Astro Round 2 2025 Paper Solutions and Marking Guidelines

Note for markers:

- Answers to two or three significant figures are generally acceptable. The solution may give more in order to make the calculation clear. Units should be present on final answers when appropriate.
- There are multiple ways to solve some of the questions; please accept all good solutions that arrive at the correct answer. Students getting the answer in a $\boxed{\text{box}}$ will get all the marks available for that calculation / part of the question (as indicated in red), so long as there are no unphysical / nonsensical steps or assumptions made (students may not explicitly calculate the intermediate stages and should not be penalised for this so long as their argument is clear).

Q1 – Gravity Train [30 marks]

a. In this part of the question, you should ignore any effects of the Earth's rotation. Assume the gravity train always starts from rest at point A.

i. Show that the gravity train using the route AOB would undergo simple harmonic motion (SHM) and hence that the time t_{AB} to get from A to B is a function of g_0 and $R_\oplus$, where g_0 is the gravitational field strength at the surface of the Earth. Hence show t_{AB} is about 42 minutes.

Given a uniform density Earth, the mass enclosed by a sphere of radius r (with $r \leq R_\oplus$)

$$M_{enclosed} = M_\oplus \frac{r^3}{R_\oplus^3}$$

Applying Newton's shell theorem to get that the acceleration experienced is $g(r)$

$$a = -\frac{GM_{enclosed}}{r^2} = -\frac{G}{r^2}\frac{M_\oplus r^3}{R_\oplus^3} = -\frac{GM_\oplus}{R_\oplus^3} r \quad \therefore \text{SHM with } \omega^2 = \frac{GM_\oplus}{R_\oplus^3} = \frac{g_0}{R_\oplus} \quad [1] \qquad [1]$$

[Expect some comparison to the general SHM equation ($a = -\omega^2 r$) or written argument with the SHM definition to get this mark. Do not penalise students for dropping the $\oplus$ symbol. Allow ω or ω^2 expressed in any valid combination of constants]

We know that the period of SHM will be $T = \frac{2\pi}{\omega}$ and hence since A→B is just one-way

$$t_{AB} = \frac{T}{2} = \frac{\pi}{\omega} = \boxed{\pi\sqrt{\frac{R_\oplus}{g_0}}} \qquad [1] \qquad [1]$$

[Must be as a function of $R_\oplus$ and g_0 for this mark. If they leave it as $t_{AB} = \pi\sqrt{\frac{R_\oplus^3}{GM_\oplus}}$ or in the form needed for part iv. ($t_{AB} = \pi\sqrt{\frac{3}{4G\pi\rho}}$) then award 0.5 marks]

Substituting in the numbers

$$t_{AB} = \pi\sqrt{\frac{6.37\times10^6}{9.81}} = 2532 \text{ s} = \boxed{42.2 \text{ mins}} \qquad (= 42 \text{ mins } 12 \text{ secs}) \quad [1] \qquad [1]$$

[Must be in minutes (or mins and seconds) for this mark and $\geq$ 3 s.f (in mins). Accept answers that use an alternative expression for t_{AB} or round to 42.2 minutes or have a remainder that rounds to 11-12 seconds. If left in seconds award 0.5 marks]

ii. Compare this to the time taken by a satellite to go from A to B when orbiting in a circular orbit of radius R⊕.

Using the speed of an object in a circular orbit and half the circumference of the Earth

$$t = \frac{s}{v} = \frac{\pi R_\oplus}{\sqrt{\frac{GM_\oplus}{R_\oplus}}} = \pi\sqrt{\frac{R_\oplus^3}{GM_\oplus}} \text{ (= 42.2 minutes)} \quad \therefore \boxed{t \text{ is the same}} \quad [1] \quad [1]$$

[Do not penalise them for finding a few seconds discrepancy, depending on the values of constants they are using, and so simply stating the times are very close / very similar.

Allow use of Kepler's 3rd Law as an alternative route to get $t = \frac{T}{2} = \frac{1}{2}\sqrt{\frac{4\pi^2}{GM_\oplus}R_\oplus^3} = \pi\sqrt{\frac{R_\oplus^3}{GM_\oplus}}$.

Allow ecf on this question if their t is right but is different to their t_{AB} from part i.]

iii. What is the largest speed achieved by the gravity train? What is this the same as?

Given it follows SHM

$$v_{max} = \omega R_\oplus = \sqrt{\frac{GM_\oplus}{R_\oplus^3}} \times R_\oplus = \sqrt{\frac{GM_\oplus}{R_\oplus}} \left(= \sqrt{g_0 R_\oplus}\right) = \boxed{7.91 \text{ km s}^{-1}} \quad [1] \quad [1]$$

[Accept answers that round to 7.90 km s^{-1} or 7.91 km s^{-1}, based upon the values of constants being used. Allow use of energy conservation where $GPE + KE = E_{tot}$ and so (following the method of Astro R1 2024 Q2 for the GPE) you have the expression $\left(-\frac{3GM_\oplus m}{2R_\oplus} + \frac{GM_\oplus m}{2R_\oplus^3}r^2\right) + \frac{1}{2}mv^2 = -\frac{GM_\oplus m}{R_\oplus}$ which you evaluate at $r = 0$. An alternative energy conservation route is that the work done by gravitational forces is equal to the change in KE of the gravity train, and so (using that $g \propto r$ inside a uniform density planet) the work done is $F_{G,average}R_\oplus = mg_{average}R_\oplus = \frac{1}{2}mg_0R_\oplus$ etc.]

This is the same as $\boxed{\text{the circular orbital velocity for an orbit with radius } R_\oplus}$ [1] [1]

[Allow ecf with this mark e.g. if a student finds $v_{max} = \sqrt{\frac{2GM_\oplus}{R_\oplus}} = 11.2 \text{ km s}^{-1}$, and correctly identifies it as the escape velocity, give this mark]

iv. Gravity trains could potentially be used in the future to mine the asteroid belt, as a way to haul raw material to a central refining point or port for launch back to Earth. The dwarf planet Ceres has an average density of 2.16 g cm^{-3}. What would be t_{AB} on Ceres?

