Solutions

Solution 1:

Schwarzschild radius of a black hole with mass M is

$$R = \frac{2GM}{c^2}$$ (4 points)

Then the mass density can be estimated as

$$\rho = \frac{M}{\frac{4}{3}\pi\frac{8G^3M^3}{c^6}} = \frac{3c^6}{32\pi}\frac{1}{G^3M^2}$$ (2 points)

$$= \frac{3 \times (3.00 \times 10^8)^6}{32 \times 3.1416}\frac{1}{(6.67 \times 10^{-11})^3(10^8 \times 1.99 \times 10^{30})^2}$$

$$= 1.85 \times 10^3\ kg\ m^{-3}$$ (4 points)

Solution 2:

To calculate flux of a $m = 6$ star we use the Sun as standard candle

$$m_1 - m_2 = -2.5\log\frac{f_1}{f_2}$$ (1 point)

$$6 - (-26.8) = -2.5\log\frac{f_1}{1.37 \times 10^3}$$

$$f_1 = 1.04 \times 10^{-10}(w/m^2)$$ (3 points)

We need to know how much energy a visual photon has (at 550 nm)

$$E_p = h\nu = \frac{hc}{\lambda}$$

$$= \frac{6.63 \times 10^{-34} \times 3.00 \times 10^{8}}{550 \times 10^{-9}}$$

(3 points)

$$= 3.62 \times 10^{-19}\, J$$

Then number of photon which arrive to our eye per second is

(3 points)

$$N = \frac{f_1 \pi r_e^2}{E_p} = \frac{1.04 \times 10^{-10}}{3.62 \times 10^{-19}} \times 3.1416 \times 0.003^2 = 8 \times 10^3 \ s^{-1}$$

Solution 3:

We must compare the jumping speed of a normal human with escape velocity of the planet .
A normal human can jump up to 50 cm then his initial velocity is

$$v = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 0.5}$$
$$v = 3.13\, ms^{-1}$$

(3 points)

Comparing this velocity with escape velocity of the planet

$$v = \sqrt{\frac{2GM}{R}}$$

(3 points)

$$v^2 = \frac{2GM}{R} = \frac{2G}{R}\frac{4\pi}{3}\rho R^3$$

$$= \frac{8\pi G}{3}\rho R^2$$ (2 points)

$$R^2 = \frac{3v^2}{8\pi G\rho}$$

$$R = 2 \times 10^3\ m$$ (2 points)

Solution 4:

The zenith angle of the sun at summer solstice will be

$$z_s = \phi - 23.5 = 12.5$$ (1.5 points)

And in the winter solstice

$$z_w = \phi + 23.5 = 59.5$$ (1.5 points)

Figures shows that in summer solstice we have

$$\tan(z_s) = \frac{x}{h} = 0.22$$ (1.5 points)

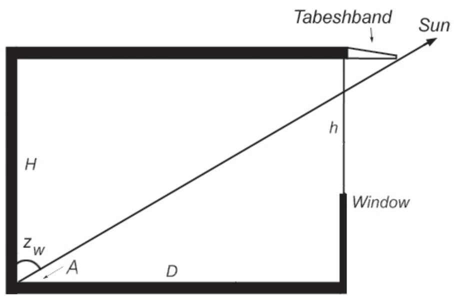
And in the winter solstice

$$\tan(z_w) = \frac{D + x}{H} = 1.70$$ (1.5 points)

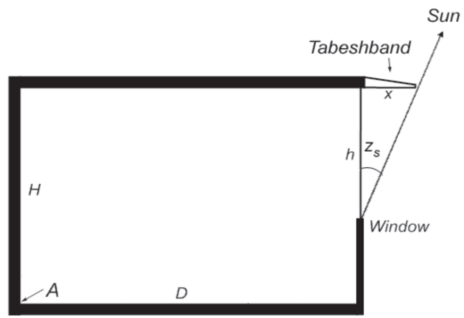
Then

$$x = 1.70H - D = 0.60\ m$$ (2 points)

$$h = 2.73\ m$$ (2 points)



Winter solstice



Summer solstice

Solution 5:

To calculate accurate value for minimum declination for circumpolar stars two major effect must be considered.

