

**BAAO**  
British Astronomy and  
Astrophysics Olympiad

## British Astronomy and Astrophysics Olympiad 2024-2025

### Astro Round 2

Tuesday 25<sup>th</sup> February 2025

**This question paper must not be photographed or taken out of the exam room**

### Instructions

**Time:** 3 hours (~ 50 minutes for Q1, ~ 65 minutes for Q2 and ~ 65 minutes for Q3).

**Questions:** All three questions should be attempted. Each question contains independent parts so that later parts can be attempted even if earlier parts are incomplete. **The questions build in difficulty.**

**Solutions:** Answers and calculations are to be written on loose paper. Students should ensure their **name** and **school** is clearly written on the **first** answer sheet and that **all** pages are numbered. A standard formula booklet may be used. **START EACH QUESTION ON A NEW PAGE.**

**Clarity:** Solutions must be written legibly, in black or blue pen, and working down the page. Scribble will not be marked and overall clarity is an important aspect of this exam.

**Instructions:** To accommodate students sitting the paper at different times, please **do not discuss** any aspect of the paper on the internet until 8 am Thursday 6<sup>th</sup> March.

**Calculators:** Any standard calculator may be used, but calculators cannot be programmable and must not have symbolic algebra capability.

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### Training Dates and the IOAA (Mumbai, India, 11<sup>th</sup> - 21<sup>st</sup> August 2025)

*The best students taking this paper eligible to represent the UK at the IOAA will be invited to attend the **Selection Camp** to be held in Oxford from **Monday 7<sup>th</sup> to Friday 11<sup>th</sup> April 2025**. At the camp, problem solving skills and observational skills (including with telescopes) will be developed, and students will sit a Data Analysis exam along with the Round 3 paper. From this, a team of five students (plus one reserve) will be selected for further training, including additional camps in the summer.*

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## Important Constants

Constant	Symbol	Value
Speed of light	$c$	$3.00 \times 10^8 \text{ m s}^{-1}$
Earth's rotation period	1 day	24 hours
Earth's orbital period	1 year	365.25 days
parsec	pc	$3.09 \times 10^{16} \text{ m}$
Astronomical Unit	au	$1.50 \times 10^{11} \text{ m}$
Radius of the Sun	$R_{\odot}$	$6.96 \times 10^8 \text{ m}$
Radius of the Earth	$R_{\oplus}$	$6.37 \times 10^6 \text{ m}$
Mass of the Sun	$M_{\odot}$	$1.99 \times 10^{30} \text{ kg}$
Mass of the Earth	$M_{\oplus}$	$5.97 \times 10^{24} \text{ kg}$
Luminosity of the Sun	$L_{\oplus}$	$3.83 \times 10^{26} \text{ W}$
Absolute magnitude of the Sun	$\mathcal{M}_{\odot}$	4.74
Hubble constant	$H_0$	$70 \text{ km s}^{-1} \text{ Mpc}^{-1}$
Stephan-Boltzmann constant	$\sigma$	$5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
Gravitational constant	$G$	$6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Boltzmann constant	$k_B$	$1.38 \times 10^{-23} \text{ J K}^{-1}$
Permittivity of free space	$\epsilon_0$	$8.85 \times 10^{-12} \text{ F m}^{-1}$
Permeability of free space	$\mu_0$	$4\pi \times 10^{-7} \text{ H m}^{-1}$
Planck's constant	$h$	$6.63 \times 10^{-34} \text{ J s}$
Elementary charge	$e$	$1.60 \times 10^{-19} \text{ C}$
Proton rest mass	$m_p$	$1.67 \times 10^{-27} \text{ kg}$
Electron rest mass	$m_e$	$9.11 \times 10^{-31} \text{ kg}$
Wien's displacement law	$\lambda_{\text{max}}T$	$2.90 \times 10^{-3} \text{ m K}$
Avagadro's constant	$N_A$	$6.02 \times 10^{23} \text{ mol}^{-1}$

### Basic calculus formulae:

Chain rule  $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Product rule  $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$

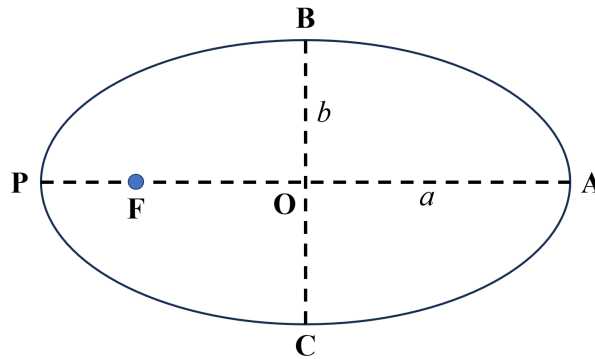
Quotient rule  $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

Integration by parts  $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

Standard integral  $\int \frac{1}{x} dx = \ln|x| + C$

## Important Formulae

You might find the diagram of an elliptical orbit below useful in solving some of the questions:



**Elements of an elliptic orbit:**

- $a = \text{OA} (= \text{OP})$  semi-major axis
- $b = \text{OB} (= \text{OC})$  semi-minor axis
- $e = \sqrt{1 - \frac{b^2}{a^2}}$  eccentricity
- F** focus
- $\text{PF} = a(1 - e)$  periapsis distance (shortest distance from **F**)
- $\text{AF} = a(1 + e)$  apoapsis distance (longest distance from **F**)

**Kepler's Third Law:**

$$T^2 = \frac{4\pi^2}{GM} a^3$$

**Vis-Viva Equation:**

$$v^2 = GM \left( \frac{2}{r} - \frac{1}{a} \right)$$

**Wien's Displacement Law:**

$$\lambda_{\text{max}} T = \text{constant}$$

**Stephan-Boltzmann Law:**

$$L = 4\pi R^2 \sigma T^4$$

**Brightness (Intensity):**

$$b = \frac{L}{4\pi d^2}$$

**Magnitudes:**

$$\frac{b_1}{b_0} = 10^{-0.4(m_1 - m_0)}$$

$$m - \mathcal{M} = 5 \log \left( \frac{d}{10} \right)$$

**Distance-Parallax Relation:**

$$d = \frac{1}{p}$$

**Rayleigh Criterion:**

$$\theta = \frac{1.22\lambda}{D}$$

**Redshift:**

$$z = \frac{\Delta\lambda}{\lambda_{\text{emit}}} \approx \frac{v}{c}$$

**Hubble's Law:**

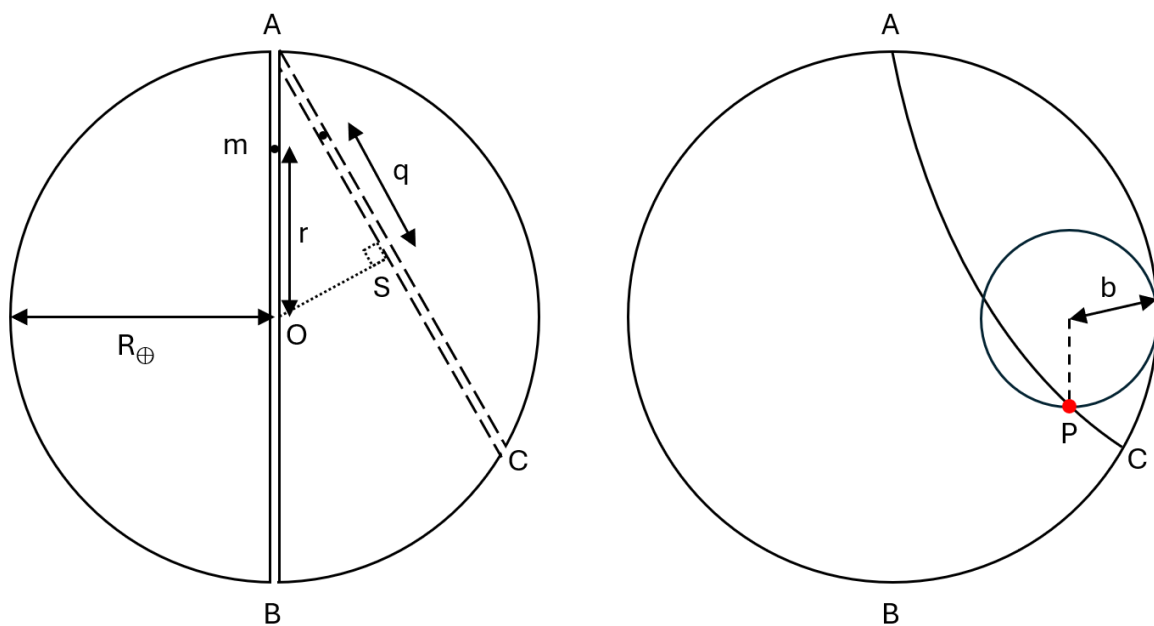
$$v = H_0 d$$

## Qu 1. Gravity Train

First proposed by Robert Hooke in a conversation with Isaac Newton, and later explored mathematically by Paul Cooper in the 1960s and several science fiction authors thereafter, a ‘gravity train’ involves drilling a tunnel through the Earth to connect two locations that are often far apart and using only gravitational forces to accelerate and decelerate the train on its journey. In this question we will investigate the physics of this novel form of transportation.

You may wish to make use of Newton’s Shell Theorem, which states that if an object is at a distance  $r$  from the centre of mass of a sphere of radius  $R$ , the gravitational force it experiences is identical to that from a point mass at the location of the centre of mass with a mass equal to the mass enclosed by a shell of radius  $r$ .

Throughout this question we will assume that the Earth is a solid sphere of uniform density,  $\rho$ , and that all tunnels are maintained in vacuum.



