Astro Round 2 2024 Paper Solutions and Marking Guidelines

Note for markers:

- Answers to two or three significant figures are generally acceptable. The solution may give more in order to make the calculation clear. Units should be present on final answers when appropriate.
- There are multiple ways to solve some of the questions; please accept all good solutions that arrive at the correct answer. Students getting the answer in a $\boxed{\text{box}}$ will get all the marks available for that calculation / part of the question (as indicated in red), so long as there are no unphysical / nonsensical steps or assumptions made (students may not explicitly calculate the intermediate stages and should not be penalised for this so long as their argument is clear).

Q1 – Maximum Payload to Mars [35 marks]

a. The spacecraft leaves Earth at time t_1 and travels on the elliptical transfer orbit until it reaches its aphelion at time t_2, where it intercepts Mars, which is assumed to have a circular orbit with semi-major axis 1.52 au. Assume the Earth's orbit is also circular and all orbits are coplanar.

i. Calculate the transfer time $\Delta t = t_2 - t_1$ in years, and hence calculate the Earth-Sun-Mars angle, φ, at $t = t_1$ required for a successful transfer.

Calculating the semi-major axis of the transfer orbit by taking the average of the perihelion distance (at Earth) and the aphelion distance (at Mars)

$$a_{HTO} = \frac{1}{2}\left(a_\oplus + a_{Mars}\right) = \frac{1}{2}(1 + 1.52) = 1.26\ \text{au}$$ [1]

Hence, using the simplified version of Kepler's 3rd Law for use in the Solar system, we can work out the period of the orbit and know that the transfer time is half that (since the probe only travels half an ellipse)

$$\Delta t = \frac{1}{2}a_{HTO}^{3/2} = \frac{1}{2} \times 1.26^{3/2} = \boxed{0.707\ \text{years}}$$ (must be in years) [1] [2]

[Accept using the SI units version of Kepler's 3rd Law, which with the given precision and values of constants leads to $\Delta t = \frac{1}{2}\sqrt{\frac{4\pi^2}{GM_\odot}a_{HTO}^3} = \boxed{0.710\ \text{years}}$]

The phase angle we are after must be the angle that Mars travels through in time Δt subtracted from π radians (or 180°)

$$\varphi = \pi - 2\pi \times \frac{\Delta t}{T_{Mars}} = \pi - 2\pi \times \frac{0.707}{1.52^{3/2}} = \boxed{0.771\ \text{rad}} \quad (= 44.1°)$$ [1] [1]

[This value should be the same whichever version of Kepler's third law they used since the corrective factor between the two methods cancels out if they used the same method to get the period of Mars]

ii. Calculate the time between consecutive occurrences of this phase angle, known as the synodic period T_{syn}.

Looking at the difference in angular velocities

$$\Delta\omega = \omega_\oplus - \omega_{Mars} = \frac{2\pi}{1} - \frac{2\pi}{1.52^{3/2}} = 2.93\ \text{rad year}^{-1}$$ [1]

$$\therefore T_{syn} = \frac{2\pi}{\Delta\omega} = \frac{2\pi}{2.93} = \boxed{2.14\ \text{years}}$$ [1] [2]

[Alternative method if they use the known formula for synodic period:

$$\frac{1}{T_{syn}} = \frac{1}{T_{\oplus}} - \frac{1}{T_{Mars}} = \frac{1}{1} - \frac{1}{1.52^{\frac{3}{2}}} = 0.466\ year^{-1} \qquad [1]$$

$$\therefore T_{syn} = \frac{1}{0.466} = \boxed{2.14 \text{ years}} \qquad [1]$$

If no other marks scored, they can get the first mark for the period of Mars as 1.87 years (if using simplified Kepler 3) or 1.88 years (if using the SI units version with given constants) – this may have been done in the previous part of the question]

iii. Calculate the perihelion velocity, v_p, of the transfer orbit and the orbital velocity of the Earth, $v_{\oplus}$ (these are both in the Sun's rest frame) and hence show that the required increase in the velocity of the spacecraft, $\Delta v_{HTO,\oplus}$ (parallel to the Earth's orbit) in the Earth's rest frame is $\Delta v_{HTO,\oplus} = v_p - v_{\oplus} \approx 3 \text{ km s}^{-1}$.