Since $\rho = \frac{M}{\frac{4}{3}\pi R^3}$ we can express t_{AB} as a function of ρ

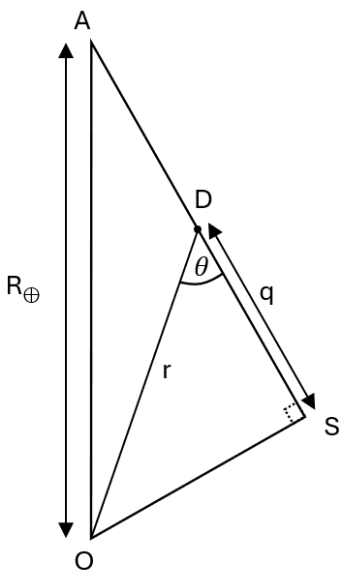
$$t_{AB} = \pi\sqrt{\frac{R^3}{GM}} = \pi\sqrt{\frac{R^3}{G \times \frac{4}{3}\pi R^3 \rho}} = \pi\sqrt{\frac{3}{4G\pi\rho}} \quad [1]$$

Substituting in the numbers using that 2.16 g cm^{-3} = 2160 kg m^{-3}

$$t_{AB} = \pi\sqrt{\frac{3}{4\pi \times 6.67 \times 10^{-11} \times 2160}} = \boxed{4044 \text{ s}} \text{ (= 67.4 mins = 1.12 hours)} \quad [1] \quad [2]$$

[Since unit not specified in the question, accept answer in seconds, minutes or hours. Accept students showing that $t_{AB} \propto \rho^{-1/2}$ and then using $\rho_\oplus = 5.514 \text{ g cm}^{-3}$ and their earlier answer in a scaling relationship]

v. The route ASC is an arbitrary chord through the planet. Given the gravity train is constrained to travel along frictionless rails, determine the time t_{AC} to go along this route. Compare your answer to t_{AB}. You may wish to make use of the notation in Fig 1.



Consider when the gravity train is at point D, which is a distance r from O and q from S (using the Figure 1 notation)

The gravitational acceleration the gravity train experiences is the component of g in the direction of S

$a = -g\cos\theta = -\frac{GM_{enc}}{r^2}\cos\theta = -\frac{g_0}{R_\oplus} r\cos\theta$ [1]

But since $\cos\theta = \frac{q}{r} \therefore a = -\frac{g_0}{R_\oplus}q$

This is SHM with the same ω as for the route AB

$\therefore \boxed{t_{AC} = t_{AB}}$ [1] [2]

[Allow any valid method that demonstrates this equivalence, or if a student uses numerical comparisons. Don't penalise if they just keep the algebraic expressions to magnitudes]

The article by Cooper was called "To Everywhere in 42 Minutes" and celebrated this fact that the smaller acceleration exactly balanced the shorter distance to travel. This only holds on the assumption of a uniform density Earth – using a realistic model for the density distribution of the Earth the time t_{AB} becomes $\sim$ 38 minutes and the position of C matters such that $38\text{ mins} < t_{AC} < 42\text{ mins}$ (with shorter chords having the longer times). Coincidentally, this shorter time for t_{AB} is very similar to what you would get if you assumed $a = g_0$ throughout and applied simple SUVAT to get $t_{AB} = 2\sqrt{\frac{2R_\oplus}{g_0}}$.

b. Cooper showed that the time taken to travel along a hypocycloid made by a circle of radius b was t_{hyp}. For all points A and C, $t_{hyp} \leq t_{AC}$. Again, ignore the Earth's rotation and assume the gravity train starts from rest at point A.

i. When $t_{hyp} = t_{AC}$, what is the value of b? What situation does this correspond to?

Given that $t_{AC} = t_{AB}$ we are looking at the situation

$$\frac{2\pi b}{R_\oplus}\sqrt{\frac{R_\oplus(R_\oplus - b)}{b g_0}} = \pi\sqrt{\frac{R_\oplus}{g_0}} \quad \therefore \boxed{b = \frac{R_\oplus}{2}}$$ [1] [1]

This means that the chord will have to pass through the centre

$\therefore \boxed{\text{the path is along a diameter}}$ (i.e. $AC = AB$) [1] [1]

ii. Consider a chord between A and C with a maximum depth below the Earth's surface of d, and the hypocycloid has a maximum depth of αd where $\alpha > 1$. If C is chosen such that $t_{hyp} = \frac{1}{2} t_{AC}$, find the latitude, φ, of C if A is the North Pole (for which $\varphi = 90°$) and hence find α.

Modifying what we did in the previous part to work out the new b

$$\frac{2\pi b}{R_\oplus}\sqrt{\frac{R_\oplus(R_\oplus - b)}{b g_0}} = \frac{\pi}{2}\sqrt{\frac{R_\oplus}{g_0}}$$

Cancelling the π and g_0 and rearranging gets the quadratic

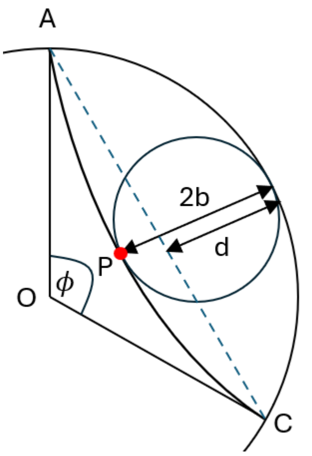
$$\therefore 16b^2 - 16R_\oplus b + R_\oplus^2 = 0$$ [1]

The solutions are

$$\frac{b}{R_\oplus} = \frac{16 \pm \sqrt{16^2 - 4\times16\times1}}{32} = \frac{1}{2} \pm \frac{\sqrt{192}}{32} = \frac{1}{2} \pm \frac{8\sqrt{3}}{32} = \frac{1}{2} \pm \frac{1}{4}\sqrt{3}$$ [0.5]