**Figure 1:** *Left:* A tunnel through the Earth along the diameter connecting point  $A$  and point  $B$ , as well as a straight line chord connecting point  $A$  to point  $C$ . The midpoint of  $AB$  is indicated  $O$ , and the midpoint of  $AC$  is indicated  $S$ . The distance of the gravity train (represented as a point mass of mass  $m$ ) from  $O$  is given by  $r$ , and from  $S$  is given by  $q$ .

*Right:* An alternative route from  $A$  to  $C$  using a hypocycloid. This special shape is generated by the trace of a fixed point  $P$  on a small circle of radius  $b$  that rolls within the larger circle.

Credit: Alex Calverley.

Figure 1 shows possible routes for the gravity train as explored by Paul Cooper. The simplest case is between two points,  $A$  and  $B$ , that are on completely opposite sides of the planet, and so line  $AOB$  is a diameter of the Earth, with  $O$  at the centre of the Earth. A more convenient case is along an arbitrary chord to point  $C$ , with  $S$  as the midpoint, and Cooper showed that the quickest route between two arbitrary points on the Earth’s surface is along the hypocycloid curve connecting the two - this is a special type of curve, generated by the trace of a fixed point  $P$  on a small circle of radius  $b$  that rolls within the larger circle.

- a. In this part of the question you should ignore any effects of the Earth's rotation. Assume the gravity train always starts from rest at point  $A$ .
- Show that the gravity train using the route  $AOB$  would undergo simple harmonic motion (SHM) and hence that the time  $t_{AB}$  to get from  $A$  to  $B$  is a function of  $g_0$  and  $R_{\oplus}$ , where  $g_0$  is the gravitational field strength at the surface of the Earth. Hence show  $t_{AB}$  is about 42 minutes.
  - Compare this to the time taken by a satellite to go from  $A$  to  $B$  when orbiting in a circular orbit of radius  $R_{\oplus}$ .
  - What is the largest speed achieved by the gravity train? What is this the same as?
  - Gravity trains could potentially be used in the future to mine the asteroid belt, as a way to haul raw material to a central refining point or port for launch back to Earth. The dwarf planet Ceres has an average density of  $2.16 \text{ g cm}^{-3}$ . What would be  $t_{AB}$  on Ceres?
  - The route  $ASC$  is an arbitrary chord through the planet. Given the gravity train is constrained to travel along frictionless rails, determine the time  $t_{AC}$  to go along this route. Compare your answer to  $t_{AB}$ . You may wish to make use of the notation in Figure 1.
- b. Cooper showed that the time taken to travel along a hypocycloid made by a circle of radius  $b$  is

$$t_{hyp} = \frac{2\pi b}{R_{\oplus}} \sqrt{\frac{R_{\oplus}(R_{\oplus} - b)}{bg_0}}.$$

For all points  $A$  and  $C$ ,  $t_{hyp} \leq t_{AC}$ . Again, ignore the Earth's rotation and assume the gravity train starts from rest at point  $A$ .

- When  $t_{hyp} = t_{AC}$ , what is the value of  $b$ ? What situation does this correspond to?
- Consider a chord between  $A$  and  $C$  with a maximum depth below the Earth's surface of  $d$ , and the hypocycloid has a maximum depth of  $\alpha d$  where  $\alpha > 1$ . If  $C$  is chosen such that  $t_{hyp} = \frac{1}{2}t_{AC}$ , find the latitude,  $\varphi$ , of  $C$  if  $A$  is the North Pole (for which  $\varphi = 90^\circ$ ) and hence find  $\alpha$ .

So far we have ignored the Earth's rotation, and this would be fine for a tunnel travelling from the North Pole to the South Pole, however if we had instead tried to go between two points anywhere else (whether a diameter or a chord), other forces would have to be taken into account. Moving into a rotating co-ordinate system (of constant angular velocity) means that in order for Newton's laws of motion to hold, we have to introduce two 'fictitious forces': a centrifugal force,  $F_{Cent}$ , that acts radially and a Coriolis force,  $F_{Cor}$  that acts tangentially (perpendicular to both the particle's velocity and to the Earth's axis of rotation).

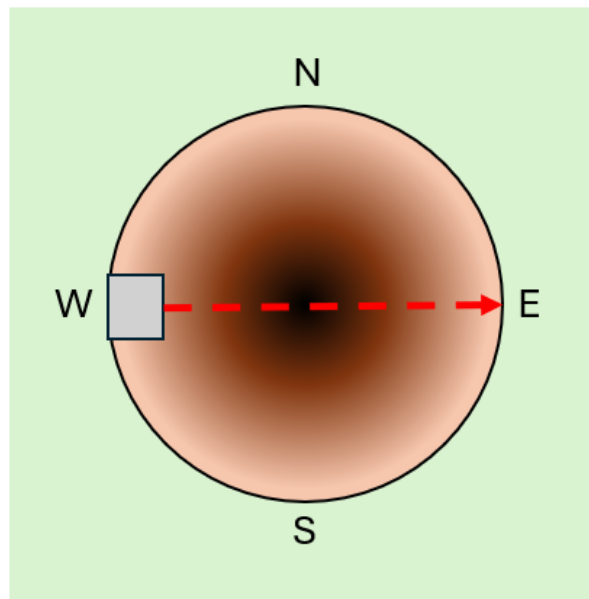
The magnitude of the centrifugal force is equal to that of the centripetal force, but pointed away from the rotational axis. The magnitude of the Coriolis force is given by