1. Refraction in earth atmosphere, which is 34′ at horizon. (3 points)
2. Horizon depression which is

$$\text{Cos}\,\theta = \frac{R}{R+h} = \frac{6370.8}{6370.8+5.6} \Longrightarrow \theta = 2°24'$$ (3 points)

Then

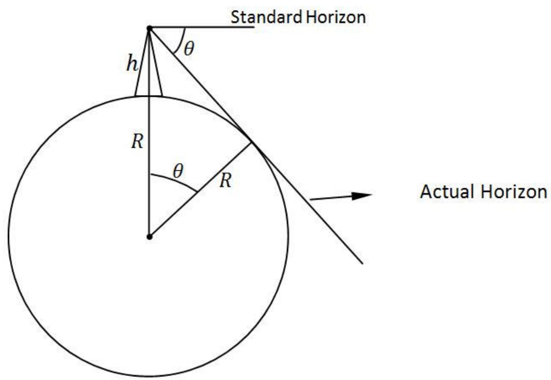
$$\delta_{min} = 90 - Latitude - Refraction - Horizon\ depression$$

(2 points)

$$= 90 - 35°57' - 34' - 2°24'$$

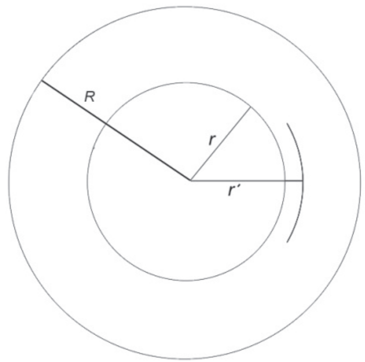
(2 points)

$$\Longrightarrow \delta_{min} = 51°5'$$



Solution 6:

To solve this problem we must calculate gravitational potential at the center of the cloud, letting $\phi(\infty) = 0$. For a uniform density and spherical mass distribution we have



$$\frac{1}{2}\,m\,v^2(r=R) - \frac{GMm}{R} = E = \frac{1}{2}mv^2(r=0) + \varphi(0)$$ (4 points)

$$v(r=R) = 0 \qquad \& \qquad v(r=0) = \sqrt{\frac{GM}{R}}$$ (1 point)

$$\varphi(0) = -\frac{1}{2}mv_0^2 - \frac{GmM}{R}$$ (1 point)

$$\varphi(0) = \frac{-1}{2}m\left(\sqrt{\frac{GM}{R}}\right)^2 - \frac{GmM}{R}$$ (2 points)

$$\varphi(0) = \frac{-3}{2} \times \frac{GMm}{R}$$

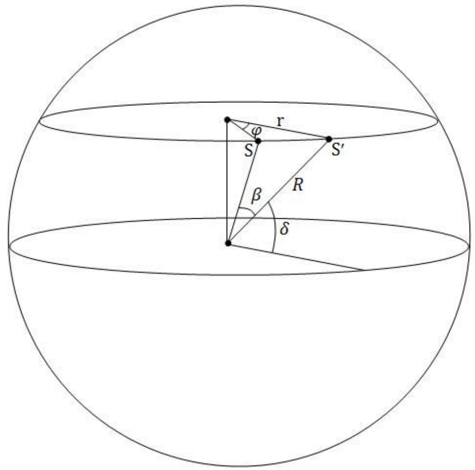
To escape from the cloud, the particle should have total energy equal to zero

$$E = \frac{1}{2}mv_e{}^2 + \phi(r=0) = \frac{1}{2}mv_e{}^2 - \frac{3}{2}\left(\frac{GMm}{R}\right) = 0$$

$$v_e{}^2 = \frac{3GM}{R} \quad \rightarrow \quad v_e = \sqrt{\frac{3GM}{R}}$$ (2 points)

Solution 7:

Figure shows that if FOV of telescope is β then we have :



$$\beta = \phi cos\delta$$ (4 points)