We know that $b < \frac{1}{2}R_\oplus$ and so we need the **negative** solution

$$\therefore b = \left(\frac{1}{2} - \frac{1}{4}\sqrt{3}\right)R_\oplus \ (= 426.7 \text{ km})$$ [0.5]



It is clear that the arc length AC is equal to the circumference of the small circle

$$\therefore 2\pi b = R_\oplus \phi$$ [1]

$$\therefore \phi = \frac{2\pi b}{R_\oplus} = 2\pi\left(\frac{1}{2} - \frac{1}{4}\sqrt{3}\right) = 0.42 \text{ rad} = 24.1°$$ [0.5]

$$\therefore \varphi = 90° - 24.1° = \boxed{65.9° \text{ N}}$$ [0.5] [4]

[No penalty if they exclude the N so long as their value is positive]

The deepest part of both routes occurs at the midway point

$$\therefore d = R_\oplus - R_\oplus \cos\left(\frac{\phi}{2}\right) \ (= 140.5 \text{ km})$$ [1]

[Accept equivalent expressions e.g. $d = 2R_\oplus \sin^2\left(\frac{\phi}{4}\right)$]

Considering the geometry

$$2b = \alpha d$$ [1]

[If they forget that b is the radius not the diameter and so use $b = \alpha d$ then award 0.5 marks]

$$\therefore \alpha = \frac{2b}{d} = \frac{1 - \frac{1}{2}\sqrt{3}}{1 - \cos\left(\frac{\phi}{2}\right)} = \boxed{6.07}$$ [1] [3]

[Allow ecf from their chosen geometry e.g. if they get $\alpha = 3.04$ due to confusing diameter and radius in MP6]

c. Suppose the tunnel along the diameter AOB is now in the plane of the Earth's equator, and so rotational effects will have to be considered.

i. The centrifugal acceleration is greatest at the surface of the Earth. Show that it is < 1% of g_0, and so we are justified in ignoring it.

We can evaluate a_{Cent} at the Earth's surface and compare it to g_0

$$\frac{a_{Cent}}{g_0} = \frac{\Omega^2 R_\oplus}{g_0} = \frac{\left(\frac{2\pi}{60\times60\times24}\right)^2 \times 6.37\times10^6}{9.81} = \frac{0.0337}{9.81} = 3.43\times10^{-3} = \boxed{0.34\%} \quad [1] \quad [1]$$

[If they only calculate $a_{Cent} = 0.0337\ \mathrm{m\ s^{-2}}$ or if they forget to convert from fraction to percentage (i.e. leave their answer as $3.43\times10^{-3}\%$) then award only 0.5 marks. Expect to see a clear comparison made with 1%, whether by stating that $0.34\% < 1\%$ or that $0.0337\ \mathrm{m\ s^{-2}} < 0.0981\ \mathrm{m\ s^{-2}}$ (i.e. 1% of g_0), but no penalty if absent]

ii. Sketch how the Coriolis acceleration experienced by the gravity train varies as a function of r (as defined in Figure 1 as the distance from O) indicating numerical values on the y-axes that confirm we cannot treat it as negligible. You may find it easiest to reverse your r-axis so it goes from $R_\oplus$ (at A) to 0 to $-R_\oplus$ (at B). Treat eastwards as positive.

Using the given formula for the Coriolis force and that v varies with r following SHM

$$a_{Cor} = 2\Omega v = 2\Omega\omega\sqrt{R_\oplus^2 - r^2} \quad [1] \quad [1]$$

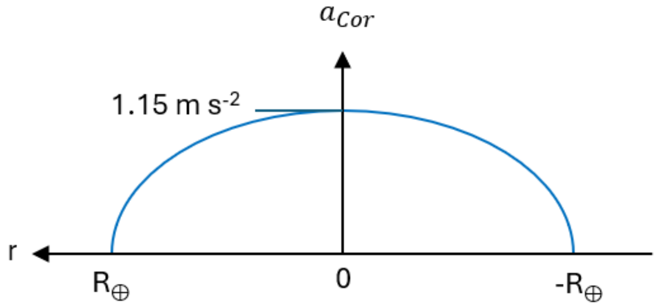
[Accept any equivalent form using $\omega = \sqrt{\frac{g_0}{R_\oplus}}$ or alternatives from a) i.]

$$\therefore \text{maximum value is } a_{Cor} = 2\Omega v_{max}$$

$$= 2\times\left(\frac{2\pi}{60\times60\times24}\right)\times 7905 = \boxed{1.15\ \mathrm{m\ s^{-2}}} \quad [1] \quad [1]$$

[This is about 12% of g_0 so we need to take it into consideration]

Sketch is of the upper half of an **ellipse** [Accept semicircle] [1] [1]



[No penalty if they decide not to reverse the r axis. If their sketch is two quarter ellipses with a discontinuity at $r = 0$, or is the bottom half of an ellipse, award 0.5 marks]

iii. If the train was originally on the West side of the tunnel and became detached from its rails the moment the journey started, find an expression for the West-East displacement as a function of time.

Again, taking advantage of the properties of SHM

$$a_{Cor} = 2\Omega v = 2\Omega\omega R_\oplus \sin\omega t \quad [1]$$

$$\therefore v_{Cor} = \int_0^t a_{Cor}\,\mathrm{d}t = 2\Omega R_\oplus(1-\cos\omega t) \quad [1]$$

$$\therefore s_{Cor} = \int_0^t v_{Cor}\,\mathrm{d}t = \boxed{2\Omega R_\oplus\left(t - \frac{1}{\omega}\sin\omega t\right)} \quad [1] \quad [3]$$

[Accept any algebraically equivalent expressions]

iv. If the distance between the top of the gravity train and the other side of the tunnel is 500 m (it's a very wide tunnel!), how much time will elapse before it crashes into the wall?