$$F_{Cor} = 2m\Omega v_{\perp}$$

where  $m$  is the particle's mass,  $v_{\perp}$  is its velocity component perpendicular to the Earth's axis of rotation, and  $\Omega$  is the Earth's angular velocity. A consequence of the Coriolis force is that the gravity train, unless constrained on rails, will drift eastwards and crash into the side. For a tunnel starting at the equator with the train on the West wall, this is movement to the right (see Figure 2).

- c. Suppose the tunnel along the diameter  $AOB$  is now in the plane of the Earth's equator, and so rotational effects will have to be considered.
- The centrifugal acceleration  $\left(\frac{F_{Cent}}{m}\right)$  is greatest at the surface of the Earth. Show that it is  $< 1\%$  of  $g_0$ , and so we are justified in ignoring it.

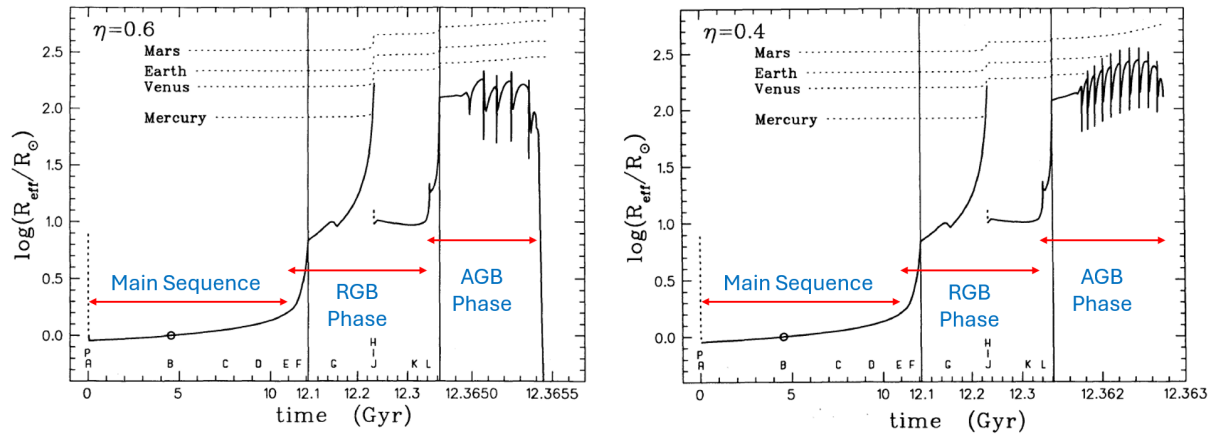
- (ii) Sketch how the Coriolis acceleration  $\left(\frac{F_{Cor}}{m}\right)$  experienced by the gravity train varies as a function of  $r$  (as defined in Figure 1 as the distance from  $O$ ) indicating numerical values on the  $y$ -axes that confirm we cannot treat it as negligible. You may find it easiest to reverse your  $r$  axis so it goes from  $R_{\oplus}$  (at A) to 0 to  $-R_{\oplus}$  (at B). Treat eastwards as positive.
- (iii) If the train was originally on the West side of the tunnel (as shown in Figure 2) and became detached from its rails the moment the journey started, find an expression for the West-East displacement as a function of time.
- (iv) If the distance between the top of the gravity train and the other side of the tunnel is 500 m (it's a very wide tunnel!), how much time will elapse before it crashes into the wall? (You may wish to use the small angle approximation for  $\sin x$  that  $\sin x \approx x - x^3/6$ )
- (v) How far below the surface did the train crash? Give your answer as a percentage of  $R_{\oplus}$ .



**Figure 2:** Looking directly into a tunnel positioned at the equator from above it, with the train (in grey) on rails on the West wall. Without the rails, the Coriolis force would mean it drifts eastwards (to the right) along the tunnel's cross-sectional diameter until it hits the East wall. Credit: Alex Calverley.

## Qu 2. The Fate of the Earth

When the Sun finishes its time on the main sequence it will expand into a red giant. As a result, both its radius and luminosity will increase dramatically, while its mass will decrease. It has been a subject of debate for decades whether the planet Earth will survive the expansion of the Sun's radius, or become engulfed by it. A classic paper on the topic is Sackmann, Boothroyd & Kraemer (1993) where they looked at the fate of the inner planets for two different scenarios, shown in Figure 3 below.