As the earth rotate, Vega moves through the FOV with constant angular velocity of the earth

$$\omega = \frac{2\pi}{86164} = 7.29 \times 10^{-5}(rad/s)$$ (2 points)

$$\phi = \omega t = 7.29 \times 10^{-5} \times 5.3 \times 60 = 0.023\ (\text{rad})$$

$$FOV = \beta = \phi cos\delta = 0.023cos39^{\circ} = 0.018\ (\text{rad}) \simeq 62\ min$$ (4 points)

Solution 8:

$$\frac{1}{2}Mv_{esc}^2 = \frac{1}{2}M\frac{2GM}{R}$$ (4 points)

The velocity must be smaller than the escape velocity ($v_{esc} \approx \sqrt{2}v_{rms}$) and since the problem is an estimation, any velocities smaller than escape velocity is accepted and therefore the escape velocity can be replaced by v_{rms}.

$$Mv_{rms}^2 = \frac{GM^2}{R}$$ (2 points)

$$M = \frac{Rv_{rms}^2}{G} = \frac{20 \times 3.09 \times 10^{16} \times 9 \times 10^6}{6.67 \times 10^{-11}} = 8.3 \times 10^{34} Kg$$

$$M = 4.2 \times 10^4 M_\odot$$ (4 points)

Solution 9:

In the figure, S is the Sun and R_0 and V_0 are Sun distance and velocity. The distance and velocity of star P is denoted by R and $V = V_0$. The radial velocity of star P respect to the Sun is

$$V_r = V\cos\alpha - V_0 \sin l = V_0(\cos\alpha - \sin l)$$ (4 points)

In SCP triangle we have

$$\frac{\sin l}{R} = \frac{\cos\alpha}{R_0} \Rightarrow \cos\alpha = \frac{R_0}{R}\sin l$$ (1 point)

So

$$V_r = V_0\left(\frac{R_0}{R} - 1\right)\sin l$$ (2 points)

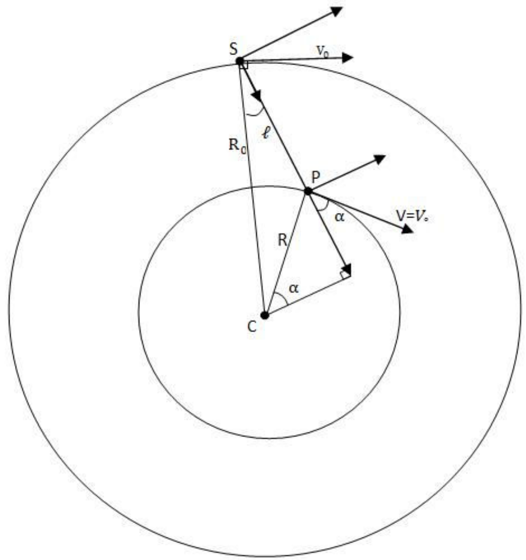
$$\frac{R_0}{R} - 1 = \frac{V_r}{V_0 sinl}$$

$$\frac{R_0}{R} = \frac{V_r + V_0 sinl}{V_0 \sin l}$$

$$R = R_0 \frac{V_0 \sin l}{V_r + V_0 sinl}$$ (2 points)

$$R = 8 \times 10^3 \frac{250 sin 15^\circ}{100 + 250 sin 15^\circ}$$

$$R = 3 \times 10^3\ pc$$ (1 point)



Solution 10:

Because of high conductivity of plasma inside the star, flux of magnetic field will be conserved through contraction then:

$$4\pi R^2 B = 4\pi R_n^2 B_n$$ (6 points)

Where R_n and B_n are radius and magnetic field of neutron star, thus:

$$B_n = \left(\frac{R}{R_n}\right)^2 B = \left(\frac{4 \times 6.96 \times 10^5}{20}\right)^2$$ (2 points)

$$B_n = 1.93 \times 10^{10}\ Gauss$$

$$= 1.93 \times 10^6\ T$$ (2 points)

Solution 11:

In a flat universe

$$\rho = \rho_c = \frac{3H^2}{8\pi G}$$ (4 points)

$$H = 75\ kms^{-1}Mpc^{-1} = 2.4 \times 10^{-18} s^{-1}$$ (2 points)

$$\rho_c = 1.1 \times 10^{-26}\ kgm^{-3}$$ (2 points)

$$n_v = \frac{0.25\ \rho_c}{10^{-5}\ m_e} = 3 \times 10^8\ m^{-3}$$ (2 points)