Using the vis-viva equation to get the perihelion velocity

$$v_p = \sqrt{GM_{\odot}\left(\frac{2}{r_p} - \frac{1}{a}\right)}$$

$$= \sqrt{6.67 \times 10^{-11} \times 1.99 \times 10^{30} \times \left(\frac{1}{1.50\times10^{11}} - \frac{1}{1.26\times1.50\times10^{11}}\right)}$$

$$= \boxed{32.67 \text{ km s}^{-1}} \qquad [1] \qquad [1]$$

The orbital velocity of the Earth

$$v_{\oplus} = \sqrt{\frac{GM_{\odot}}{r}} = \sqrt{\frac{6.67\times10^{-11}\times1.99\times10^{30}}{1.50\times10^{11}}} = \boxed{29.75 \text{ km s}^{-1}} \qquad [1] \qquad [1]$$

[Accept $v_{\oplus} = \frac{2\pi r}{T} = \frac{2\pi\times1.50\times10^{11}}{365.25\times24\times60\times60} = \boxed{29.87 \text{ km s}^{-1}}$ using given constants]

Hence the required increase in the velocity of the spacecraft is

$$\Delta v_{HTO,\oplus} = 32.67 - 29.75 = \boxed{2.925 \text{ km s}^{-1}} \quad \text{(must be at least 2 s.f.)} \quad [1] \qquad [1]$$

[Accept using the other value of $v_{\oplus}$ to give $\boxed{2.807 \text{ km s}^{-1}}$ for all the marks]

iv. What is the minimum distance from the Earth (in Earth radii) necessary for the spacecraft to be travelling at $\Delta v_{HTO,\oplus}$ (as measured by observers on Earth) and NOT still be in an elliptical orbit around the Earth?

Recognise that we want to know the distance at which this speed corresponds to escape velocity (or total energy is zero)

$$\Delta v_{HTO,\oplus} = \sqrt{\frac{2GM_{\oplus}}{r}} \quad \text{or} \quad \frac{1}{2}m\left(\Delta v_{HTO,\oplus}\right)^2 - \frac{GM_{\oplus}m}{r} = 0 \qquad [1]$$

$$\therefore d = \frac{2GM_{\oplus}}{\Delta v_{HTO,\oplus}^2} = \frac{2\times6.67\times10^{-11}\times5.97\times10^{24}}{(2.925\times10^3)^2} = 9.31 \times 10^7\ m = \boxed{14.6\ R_{\oplus}} \qquad [1] \qquad [2]$$

[Must be in units of $R_{\oplus}$ for the final mark. Allow full ecf on their value for $\Delta v_{HTO,\oplus}$. For reference, $\Delta v_{HTO,\oplus} = 2.807 \text{ km s}^{-1} \rightarrow d = \boxed{15.9\ R_{\oplus}}$ and $\Delta v_{HTO,\oplus} = 3 \text{ km s}^{-1} \rightarrow d = \boxed{13.9\ R_{\oplus}}$. If they get answers that are half these (due to assuming circular orbit) then only allow second mark]

[All of these answers are clearly much further away from Earth than even geostationary orbits and so it is apparent that we must escape the Earth's gravity with a hyperbolic orbit that then patches onto the elliptical Hohmann transfer orbit once far enough from Earth]

b. The spacecraft is initially in a circular orbit around the Earth with a radius of $r_0 = 8.00 \times 10^6$ m and orbital velocity $v_{c,\oplus}$ (all velocities in this part of the question are in the Earth's rest frame). It then makes a single, instantaneous change in velocity $\Delta v_{hyp,\oplus}$, parallel to its direction of travel, to accelerate into a hyperbolic orbit, with a perigee velocity of $v_{p,\oplus} = v_{c,\oplus} + \Delta v_{hyp,\oplus}$ at a perigee distance of r_0, and an asymptotic velocity $v_{\infty,\oplus}$ (the value that the spacecraft's velocity approaches as it gets far away from the Earth). To join the HTO we will assume $v_{\infty,\oplus} = \Delta v_{HTO,\oplus}$.

i. Calculate the total specific energy, $\varepsilon = \frac{E_{tot}}{m}$, of the desired hyperbolic orbit, where E_{tot} is the total energy of the orbit and m is the mass of the spacecraft.