**Figure 3:** *Left:* The radius of the Sun during its lifetime, indicating the periods when it is in the main sequence, its Red Giant Branch (RGB) phase, and its Asymptotic Giant Branch (AGB) phase, with a mass loss efficiency given by a Reimer's parameter of  $\eta = 0.6$ . The orbital radii of the four terrestrial planets are given as dotted lines. For this value of  $\eta$  they found that only Mercury was engulfed.

*Right:* The same plot as on the left, but for  $\eta = 0.4$ . In this situation only Mars survived the Sun's giant phase. As you can see from each panel in this figure, the RGB and AGB phases are actually rather complex, but the full details behind the shape of the graph are not necessary here.

Credit: Sackmann, Boothroyd & Kraemer (1993).

The key factor that is unknown is the amount of mass lost by the star towards the end of its life. This can be modelled by the Reimers' (1975) parameter,  $\eta$ , which is related to the efficiency of the mass-loss mechanism (larger values of  $\eta$  mean higher mass loss, whilst  $\eta = 0$  means no mass loss). In the graphs, three key areas are indicated: the Main Sequence, when we have hydrogen fusion in the core; the Red Giant Branch (RGB) phase, when we have helium fusion in the core and hydrogen fusion in a shell around it; and the Asymptotic Giant Branch (AGB) phase, when we have an inert core, helium fusion in a shell around the core, and hydrogen fusion in a second shell around the helium shell. During the RGB and AGB phases, the radius of the Sun is of order  $\sim 1$  au, and the largest value reached in each phase is called the 'tip' (i.e. the RGB tip and AGB tip).

In this question, we will investigate how the Earth's orbital radius is affected by changes in the Sun's mass and radius, closely following the method of Guo et al. (2016).

Due to the conservation of angular momentum, the orbital radii of the planets increase following the relation  $a_1 M_1 = a_0 M_0$  where  $a_1$  and  $M_1$  are the new orbital radius and stellar mass and  $a_0$  and  $M_0$  are the original values - i.e. as the Sun loses mass the planets move further away, and so the Earth may survive.

a. Show that

$$\frac{\dot{a}}{a} = -\frac{\dot{M}}{M}$$

where  $\dot{a}$  and  $\dot{M}$  denote the time derivatives of the orbital radius,  $a$ , and the stellar mass,  $M$ , respectively.

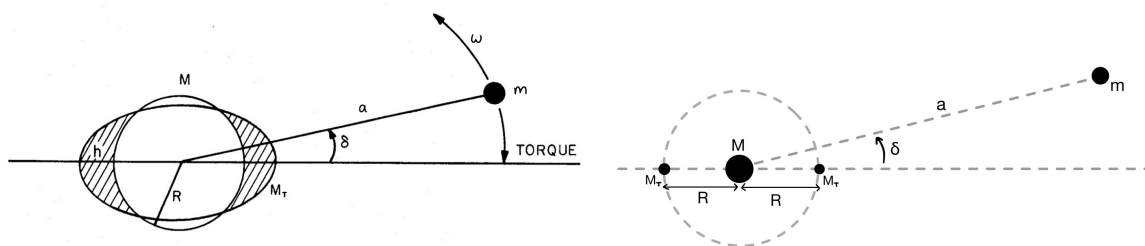
As the Sun expands, its rotation about its own axis slows drastically and the stellar surface draws much closer to the planets - this means that tidal effects begin to become important. In this section we will consider the tidal effects in a general solar system containing a star (with radius  $R$  and mass  $M$ ) and a single orbiting planet (with orbital radius  $a$  and mass  $m$ ). You may assume that  $R \ll a$  throughout: although this will become less reasonable in the later stages of the star's expansion, it will provide a good (and much more mathematically feasible!) approximation for the system's initial behaviour. You may also assume that the planet's orbit is circular at each moment, although the radius will change slowly over time.

Tides are caused by a difference in the strength of the gravitational force of the planet across each part of the star; this creates two bulges (tides) on either side of the star, as seen in Figure 4. We will model the tides as two point masses positioned on the surface of the star, a distance  $R$  from its centre, each with a mass

$$M_T = \frac{1}{2} k m \frac{R^3}{a^3},$$

where  $k$  is the *apsidal motion constant*.

Due to the inelasticity within the star, the tides actually 'lag behind' the planet by an angle  $\delta$ . This creates a torque,  $\tau$ , on the planet which ultimately causes its orbital radius to decrease, and thus somewhat counteracts the effect of the Sun's mass loss. Torque is the angular counterpart to force, and has magnitude  $\tau = rF \sin \theta$ , where  $r$  denotes the radius from the centre of rotation (in this case  $a$ ),  $F$  denotes force on the planet, and  $\theta$  denotes the angle between the radius and force vectors.



**Figure 4:** *Left:* The tidal bulges on the star lag behind the position of the planet by angle  $\delta$ .

Credit: Lecar, Wheeler & McKee (1975).

*Right:* The tides can be approximated as diametrically opposed point masses on the surface of the star.

Credit: Charlotte Stevenson.