Solution 12:

The rate of change of solar mass could be estimated from solar luminosity:

(2 points)

$$L_\odot = -\frac{\Delta E}{\Delta t} = -\frac{\Delta M c^2}{\Delta t} = -\dot{M} c^2$$

$$\dot{M} = -\frac{L_\odot}{c^2} = -\frac{3.83 \times 10^{26}}{(3.00 \times 10^8)^2} = -4.26 \times 10^9 (kgs^{-1})$$

Newton second law will give us:

$$\frac{v^2}{r} = \frac{GM}{r^2} \Longrightarrow v^2 = \frac{GM}{r}$$ (2 points)

where v and r are orbital velocity and orbital radius of the Earth.
From conservation of angular momentum:

$$l = rmv \Longrightarrow v = \frac{l}{mr}$$ (2 points)

Where m and l are the Earth mass and angular momentum which are constant:

$$\frac{l^2}{m^2r^2} = \frac{GM}{r} \Rightarrow r = \frac{l^2}{GMm^2} \Rightarrow \dot{r} = -\frac{l^2}{Gm^2}\frac{\dot{M}}{M^2} = -\frac{r^2m^2v^2}{GM^2m^2}\dot{M}$$

$$\Rightarrow \dot{r} = -\frac{(GM/r)r^2}{G}\frac{\dot{M}}{M^2} = -r\frac{\dot{M}}{M} \Rightarrow \frac{\dot{r}}{r} = -\frac{\dot{M}}{M} \Rightarrow \dot{r} = -r\frac{\dot{M}}{M}$$

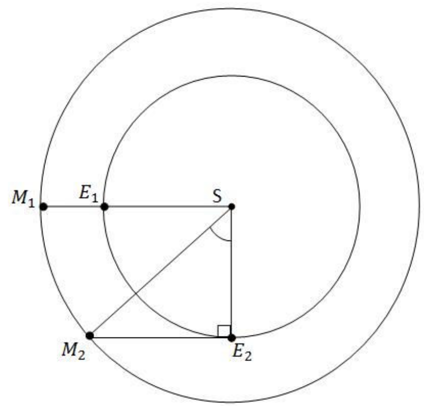
$$\Rightarrow \dot{r} = -\frac{1.50 \times 10^{11} \times 4.26 \times 10^9}{1.99 \times 10^{30}} = 3.21 \times 10^{-10} (m/s)$$

(2 points)

$$\Delta r = 3.19 \times 10^{-10} \times 100 \times 86400 \times 365.24 = 1.01 m \quad (for\ 100\ years)$$ (2 points)

Solution 13:

We know the orbital period of Mars then the angle $\angle M_1SM_2$ can be determined simply



$$\angle M_1SM_2 = \frac{106}{687} \times 360 = 55.5°$$ (2 points)

By the same way we determine $\angle E_1SE_2$:

$$\angle E_1SE_2 = \frac{106}{365} \times 360 = 104.5°$$ (2 points)

Then angle $\angle M_2SE_2$ is:

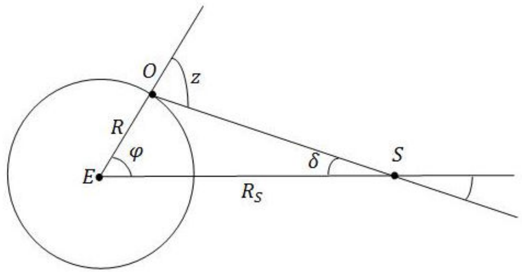
$$\angle M_2SE_2 = 104.5 - 55.5 = 49.0°$$ (2 points)

$$\cos(\angle M_2SE_2) = \frac{SE_2}{SM_2}$$ (2 points)

$$r_{mars} = \frac{SM_2}{SE_2} = \frac{1}{\cos(\angle M_2SE_2)} = \frac{1}{\cos(49°)} = 1.52\ AU$$ (2 points)

Solution 14:

In the figure, the observer is at *O* and the satellite is in *S* the angle $\angle EOS$ will be



$$\angle EOS = 180 - z = 180 - 46.0 = 134°$$ (2 points)

$$\delta = 180 - \varphi - \angle EOS = 10.4°$$ (3 points)

In *EOS* triangle we have:

$$\frac{R_S}{\sin(\angle EOS)} = \frac{R}{\sin\delta}$$ (3 points)

$$\frac{R_S}{R} = \frac{\sin(\angle EOS)}{\sin\delta} = 3.98$$ (2 points)

Solution 15:

Ignoring temperature variation on the stars surface, the brightness of star system will be proportional to projected surface of both stars on plane of the sky. Maximum brightness will occur when two stars are seen like figure 1 and minimum light will happen when one of the stars is in total eclipse and projected surface of the other is a circle with radius *b* (Figure 2). In maximum brightness

$$I_{max} \propto 2\pi ab$$ (3 points)

In minimum brightness

$$I_{min} \propto \pi b^2$$ (3 points)

So

$$\Delta m = -2.5\log\frac{I_{max}}{I_{min}} = -2.5\log\frac{2\pi ab}{\pi b^2} = -2.5\log 4$$ (2 points)

$$\Delta m = -1.5$$ (2 points)

Solution 16:

a) Total energy of the projectile is

$$E = \frac{1}{2}mv_\circ^2 - \frac{GMm}{R} = -\frac{GMm}{2R} < 0$$

$E < 0$ means that orbit might be ellipse or circle. As $\theta > 0$, the orbit is an ellipse.
Total energy for an ellipse is

$$E = -\frac{GMm}{2a}$$

Then

$$a = R$$ (7 points)

b) In figure (1) we have

$$OA + O'A = 2a$$

$$O'A = a$$

So

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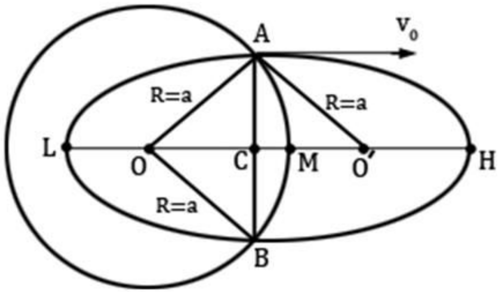


Figure (1)

In OAO' triangle it is obvious that

$$OC = CO'$$

Then C must be the center of the ellipse with the initial velocity vector $v_\circ$ parallel to the ellipse major-axis (LH).

In figure (2)

$$HM = CH - CM = a - (R - R\sin\theta) = R - R + R\sin\theta = R\sin\theta = \frac{R}{2}$$ (15 points)

c) Range of the projectile is $\widehat{AB}$

$$\widehat{AB} = 2\left(\frac{\pi}{2} - \theta\right)R = (\pi - 2\theta)R = \frac{2\pi}{3}R$$

(6 points)

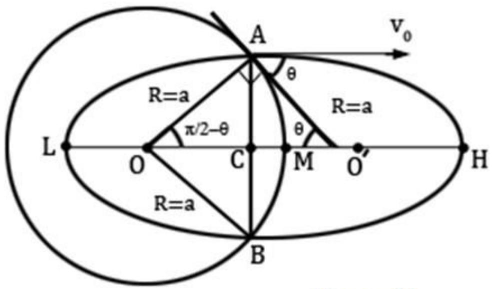


Figure (2)

d) Start with ellipse equation in polar coordinates

$$r = \frac{a(1-e^2)}{1+e\cos\varphi}$$

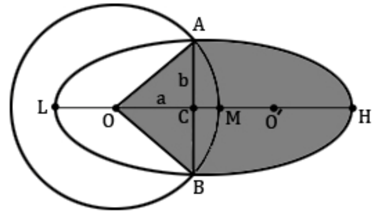
For point A

$$R = \frac{R(1-e^2)}{1-e\cos(\frac{\pi}{2}+\theta)}$$

$$e = \sin\theta = \frac{1}{2}$$

(5 points)

e) Using Kepler's second law



$$\frac{\Delta S}{S_0} = \frac{\Delta T}{T}$$

$$\Delta S = S_{AOBH} = S_{\Delta AOB} + \frac{S_0}{2}$$

$$= 2 \times \frac{\text{bae}}{2} + \frac{\pi\text{ab}}{2} = \text{bae} + \frac{\pi\text{ab}}{2}$$

$$\frac{\Delta S}{S} = \frac{\text{bae} + \frac{\pi\text{ab}}{2}}{\pi ab} = \frac{e + \frac{\pi}{2}}{\pi} = \frac{0.5 + \frac{\pi}{2}}{\pi}$$