For a hyperbolic orbit, the total energy will be positive and so will be equal to the KE when infinitely far from the Earth (so GPE = 0)

$$E_{tot} = GPE + KE = -\frac{GMm}{r} + \frac{1}{2}mv^2 = 0 + \frac{1}{2}mv_{\infty,\oplus}^2 \qquad [1]$$

$$\therefore \varepsilon = \frac{E_{tot}}{m} = \frac{1}{2}v_{\infty,\oplus}^2 = \frac{1}{2}\left(\Delta v_{HTO,\oplus}\right)^2 = \frac{1}{2}(2.925 \times 10^3)^2$$

$$= \boxed{4.279 \times 10^6 \text{ J kg}^{-1}} \qquad [1] \qquad [2]$$

[Allow full ecf for their value for $\Delta v_{HTO,\oplus}$. For reference, $\Delta v_{HTO,\oplus} = 2.807 \text{ km s}^{-1} \rightarrow \varepsilon = \boxed{3.940 \times 10^6 \text{ J kg}^{-1}}$ and $\Delta v_{HTO,\oplus} = 3 \text{ km s}^{-1} \rightarrow \varepsilon = \boxed{4.5 \times 10^6 \text{ J kg}^{-1}}$.]

ii. Find an expression for $\Delta v_{hyp,\oplus}$ in terms of ε, r_0, G and $M_\oplus$ and hence calculate its value.

Using energy conservation, we can get an expression for the perigee velocity

$$\varepsilon = -\frac{GM_\oplus}{r_0} + \frac{1}{2}v_{p,\oplus}^2 \qquad \therefore v_{p,\oplus} = \sqrt{2\left(\varepsilon + \frac{GM_\oplus}{r_0}\right)} \qquad [1]$$

Hence the expression for $\Delta v_{hyp,\oplus}$ is

$$\Delta v_{hyp,\oplus} = v_{p,\oplus} - v_{c,\oplus} = \boxed{\sqrt{2\left(\varepsilon + \frac{GM_\oplus}{r_0}\right)} - \sqrt{\frac{GM_\oplus}{r_0}}} \qquad [1] \qquad [2]$$

[Accept any algebraically equivalent forms]

Evaluating it with our calculated value of ε

$$\Delta v_{hyp,\oplus} = \sqrt{2\left(4.279 \times 10^6 + \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{8.00 \times 10^6}\right)} - \sqrt{\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{8.00 \times 10^6}}$$

$$= \boxed{3.34 \text{ km s}^{-1}} \qquad [1] \qquad [1]$$

[Allow full ecf from previous values of ε. If using $\varepsilon = 3.940 \times 10^6 \text{ J kg}^{-1} \rightarrow \Delta v_{hyp,\oplus} = \boxed{3.31 \text{ km s}^{-1}}$ and $\varepsilon = 4.5 \times 10^6 \text{ J kg}^{-1} \rightarrow \Delta v_{hyp,\oplus} = \boxed{3.36 \text{ km s}^{-1}}$]

[This is reassuringly a little bigger than what we calculated for $\Delta v_{HTO,\oplus}$ in the naive case where we ignored having to escape Earth's gravity – although it might surprise you to see that it's not that much bigger (only about 10%); this is because a lot of work has already been done by the rocket getting the spacecraft into that low Earth orbit to begin with.]

iii. Hence calculate the specific angular momentum $h = rv_\perp$ of the hyperbolic orbit, where r is the spacecraft's distance from the focus of the hyperbolic orbit (in this case the centre of the Earth) and $v_\perp$ is the component of its velocity perpendicular to that distance. Note that h is conserved over the orbit.

Since h is conserved over the whole orbit, the obvious place to choose is where we know both r and v – this is the perigee, where also very conveniently $v_\perp = v_{p,\oplus}$

$$h = rv_\perp = r_0 v_{p,\oplus}$$ [1]

$$\therefore h = r_0\sqrt{2\left(\varepsilon + \frac{GM_\oplus}{r_0}\right)} = 8.00 \times 10^6 \times \sqrt{2\left(4.279 \times 10^6 + \frac{6.67\times10^{-11}\times5.97\times10^{24}}{8.00\times10^6}\right)}$$

$$= \boxed{8.32 \times 10^{10}\ \mathrm{m^2\ s^{-1}}}$$ [1] [2]

[Allow full ecf from previous values of ε. If using $\varepsilon = 3.940 \times 10^6\ \mathrm{J\ kg^{-1}} \rightarrow h = \boxed{8.29 \times 10^{10}\ \mathrm{m^2 s^{-1}}}$ and $\varepsilon = 4.5 \times 10^6\ \mathrm{J\ kg^{-1}} \rightarrow h = \boxed{8.33 \times 10^{10}\ \mathrm{m^2 s^{-1}}}$]

iv. Let θ be the perigee-Earth-spacecraft angle, which increases with time t as the spacecraft moves along the hyperbolic orbit. Show that $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h}{r^2}$, derive an expression for $\frac{\mathrm{d}r}{\mathrm{d}t}$ (in terms of h, r, ε, and any relevant constants), and hence find an expression for $\frac{\mathrm{d}r}{\mathrm{d}\theta}$.