Using the given hint for the small angle approximation for $\sin x$

$$s_{Cor} = 2\Omega R_{\oplus}\left(t - \frac{1}{\omega}\left(\omega t - \frac{(\omega t)^3}{6}\right)\right) = 2\Omega R_{\oplus}\left(\frac{\omega^2}{6}t^3\right) = \frac{\Omega R_{\oplus}\omega^2 t^3}{3}$$

$$\therefore t = \sqrt[3]{\frac{3s_{Cor}}{\Omega R_{\oplus}\omega^2}} \quad \left(= \sqrt[3]{\frac{3s_{Cor}}{\Omega g_0}}\right)$$ [1]

$$\therefore t = \sqrt[3]{\frac{3\times 500}{\left(\frac{2\pi}{60\times 60\times 24}\right)\times 9.81}} = \boxed{128\text{ s}}$$ [1] [2]

[The first marking point is for a correctly rearranged equation for t OR for a successful usage of the small angle approximation]

v. How far below the surface did the train crash? Give your answer as a percentage of $R_{\oplus}$.

Finding the fraction of the distance travelled

$$\frac{R_{\oplus}-r}{R_{\oplus}} = \frac{R_{\oplus}-R_{\oplus}\cos\omega t}{R_{\oplus}} = 1 - \cos\omega t$$ [1]

$$\therefore \frac{R_{\oplus}-r}{R_{\oplus}} = 1 - \cos\left(\sqrt{\frac{9.81}{6.37\times 10^6}} \times 128\right) = 0.0126 = \boxed{1.26\%}$$ [1] [2]

[If they forget to put their calculator into radians they will get 0.000385% - in this case give only 1.5 marks total]

(2) a) $aM = k$ (constant)

$$\dot{a}M + a\dot{M} = 0$$

$$\dot{a}M = -a\dot{M}$$

$$\frac{\dot{a}}{a} = -\frac{\dot{M}}{M}$$

1 for recognising aM = constant.
1 for differentiating + rearranging
(2 for other convincing methods)
1 for semi-convincing arguments

(3) b) i) $$\frac{GM}{(a-R)^2} - \frac{GM}{a^2} = \frac{GM}{a^2}\left[\left(1-\frac{R}{a}\right)^{-2} - 1\right] \approx \frac{GM}{a^2}\left(1 + \frac{2R}{a} - 1\right)$$

1 for correct set-up

1 for correct expansion

(0.5 for $\frac{1}{1-\frac{2R}{a}}$)

$$= \frac{2GMR}{a^3}$$

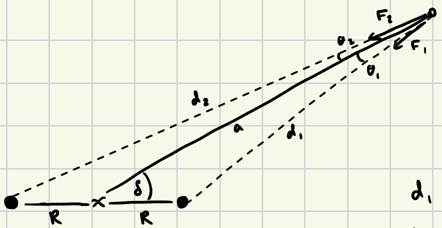
1 for simplification

2 for $\frac{2GMR}{a(a-R)^2}$

−0.5 for force (GMm)

(1) b) ii) More mass at edges of star OR larger bulges / more elliptical shape

(8) b) iii) $$M_T = \frac{kmR^3}{2a^3}$$



1 for correct τ eq (WITH CORRECT a, θ.)

$$\tau_1 = F_1 a \sin\theta_1 = \frac{GM_T m}{d_1^2} a \sin\theta_1$$

$$\frac{\sin\theta_1}{R} = \frac{\sin(\delta)}{d_1}$$ } 1 for correct θ expression

$$d_1 = \sqrt{R^2 + a^2 - 2Ra\cos\delta}$$ } 1 for correct d expression

$$\tau_1 = \frac{GM_T m R a \sin\delta}{(R^2+a^2-2Ra\cos\delta)^{3/2}} \approx \frac{GM_T m R a \sin\delta}{a^3\left(\frac{R^2}{a^2} + 1 - 2\frac{R}{a}\cos\delta\right)^{3/2}}$$

$$= \frac{GM_T m R \sin\delta}{a^2}\left(1 + 3\frac{R}{a}\cos\delta\right)$$

- 1 for getting rid of $\frac{R^2}{a^2}$ term (some attempt to simplify)
- 1 for Taylor expansion

$$\tau_2 = \frac{GM_T m R a \sin\delta}{(R^2+a^2+2Ra\cos\delta)^{3/2}} \approx \frac{GM_T m R \sin\delta}{a^2}\left(1 - 3\frac{R}{a}\cos\delta\right)$$ } 1 for τ_2

1 for this

$$\tau = \tau_2 - \tau_1 = -\frac{6GM_T m R^2 \sin\delta\cos\delta}{a^3} = -\frac{3GmR^2\sin(2\delta)}{a^3} \cdot \frac{kmR^3}{2a^3}$$

$$= -\frac{3Gm^2kR^5\sin(2\delta)}{2a^6}$$ } 1 for final answer (equivalent forms okay) −0.5 for with M_T

LENIENCE FOR OTHER EXPRESSIONS SHOULD BE GIVEN!

↳ but dock marks if wrong dimensions, or not simplified

MAX 4 IF CORRECT FOR PLANET ON STAR: $\sim \frac{Gkm^2R^3\sin\delta}{a^4}$

(5) b) iv) $L = rp\sin\phi = \underbrace{m\omega a^2}$

1 for realising $L = mvr = m\omega r^2$ (ie. $\phi = 90°$)

$$\omega^2 a = \frac{GM}{a^2} \Rightarrow \omega = \sqrt{\frac{GM}{a^3}}$$ } 1 for use of this to get rid of v/ω

$$L = m\sqrt{GMa}$$

$$\frac{dL}{dt} = \frac{m\sqrt{GM}}{2\sqrt{a}}\frac{da}{dt} = -\frac{3kGm^2R^5\delta}{a^6}$$ 1 for correct $\frac{dL}{dt}$