- b. (i) Find an expression for the difference in the gravitational field strength of the gravitational field of the planet at the star's centre and the closest point on the (spherical) star's surface to the planet. Simplify this using the fact that  $R \ll a$ . You may wish to use the fact that  $(1+x)^a \approx 1+ax$  for  $x \ll 1$ .
- (ii) How would the structure of the star differ between smaller and larger values of  $k$ ?
- (iii) Calculate the resultant torque on the planet in terms of  $k, m, R, a, \delta$ , assuming again that  $R \ll a$ .
- (iv) Torque can also be defined as the *rate of change of angular momentum*  $L$  with  $L = rp \sin \phi$ , where  $\phi$  is the angle between the radius vector  $\vec{r}$  and the planet's momentum vector  $\vec{p} = m\vec{v}$ . Assuming  $\delta$  to be small and constant, the star's mass constant, and the planet's orbit to be circular at each instant, show that

$$\frac{\dot{a}}{a} = -6 \frac{k}{T_E} \frac{m}{M} \left( \frac{R}{a} \right)^8$$

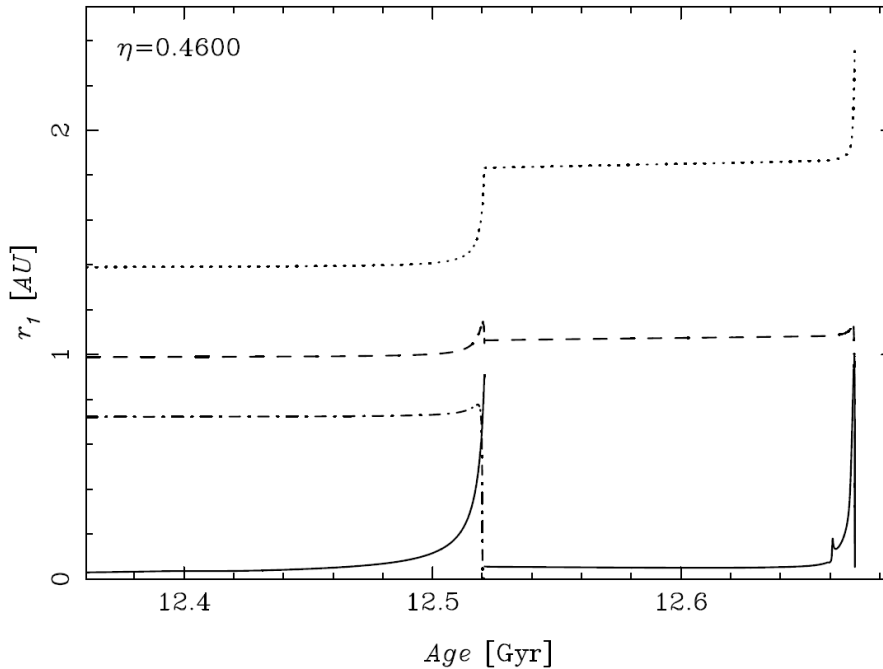
where  $T_E = \frac{R^3 \omega}{GM\delta}$  (the *eddy turnover time*), and  $\omega$  is the instantaneous angular velocity.

One factor we have not considered so far is *solar wind*. When the central star loses mass, this mass can be considered to fly out radially and isotropically. This creates a ‘wind’, some of which collides with the planet and pushes it away from the star.

- c. (i) Assuming that the mass reaches the planet at speed  $v_m$ , travels in straight lines, and is perfectly and elastically reflected from the planet’s surface upon collision, calculate the force of the wind on the planet in terms of  $M$ . Compare this to the force if the collisions were perfectly inelastic. Use  $R_P$  to denote the planet’s radius. (*Hint: split up the surface of the planet into thin rings and integrate over the momentum transfer rate for each.*)
- (ii) Since the solar wind force is proportional to  $\frac{1}{a^2}$ , we can consider the combined effect of the star’s mass and solar wind as one combined gravitational force with a reduced solar *effective mass*  $M_{eff}$ . Using the equation you derived in part a., find an expression for  $\frac{\dot{a}}{a}$  which takes into account solar mass loss and solar wind effects in terms of  $M$  and its derivatives (you may ignore tidal effects, and assume that  $v_m$  can be approximated as constant).
- (iii) Suppose the mass of the Sun as a function of time throughout the RGB phase can be approximated by

$$M(t) \approx M_0 \left( 1 - 0.25 \left( \frac{t}{t_{RGB}} \right)^{10} \right)$$

where  $M_0$  denotes the mass of the Sun at the start of the RGB phase,  $t$  denotes the time since the start of the RGB phase, and  $t_{RGB}$  is the approximate total length of the RGB phase ( $10^9$  years). Using this expression, find a rough order of magnitude estimate for the percentage difference in the value of  $\frac{\dot{a}}{a}$  for the Earth when the Sun is at the RGB tip, for the value obtained if one does account for solar wind compared to the value if they do not. You may once again ignore tidal effects, and assume that  $v_m \approx 1000 \text{ ms}^{-1}$ . Is the effect of solar wind significant?