Kepler's third law

$$T = \sqrt{\frac{4\pi^2 R^3}{GM}} = 84.5\, min$$

$$\Delta T = T \times \frac{0.5 + \frac{\pi}{2}}{\pi} = 55.7\, min$$

(12 points)

Solution 17:

a) Relation between the apparent and absolute magnitude is given by

$$m = M + 5log\left(\frac{d}{10}\right)$$ (3 points)

where d is in terms of parsec. Substituting $m = 18$ and $M = -0.2$, results in

(5 points)

$$d = 4.37 \times 10^4\ pc$$

b) Adding the term for the extinction, changes the magnitude distance relation as follows

$$m = M + 0.7x + 5\log(100x)$$

where x is given in terms of kilo parsec. To have a rough value for x, after substituting m and M, this equation reduces to

$$8.2 = 0.7x + 5\log(x)$$ (6 points)

To solve this equation, we examine

$$x = 5, 5.5, 6, 6.5$$

where the best value is obtained roughly $x \cong 6.1\ kpc$. (8 points)

c) For a solid angle Ω, the number of observed red clump stars at the distance in the range of x and $x + \Delta x$ is given by

$$\Delta N = \Omega\, x^2\, n(x) f\, \Delta x$$

So the number of stars observed in Δx is given by

$$\frac{\Delta N}{\Delta x} = \Omega\, x^2\, n(x) f$$

(6 points)

From the relation between the distance and apparent magnitude we have

$$m_1 = M + 5log\left(\frac{x}{10}\right)$$

$$m_2 = M + 5log\left(\frac{x + \Delta x}{10}\right)$$

$$\Delta m = 5log\left(\frac{x + \Delta x}{x}\right)$$

$$\Delta m = 5log\left(1 + \frac{\Delta x}{x}\right)$$

$$\Delta m = \frac{5}{ln10}\, ln\left(1 + \frac{\Delta x}{x}\right)$$

$$\Delta m = \frac{5}{ln10}\left(\frac{\Delta x}{x}\right)$$

Replacing Δx with Δm, results in

$$\frac{\Delta N}{\Delta m} = \frac{\Delta N}{\Delta x} \times \frac{\Delta x}{\Delta m}$$

So the number of stars for a given magnitude is obtained by (5 points)

$$\frac{\Delta N}{\Delta m} = \frac{\Omega ln10}{5} n(x) x^3 f$$

Finally we substituting x in terms of apparent magnitude using $\mathrm{x} = 10^{\frac{\mathrm{m}+5.2}{5}}$.

In the case of no extinction, we are able to observe the Galaxy beyond the center. So $\frac{dN}{dm}$ has two terms in

$x < R_0$ and $x > R_0$. The relation between x and r for these two cases are

$$x = R_0 - r \qquad x < R_0$$

(6 points)

and

$$x = R_0 + r \qquad x > R_0$$

So in general we can write $\frac{\Delta N}{\Delta m}$ as

(6 points)

$$\frac{\Delta N}{\Delta m} = \frac{\Omega ln10}{5} n_0 \exp\left(\frac{10^{\frac{m-5.2}{5}}}{R_d}\right) \times 10^{\frac{3(m-5.2)}{5}} f \qquad x < R_0$$

$$\frac{\Delta N}{\Delta m} = \frac{\Omega ln10}{5} n_0 \exp\left(\frac{2R_0}{R_d}\right) \exp\left(-\frac{10^{\frac{m-5.2}{5}}}{R_d}\right) \times 10^{\frac{3(m-5.2)}{5}} f\, \Theta(x_0 - x) \quad x > R_0$$

where $\Theta(x)$ is the step function and x_0 is the maximum observable distance.