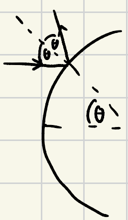
$$\Rightarrow \frac{1}{a}\frac{da}{dt} = -\frac{6\sqrt{G}kmR^5\delta}{\sqrt{M}\,a^{6.5}}$$ } 1 for correct $\sin 2\delta \approx 2\delta$ (or any small angle approx)

$$\delta = \frac{R^3}{T_E}\cdot\frac{1}{\sqrt{GMa^3}}$$

$$T_E = \frac{R^3\omega}{GM\delta} = \frac{R^3}{GM\delta}\sqrt{\frac{GM}{a^3}} = \frac{R^3}{\delta}\frac{1}{\sqrt{GMa^3}}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dt} = -\frac{6kmR^8}{T_E M a^8}$$ } 1 for correct insertion of T_E

(8) c) i)



$$F = \underbrace{\frac{-\dot{M}v_m}{4\pi a^2}}_{\text{1: correct incoming momentum rate per area}}\int_0^{\theta=\pi/2}\underbrace{2\pi R_p^2\sin\theta}_{\text{1: correct 'area' of ring}}\underbrace{\cos\theta}_{\text{1: correct projection factor}}\underbrace{(1+\cos(2\theta))}_{\text{1: correct momentum}}\,d\theta$$

(1 for right limits)

$$= -\frac{\dot{M}R_p^2v_m}{2a^2}\int_0^{\pi/2}\frac{1}{2}\sin(2\theta) + \frac{1}{4}\sin(4\theta)\,d\theta$$ } 1 for trig rearrangement that can be integrated in 1 step

$$= -\frac{\dot{M}R_p^2v_m}{2a^2}\left[-\frac{1}{4}\cos(2\theta) - \frac{1}{16}\cos(4\theta)\right]_0^{\pi/2}$$

$$= \frac{-\dot{M}R_p^2v_m}{2a^2}\cdot\frac{1}{2} = -\frac{\dot{M}R_p^2v_m}{4a^2}$$ } 1 for final answer (IF DIM. CORRECT) −0.5 FOR WRONG SIGN

If the collisions were inelastic, $F = -\dot{M}v_m\frac{\pi R_p^2}{4\pi a^2} = \underbrace{\frac{-\dot{M}R_p^2v_m}{4a^2}}_{1}$, so the same.

2 for $-\frac{\dot{M}R_p^2v_m}{2a^2} \Longleftrightarrow 0$ case.

(3) ii) $$F_{total} = -\frac{GMm}{a^2} - \frac{\dot{M}R_p^2v_m}{4a^2} = -\frac{GM_{eff}m}{a^2}$$ } 1 for getting what M_{eff} is

$$\Rightarrow M_{eff} = M + \frac{\dot{M}R_p^2v_m}{4Gm}$$ } 1 for M_{eff} −0.5 for using inelastic eqn.

$$\frac{\dot{a}}{a} = -\frac{\dot{M}_{eff}}{M_{eff}} = -\frac{\dot{M} + \frac{\dot{M}R_p^2v_m}{4Gm}}{M + \frac{\dot{M}R_p^2v_m}{4Gm}}$$ } 1 for final answer.

(5) iii)

$$\alpha = \frac{R_p^2 \, v_m}{4Gm} = 25.5$$

$$\text{change} = \frac{\left(\frac{\dot{a}}{a}\right)_{solar}}{\left(\frac{\dot{a}}{a}\right)_{no\ solar}} - 1 = \frac{M}{\dot{M}}\left(\frac{\dot{M} + \alpha \ddot{M}}{M + \alpha \dot{M}}\right) - 1 = \underbrace{\frac{\alpha \frac{\ddot{M}}{\dot{M}} - \alpha \frac{\dot{M}}{M}}{1 + \alpha \frac{\dot{M}}{M}}}_{1}$$

$$M = M_0 \left(1 - 0.25 \left(\frac{t}{t_{RGB}}\right)^{10}\right)$$

$$\dot{M} = \underbrace{-\frac{2.5\, M_0}{10^9 \text{yrs}}\left(\frac{t}{t_{RGB}}\right)^9}_{0.5} = -\frac{2.5\, M_0}{10^9 \text{yrs}} = -7.92 \times 10^{-17} \, M_0 \, s^{-1} \rightarrow 1$$

$$\ddot{M} = \underbrace{-\frac{22.5\, M_0}{(10^9 \text{yrs})^2}\left(\frac{t}{t_{RGB}}\right)^9}_{0.5} = -\frac{22.5\, M_0}{(10^9 \text{yrs})^2} = -2.26 \times 10^{-32} \, M_0 \, s^{-2} \rightarrow 1$$

change $\approx 10^{-16}$ or 10^{-14} percent $\rightarrow 1$

1 Gravitational Microlensing (Model Solution)

(a) (i) Show that the Einstein radius is given by the formula: [4]

$$\theta_E = \sqrt{\frac{4GM}{c^2} \cdot \frac{D_{LS}}{D_S D_L}}$$

By geometry and the small angle approximation for $\tan x$, we have:

$$b = \theta_E D_L \quad (1)$$

By considering the angle α shown in Figure 1, we also have:

$$\delta = \theta_E + \alpha \quad (1)$$
$$b = \alpha D_{LS} \quad (1)$$

Hence:

$$\frac{4GM}{bc^2} = \delta = \theta_E + \alpha = \left(1 + \frac{D_L}{D_{LS}}\right)\theta_E$$
$$\frac{4GM}{D_L \theta_E c^2} = \frac{D_S}{D_{LS}}\theta_E$$
$$\theta_E = \sqrt{\frac{4GM}{c^2} \cdot \frac{D_{LS}}{D_L D_S}} \quad (1)$$