**Figure 5:** In the paper from Guo et al. (2016) they found that Earth (second dashed line) survived for  $\eta \geq 0.46$ , which is close to the predicted value for the Sun, although there is still uncertainty around the value of  $\eta$  for the Sun and hence whether Earth survives is not yet completely solved. Note that in this model the Sun has about  $0.75 M_\odot$  at the RGB tip, which leads to the expected large increase in the orbit of Mars (top dotted line), but the increase in the orbit of Earth is much smaller than you would expect from this mass loss due to the strong tidal effects. Credit: Guo et al. (2016).

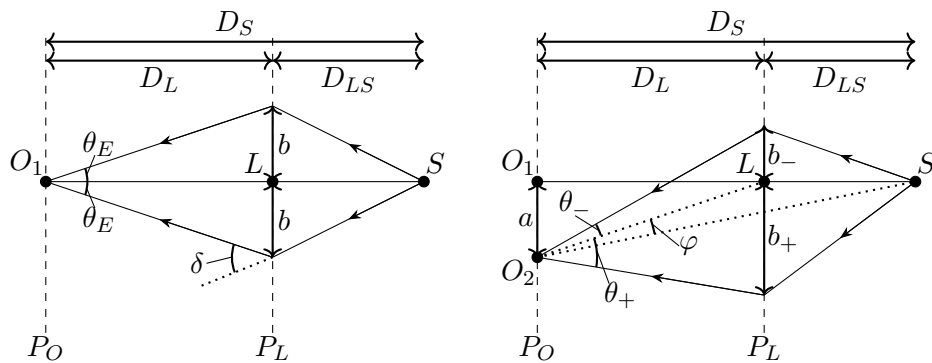
### Qu 3. Gravitational Microlensing

When a ray of light passes near to a massive body, the ray is deflected towards it by a very small angle due to the body's gravity. This effect is known as gravitational lensing. The angle of deflection  $\delta$  is given by the formula

$$\delta = \frac{4GM}{bc^2},$$

where  $b$  is the impact parameter, defined as the distance of closest approach between the ray's original trajectory and the lensing mass.

One particularly interesting example of this phenomenon is Gravitational Microlensing, where the body causing the deflection is a relatively small mass such as a star or planet. This allows us to detect objects which are too dim to observe directly, such as exoplanets and brown dwarfs. In this question, we will examine how microlensing can be used to discover exoplanets by studying the gravitational interaction of light with the planet and its host star.



**Figure 6:** The lensing setup in both the aligned (left panel) and unaligned (right panel) cases. Note that the diagrams have been greatly scaled up in the vertical direction to make them visible, and in reality all rays are almost parallel. Credit: Ben Woodrow

- a. First, we will consider the scenario where the source  $S$ , the lensing mass  $L$ , and the observer  $O_1$  are directly aligned, as shown in the left of Figure 6. This is an especially interesting case, as light from the source is able to reach the observer by passing around the source in any direction, creating a circular image known as an Einstein Ring, with angular radius  $\theta_E$ . We will assume that the lensing deflection occurs instantaneously at the point where the light passes through the lensing plane  $P_L$ . Since all the angles involved are extremely small, the impact parameter  $b$  is simply the distance between this point and  $L$ .

- (i) Show that the Einstein radius is given by the formula

$$\theta_E = \sqrt{\frac{4GM}{c^2} \cdot \frac{D_{LS}}{D_S D_L}}.$$

- (ii) Calculate the Einstein radius for a hypothetical Jupiter-sized exoplanet in a system with parallax 0.196 milliarcseconds (mas), lensing light from a source star with parallax 0.087 mas. Jupiter has a mass of  $318 M_\oplus$ .

b. We will now consider how the lensed source will appear to an observer  $O_2$ , still situated in the plane  $P_O$ , but now at a distance  $a$  from  $O_1$ . In this case, instead of seeing the full Einstein Ring around the lens, the observer will see two point images at angles  $\theta_+$  and  $\theta_-$ , one each side of the lens. The situation is shown in the right panel of Figure 6. This a much more typical scenario, as perfect alignments like the one in the previous part are extremely rare.

- (i) Find an equation which must be satisfied by each of the angles  $\theta_{\pm}$ . Hence, find show that  $\theta_+$  in terms of  $\theta_E$  and the Source-Observer-Lens angle  $\varphi$  is given by

$$\theta_+ = \frac{1}{2} \left( \varphi + \sqrt{\varphi^2 + 4\theta_E^2} \right)$$

and find an equivalent formula for  $\theta_-$ .

- (ii) Determine the maximum value of the angular displacement of the image at  $\theta_+$  from the unlensed position of the source as  $\varphi$  varies, and the corresponding value of  $\varphi$ .
- (iii) What is the smallest diameter of telescope which would be able to resolve this maximal displacement for the hypothetical exoplanet mentioned earlier?

