kg} \quad (\text{depending on accuracy}) \quad (4 \text{ points})$$

Then, using Kepler's 3<sup>rd</sup> law for *Seneca* and *Jupiter*, one gets

$$(a_{\text{Jup}}^3 / T_{\text{Jup}}^2) / (a_{\text{Sen}}^3 / T_{\text{Sen}}^2) = (M_{\text{Sun}} + M_{\text{Jup}}) / M_{\text{Sun}} = 1 + x \quad (4 \text{ points})$$

where  $x =$  is the mass ratio of *Jupiter* to the *Sun*. Then, solving for  $x$  we get (depending on the accuracy used in determining the elements of *Seneca's* orbit)

$$x = 0.0016, \text{ thus } M_{\text{Jup}} = 3.2 \times 10^{27} \text{ kg} \quad (4 \text{ points})$$

### Question 3

(1) Using the *virial theorem* for an isolated, spherical system, i.e, that  $-2\langle K \rangle = \langle U \rangle$ , where “ $K$ ” is the average kinetic energy and “ $U$ ” is the average potential energy of the system, determine an expression for the total mass of a cluster of galaxies if we know the radial velocity dispersion,  $\sigma$ , of the cluster’s galaxy members and the cluster’s radius,  $R$ . Assume that the cluster is isolated, spherical, has a homogeneous density and that it consists of galaxies of equal mass.

(2) Find the *virial mass*, i.e. the mass calculated from the *virial theorem*, of the Coma cluster, which lies at a distance of 90 Mpc from us, if you know that the radial velocity dispersion of its member galaxies is  $\sigma_{v_r} = 1000 \text{ km/s}$  and that its angular diameter (on the sky) is about  $4^\circ$ .

(3) From observations, the total luminosity of the galaxies comprising the cluster is approximately  $L = 5 \times 10^{12} L_\odot$ . If the mass to luminosity ratio,  $M/L$ , of the cluster is  $\sim 1$  (assume that all the mass of the cluster is visible mass), this should correspond to a total mass  $M \sim 5 \times 10^{12} M_\odot$  for the mass of the cluster. Give the ratio of the luminous mass to the total mass of the cluster you derived in question (2).

**Answer:**

(1) Using the virial theorem for our isolated, spherical system of  $N$  galaxies of mass  $m$ , each, we get

$$-2\langle K \rangle = \langle U \rangle \Rightarrow -\frac{2}{N} \sum_1^N \frac{1}{2} m_i u_i^2 = \langle U \rangle \Rightarrow -\frac{m}{N} \sum_1^N u_i^2 = \frac{U}{N} \quad (3 \text{ point})$$

where  $\frac{1}{N} \sum_1^N u_i^2 \approx \langle u^2 \rangle = \langle u_r^2 \rangle + \langle u_\theta^2 \rangle + \langle u_\phi^2 \rangle$ , where  $u_r$ ,  $u_\theta$  and  $u_\phi$  are the radial velocity and the two perpendicular velocities on the plane of the sky of the members of the cluster. (6 points)



Assuming that  $\langle u_r^2 \rangle \sim \langle u_\theta^2 \rangle \sim \langle u_\phi^2 \rangle$ , we have

$$-3m\langle u_r^2 \rangle = U / N = \left(-\frac{3GM^2}{5R}\right) / N = -\frac{3GMm}{5R} \Rightarrow M \approx \frac{5R\langle u_r^2 \rangle}{G} \quad (13 \text{ points})$$

(where we used the gravitational potential of a spherical homogeneous mass M enclosed within radius R)

Alternatively the student can give a rougher order of magnitude estimate

$$-2\langle K \rangle = \langle U \rangle \Rightarrow -2 \cdot \frac{1}{2} \langle u_r^2 \rangle \approx -\frac{GM}{R} \text{ etc. If they do not use the exact formula:} \quad (11 \text{ Points})$$

(2) From the result of the previous question we have  $M \approx \frac{5R\langle u_r^2 \rangle}{G}$ , where

$$\sqrt{\langle u_r^2 \rangle} = \sigma = 1000 \text{ km s}^{-1} \quad (8 \text{ points})$$

The angular diameter of the cluster is  $\phi = 4^\circ$  at a distance of  $d = 90 \text{ Mpc}$ . Therefore the diameter D of the cluster in Mpc is calculated from:

$$\tan \phi = \frac{D}{d} \Rightarrow \phi(\text{rad}) \approx \frac{D}{d} \Rightarrow D \approx 4 \times \frac{\pi}{180} \times 90 \text{ Mpc} \approx 6.3 \text{ Mpc} \Rightarrow R \approx 3 \text{ Mpc} \quad (11 \text{ points})$$

$$\text{Therefore } M \approx \frac{5R\langle u_r^2 \rangle}{G} = 3.6 \times 10^{15} M_\odot \quad (3 \text{ point})$$

$$(3) \frac{M_{\text{virial}}}{L_{\text{galaxies}}} = \frac{3.6 \times 10^{15} M_\odot}{5 \times 10^{12} L_\odot} = 720 \frac{M_\odot}{L_\odot} \text{ This is obviously much larger from } \frac{M_\odot}{L_\odot} = 1 \text{ which is found from}$$

the visible mass of the cluster. Thus  $\frac{M}{M_{\text{virial}}} = \frac{1}{720}$ . (6 points)