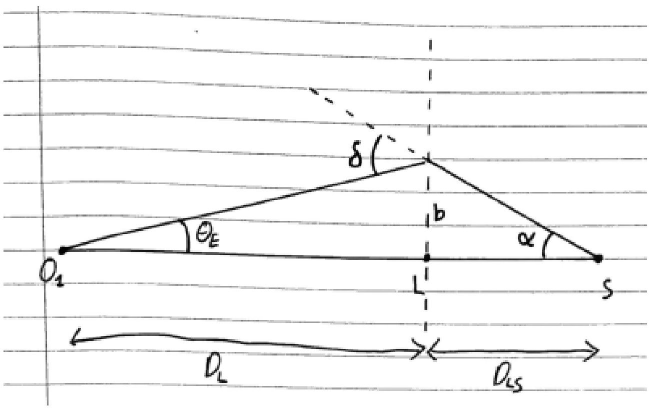


Figure 1: Diagram for (a-i)

(ii) Calculate the Einstein radius for an Earth-sized exoplanet with parallax 0.196mas, lensing light from a source star with parallax 0.087mas. [3]

Using the definition of parallax, we have:

$$D_S = \frac{1}{0.087} = 11.5\text{kpc} \qquad D_L = \frac{1}{0.196} = 5.1\text{kpc} \quad (1)$$
$$D_{LS} = D_S - D_L = 6.1\text{kpc} \quad (1)$$

$$\theta_E = \sqrt{\frac{4GM}{c^2} \cdot \frac{D_{LS}}{D_S D_L}}$$
$$= \sqrt{\frac{4 \times 6.67 \times 10^{-11} \times 318 \times 5.97 \times 10^{24}}{(3.00 \times 10^8)^2} \cdot \frac{6.4}{11.5 \times 5.1 \times 3.09 \times 10^{19}}}$$
$$= 1.41 \times 10^{-10}\text{rad} \qquad (1)$$

(b) (i) Find an equation which must be satisfied by each of the angles $\theta_\pm$. Hence, find explicit formulae for θ_+ and θ_- in terms of θ_E and the Source-Observer-Lens angle φ. [6]

By small angles:

$$b_+ = \theta_+ D_L \qquad (1)$$

Extending the ray's trajectory and considering the distance y, as shown in Figure 2, by small angle approximations:

$$D_{LS}\delta_+ = y = D_S(\theta_+ - \varphi) \qquad (2)$$
$$D_{LS} \cdot \frac{4GM}{b_+ c^2} = D_S(\theta_+ - \varphi)$$
$$D_{LS} \cdot \frac{4GM}{D_L \theta_+ c^2} = D_S(\theta_+ - \varphi) \qquad (1)$$
$$\theta_+^2 - \theta_+\varphi = \frac{4GM}{c^2} \cdot \frac{D_{LS}}{D_L D_S} = \theta_E^2$$
$$\theta_+^2 - \theta_+\varphi - \theta_E^2 = 0 \qquad (1)$$

Since we can do exactly the same construction with θ_- (using a sign convention where θ_- is negative), it satisfies the same equation. Thus the $\theta_\pm$ are the 2 roots of this equation, and from the quadratic formula, we have:

$$\theta_\pm = \frac{1}{2}\left(\varphi \pm \sqrt{\varphi^2 + 4\theta_E^2}\right) \qquad (1)$$

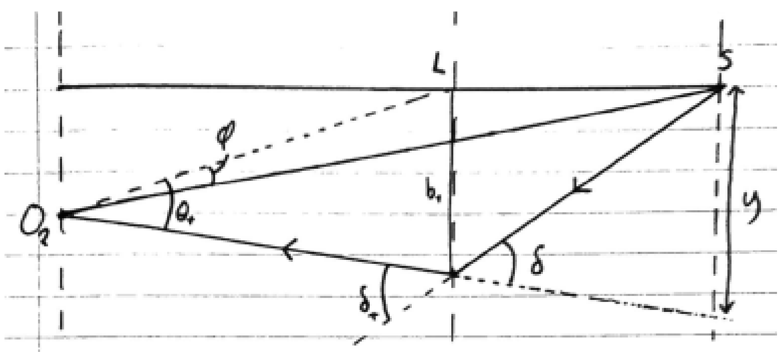


Figure 2: Diagram for (b-i)

(ii) Determine the maximum value of the angular displacement of the image at θ_+ from the unlensed position of the source as φ varies, and the corresponding value of φ. [4]

The displacement of the source from its unlensed position is given by the expression $\theta_+ - \varphi$. Differentiating, we have:

(1)

$$\frac{d}{d\varphi}(\theta_+ - \varphi) = \frac{1}{2} \cdot \frac{d}{d\varphi}\left(\sqrt{\varphi^2 + 4\theta_E^2} - \varphi\right) = \frac{1}{2}\left(\frac{\varphi}{\sqrt{\varphi^2 + 4\theta_E^2}} - 1\right) < 0 \qquad (1)$$

Thus, the displacement is decreasing with φ, and so for minimal displacement we pick $\varphi = 0$, corresponding to $\theta_+ - \varphi = \theta_E$.

(2)

(iii) What is the smallest diameter of telescope which would be able to resolve this maximal displacement for the exoplanet in (a-ii) [1]

By the Rayleigh criterion for an example wavelength of 500 nm (any wavelength corresponding to visible light is fine):

$$d = 1.22 \cdot \frac{\lambda}{\theta_E} = 1.22 \cdot \frac{5.0 \times 10^{-7}}{1.41 \times 10^{-10}} = 4300 \text{ m} \qquad (1)$$

We remark that this is an absurd size for an optical telescope, so the deflection cannot be detected by conventional means (although it may be observed using interferometry).