As you have just seen, the deflection caused by typically sized exoplanets is typically not directly observable by conventional optical means. Fortunately, the lensing also has a secondary effect of magnifying the brightness of the source, which is much more easily detected.

c. We will now attempt to calculate the size of the amplification effect caused by the lensing.

- (i) Find expressions for  $a$  and the impact parameters  $b_{\pm}$  in terms of  $\varphi$ ,  $\theta_E$  and the various  $D$ .
- (ii) Consider changing  $\varphi$  by a small increment  $d\varphi$ . Determine the corresponding changes  $da$  and  $db_{\pm}$ .
- (iii) By considering the total power incident on the annulus of points in  $P_O$  between  $a$  and  $a + da$  with and without the lensing effect, determine the brightness of each of the two images compared to the unlensed brightness of the source. Give your answer in terms of the ratio  $u = \frac{\varphi}{\theta_E}$ .
- (iv) Assuming that the observer cannot resolve the two images, show that the total amplification factor is given by

$$A = \frac{u^2 + 2}{u\sqrt{u^2 + 4}}.$$

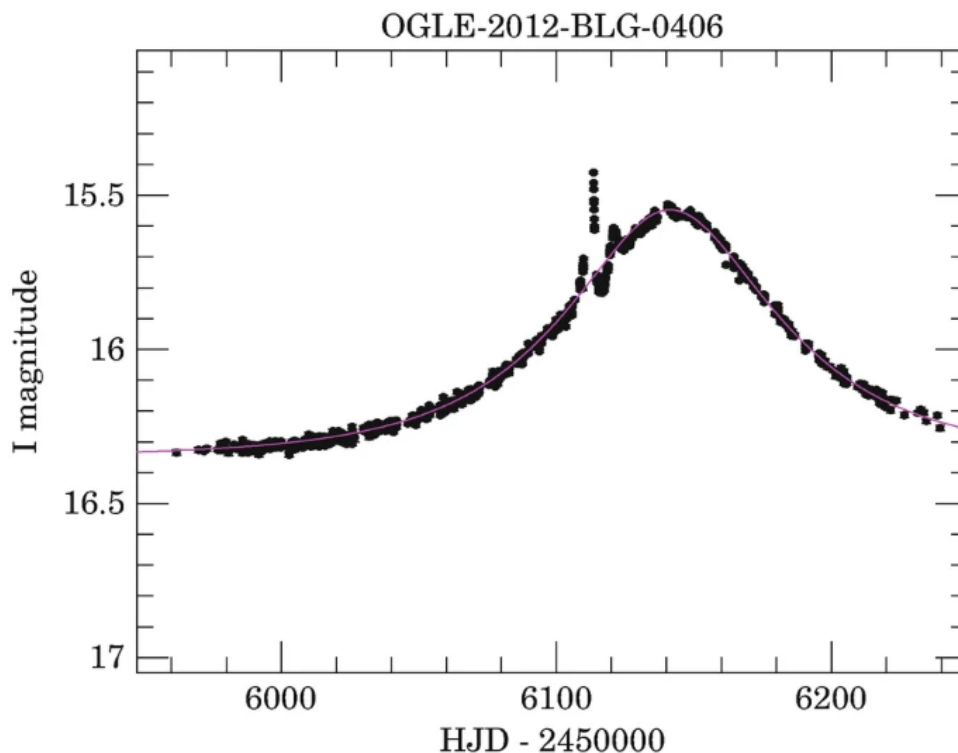
- (v) For the same hypothetical exoplanet considered earlier, calculate the percentage increase in source brightness due to the lensing effect given  $\varphi = 0.07$  mas.

This increase in brightness is much more easily detected than the change in image position angle found earlier, and so is the focus of many astronomical surveys.

*This question continues on the next page.*

d. In 2012, researchers at the Optical Gravitational Lensing Experiment (OGLE) studied a microlensing event caused by a star and its exoplanet lensing the light from a more distant star. The light curve is shown in Figure 7. Most of the curve is a typical single source microlensing event with the star as the lens, as we considered in b. earlier. The anomaly at around  $t = 6130$  is due to interaction with the exoplanet. As the source and lens move across the sky relative to each other, the value of  $u$  changes, causing the variation in the brightness of the source. We will assume that the source and lens are moving with a constant relative angular velocity.

- (i) Using values taken from Figure 7, a suitable numerical method and the formula given above for  $A$ , determine the minimum value of  $u$  over the course of this lensing interaction.
- (ii) Determine the Einstein crossing time  $t_E$ , defined as the time taken for the source and lens to move relative to each other by an angle  $\theta_E$ .



**Figure 7:** The light curve from the microlensing event OGLE-2012-BLG-0406, showing the variation in apparent magnitude against time in days. Credit: Udalski (2018).

These parameters describing the geometry of the microlensing event make it possible to fit a suitable model for a binary lens to match the anomaly seen in the data and determine the position and mass of the exoplanet.

END OF PAPER

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