(c) (i) Find expressions for a and the impact parameters $b_\pm$ in terms of φ, θ_E and the various D [3]

By small angles:

$$b_\pm = D_L\theta_\pm = \frac{D_L}{2}\left(\varphi \pm \sqrt{\varphi^2 + 4\theta_E^2}\right) \qquad (1)$$

By considering Figure 3:

$$a = D_S\beta = D_S \cdot \frac{x}{D_{LS}} = D_S \cdot \frac{x}{D_{LS}} = D_S \cdot \frac{D_L\varphi}{D_{LS}} = \frac{D_L D_S}{D_{LS}} \cdot \varphi \qquad (2)$$

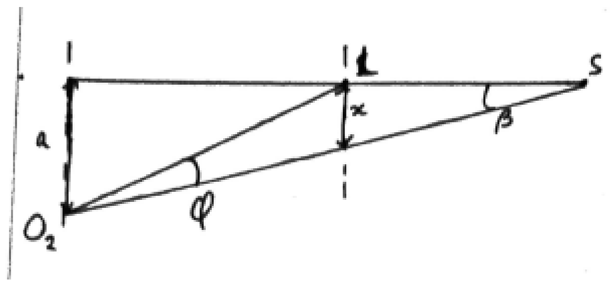


Figure 3: Diagram for (c-i)

(ii) Consider changing φ by a small increment $d\varphi$. Determine the corresponding changes da and $db_\pm$ [3]

By differentiating:

$$da = \frac{D_L D_S}{D_L S} \cdot d\varphi \qquad (1)$$

$$db_\pm = D_L \theta_\pm = \frac{D_L}{2}\left(1 \pm \frac{\varphi}{\sqrt{\varphi^2 + 4\theta_E^2}}\right) d\varphi \qquad (2)$$

(iii) By considering the total power incident on the annulus of points in P_O between a and $a + da$ with and without the lensing effect, determine the brightness of each of the two images compared to the unlensed brightness of the source. Give your answer in terms of the ratio $u = \frac{\varphi}{\theta_E}$. [6]

Let A be the annulus described above, and B the corresponding annulus of points in P_L between $b_\pm$ and $b_\pm + db_\pm$. Let P be the luminosity of the source. Without the lensing effect:

$$\text{Unlensed Power on } A = \frac{\text{Area of } A}{4\pi D_S^2} \cdot P \qquad (2)$$

Since A is a narrow strip with length $2\pi a$ and width da, it has area $2\pi a \cdot da$. Hence:

$$\text{Unlensed Power on } A = \frac{2\pi a \cdot da}{4\pi D_S^2} \cdot P \qquad (1)$$

When the lensing effect is present, the rays passing through A which contribute to the image we are considering are exactly the ones which are incident on B. Thus:

$$\begin{aligned} \text{Lensed Power on } A &= \text{Power on } B \\ &= \frac{\text{Area of } B}{4\pi D_{LS}^2} \cdot P \\ &= \frac{2\pi |b| \cdot |db|}{4\pi D_{LS}^2} \cdot P \qquad (1) \end{aligned}$$

Therefore, the ratio of powers on the annuli, which is the same as the ratio of brightnesses (since the power is constant over the annulus for small enough $d\varphi$), is given by:

$$\frac{\text{Lensed Brightness}}{\text{Unlensed Brightness}} = \frac{|b| \cdot |db|}{a \cdot da} \cdot \frac{D_S^2}{D_{LS}^2}$$

$$= \frac{\frac{D_L}{2}\left|\varphi \pm \sqrt{\varphi^2 + 4\theta_E^2}\right| \cdot \frac{D_L}{2}\left|1 \pm \frac{\varphi}{\sqrt{\varphi^2 + 4\theta_E^2}}\right| \cdot d\varphi}{\left(\frac{D_L D_S}{D_{LS}}\right)^2 \varphi \cdot d\varphi} \cdot \frac{D_S^2}{D_{LS}^2}$$

$$= \frac{1}{4}\left|1 \pm \frac{\sqrt{u^2+4}}{u}\right| \cdot \left|1 \pm \frac{u}{\sqrt{u^2+4}}\right| \qquad (1)$$

$$= \frac{1}{4}\frac{\left(\sqrt{u^2+4} \pm u\right)^2}{u\sqrt{u^2+4}}$$

$$= \frac{1}{4}\frac{\left(u^2 + 4 \pm 2u\sqrt{u^2+4} + 1\right)}{u\sqrt{u^2+4}}$$

$$= \frac{1}{2}\left(\frac{u^2+2}{u\sqrt{u^2+4}} \pm 1\right) \qquad (1)$$

(iv) Assuming that the observer cannot resolve the two images, show that the total amplification factor is given by:

[1]

$$A = \frac{u^2+2}{u\sqrt{u^2+4}}$$

$$A = A_+ + A_- = \frac{1}{2}\left(\frac{u^2+2}{u\sqrt{u^2+4}} + 1\right) + \frac{1}{2}\left(\frac{u^2+2}{u\sqrt{u^2+4}} - 1\right) = \frac{u^2+2}{u\sqrt{u^2+4}} \qquad (1)$$

(v) For the same hypothetical exoplanet considered earlier, calculate the percentage increase in source brightness due to the lensing effect given $\varphi = 0.07$ mas.

[2]

$$u = \frac{\varphi}{\theta_E} = \frac{0.07 \text{ mas}}{0.0291 \text{ mas}} = 2.4 \qquad (1)$$

$$A = \frac{u^2+2}{u\sqrt{u^2+4}} = \frac{2.4^2+2}{2.4\sqrt{2.4^2+4}} = 1.035$$

So the % increase in brightness is 3.5%

(1)

(d) i) Using a suitable numerical method, determine the minimum value of u over the course of this lensing interaction. [4]

Reading values from Figure 6, we see that the baseline magnitude of the source is $m = 16.35$, and the peak magnitude is $m = 15.55$. Thus:

(1)

$$A = 10^{0.4(16.35-15.55)} = 2.09 \quad (1)$$

We therefore need to solve for u in the equation:

$$\frac{u^2+2}{u\sqrt{u^2+4}} = 2.09$$

This could be done with Newton-Raphson iteration, or even by by rearranging into a quadratic in u^2 to solve exactly, but it is easiest to use the following fixed-point iteration scheme:

$$u_{n+1} = \frac{u_n^2+2}{2.09\sqrt{u_n^2+4}} \quad (1)$$

Applying the scheme, with $u_0 = 1$ (any sensible value will work fine), it converges in relatively few iterations to $u = 0.527$

(1)

ii) Determine the Einstein crossing time t_E, defined as the time taken for the source and lens to move relative to each other by an angle θ_E. [3]

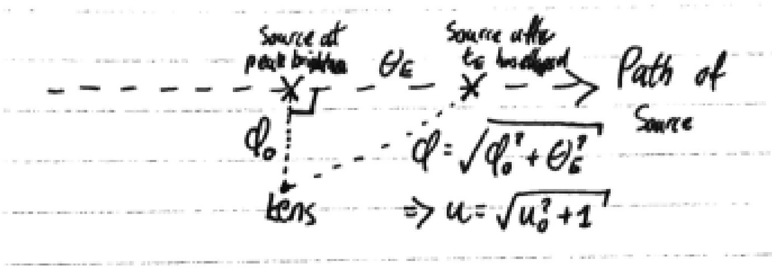


Figure 4: Diagram for (d-ii)

Figure 4 shows the movement of the source relative to the lens, as it would be seen by the observer without the lensing effect. Therefore, at a time of t_E from the peak brightness, the source will have traversed a distance of θ_E from the point of closest approach, and will therefore be at a value of $u = \sqrt{0.527^2 + 1^2} = 1.13$. Therefore:

(1)

$$A = \frac{1.13^2+1}{1.13\sqrt{1.13^2+4}} = 1.26 \quad (1)$$

$$m = m_0 - 2.5\log 1.26 = 16.05$$

From the graph, this happens at $t \approx 6210$, while the peak occurs at $t \approx 6140$. Thus, $t_E = 6210 - 6140 = 70$ days.

(1)

2 Gravitational Lensing (Marking Scheme)

(a) (i) [1] Deduces $b = \theta_E D_L$

[1] Considers the angle α and finds one of the corresponding equations **or** finds another similarly useful geometric construction and a corresponding equation

[1] Makes all necessary small angle approximations for their method

[1] Makes all necessary substitutions for their method

(If none of marks 2-4 scored, award 1 mark for substitution of expression for b or δ into any correct geometric equation. Award 3 marks for any solution only missing approximations. Must be rearranged into the final form to score full marks)

(ii) [1] Finds $D_S = 11.5$kpc or $D_L = 5.1$kpc

[1] Finds all 3 D's

[1] Calculates $\theta_E = 1.41 \times 10^{-10}$rad $(= (8.08 \times 10^{-9})^\circ = 0.0291mas)$

(Award one of the first 2 marks if the same error is made for both parallax calculations and D_{LS} is correct for their values)

(b) (i) [1] Deduces $b = D_L\theta$

[1] Helpful geometric observation which could be used to find θ and a corresponding expression

[1] Finds any equation which would reduce to the correct quadratic upon substitutions of $b = D_L\theta$ and the formulae for δ and θ_E $D_S\theta = D_S\varphi + D_{LS}\delta$ or $\frac{D_L D_S}{D_{LS}}\varphi + D_L\delta = \frac{bD_S}{D_{LS}}$)

[1] Correctly makes any 2 of the above substitutions

[1] Obtains $\theta^2 - \varphi\theta - \theta_E^2 = 0$

[1] Deduces $\theta_- = \pm\frac{1}{2}\left(\sqrt{\varphi^2 + 4\theta_E^2} - \varphi\right)$ *(Ignore sign)*

(ii) *(Allow full marks for all parts from (b-ii) to (c-iii) considering only θ_+, b_+ in place of $\theta_\pm$, $b_\pm$)*

[1] Considers the angle $\varepsilon = \theta_+ - \varphi$

[1] Differentiates ε correctly *(Award this mark for correctly differentiating θ_+ if the first mark has been lost)*

[1] Deduces ε decreasing with φ and hence minimised when $\varphi = 0$

[1] Deduces $\varepsilon_{\min} = \theta_E$

(iii) [1] Correct Rayleigh criterion calculation for a sensible wavelength. 500 nm gives 4300 m.

(c) (i) [1] Finds $b\pm$

[1] Helpful geometric observation which could be used to find a and a corresponding expression (e.g. considering the length given by φD_L and $\frac{D_{SL}}{D_S}$ and finding either one of those expressions)

[1] Finds $a = \frac{D_L D_S}{D_{LS}}\varphi$

(ii) [1] Correctly finds $da = \frac{D_L D_S}{D_{LS}}d\varphi$ (ECF their a)

[1] Attempts to differentiate their b and uses any one differentiation rule correctly

[1] Correctly finds $db = \dfrac{D_L}{2}\left(1 \pm \dfrac{\varphi}{\sqrt{\varphi^2 + 4\theta_E^2}}\right)d\varphi$ (ECF their b)

(iii) [1] Formula for area of annulus (in terms of a or b)

[1] Correct brightness on either plane (allow unnecessary terms accounting for extra distance due to a or b as long as they are correct)

[1] Determines power on any relevant annulus (ECF the above)

[1] Determines power on the other annulus (does not need to be simplified)

[1] Finds an expression for the ratio of powers in terms of only u

[1] Correctly simplifies their expression

(iv) [1] Sums brightnesses from both images to obtain A

(v) [1] Determines $u = 2.4$
[1] Determines % increase ($= 3.5\%$)

(d) (i) [1] Reads $m_0 \in [16.30, 16.36]$ or $m_{\text{peak}} \in [15.52, 15.58]$ from figure 6
[1] Determines $A = 2.09$ (ECF any sensible values for m read from the graph)
[1] Sets up an appropriate numerical method to solve for u or rearranges to a quadratic in u^2
[1] Finds $u = 0.527$ (ECF their A)

(ii) [1] Deduces that 1 Einstein time away from the peak, $u = \sqrt{u_0^2 + 1^2} = 1.13$
[1] Deduces $A = 1.26$ at t_E away
[1] Uses the graph to see that this is 70 ± 15 days away from the peak