Combining the expression for specific angular momentum with a standard one for ω

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega = \frac{v_\perp}{r} = \frac{h/r}{r} = \frac{h}{r^2}$$ [1] [1]

[Since this was a 'show that' question, the mark is not for the final result but for explicitly substituting $v_\perp = h/r$ into $\omega = v_\perp/r$]

Considering energy conservation, using Pythagoras to resolve the velocity into perpendicular and parallel components

$$\varepsilon = -\frac{GM_\oplus}{r} + \frac{1}{2}v^2 = -\frac{GM_\oplus}{r} + \frac{1}{2}\left(v_\perp^2 + v_\parallel^2\right) = -\frac{GM_\oplus}{r} + \frac{1}{2}\left(\left(\frac{h}{r}\right)^2 + v_\parallel^2\right)$$ [1]

[This mark could be awarded for just correct application of Pythagoras if the student makes no further progress]

$$\therefore \frac{\mathrm{d}r}{\mathrm{d}t} = v_\parallel = \boxed{\sqrt{2\varepsilon + \frac{2GM_\oplus}{r} - \frac{h^2}{r^2}}}$$ [1] [2]

[Accept algebraically equivalent forms]

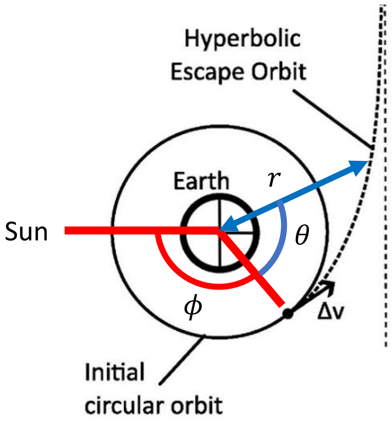
Hence, using the chain rule

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}t}{\mathrm{d}\theta}\frac{\mathrm{d}r}{\mathrm{d}t} = \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^{-1}\frac{\mathrm{d}r}{\mathrm{d}t} = \boxed{\frac{r^2}{h}\sqrt{2\varepsilon + \frac{2GM_\oplus}{r} - \frac{h^2}{r^2}}}$$ [1] [1]

[Accept algebraically equivalent forms]

v. It can be shown that solving your expression for $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ gives the result $r = \frac{p}{1+e\cos\theta}$ where p and e are constants given by $p = \frac{h^2}{GM_\oplus}$ and $e = \sqrt{1 + \frac{h^2\varepsilon}{G^2M_\oplus^2}}$. Use this to determine the required Sun-Earth-Spacecraft angle at the time when the spacecraft accelerates into its hyperbolic orbit.

Using Figure 1 from the paper for reference:



We have been given a polar form for the hyperbola, and we want to find the angle ϕ [accept any choice of notation]

Considering the asymptote, $r \to \infty$ as $\theta \to \theta_\infty$, so from the polar equation this is achieved as

$$1 + e\cos\theta_\infty = 0 \qquad [1]$$

[This mark can alternatively be scored for calculating $e = 1.09$]

$$\therefore \theta_\infty = \cos^{-1}\left(-\frac{1}{e}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{1+\frac{(8.32\times10^{10})^2\times4.279\times10^6}{(6.67\times10^{-11})^2(5.97\times10^{24})^2}}}\right)$$

$$= 2.73 \text{ rad } (= 156.5°) \qquad [1]$$

By considering the diagram

$$\phi + \theta_\infty = \frac{3\pi}{2} \text{ rad}$$

$$\therefore \phi = \frac{3\pi}{2} - \theta_\infty = \frac{3\pi}{2} - 2.73 = \boxed{1.98 \text{ rad}} \ (= 113.4°) \qquad [1] \qquad [3]$$

[Allow ecf using their values of ε and h. For reference, using the values of ε and h that come from $\Delta v_{HTO,\oplus} = 2.807 \text{ km s}^{-1} \to \phi = \boxed{1.96 \text{ rad}}$ $(= 112.5°)$ and $\Delta v_{HTO,\oplus} = 3 \text{ km s}^{-1} \to \phi = \boxed{1.99 \text{ rad}}$ $(= 113.9°)$.]

[Although not needed to solve this question, the constant p is known as the semi-latus rectum, and corresponds to the distance from the focus (at the centre of the Earth) to the hyperbola along a line perpendicular to the one between the focus and the perigee, known as the major axis. As you might have guessed, the constant e is the eccentricity of the hyperbola; for hyperbolae $e > 1$]

c. Once the spacecraft arrives at Mars, it will approach Mars on a hyperbolic orbit similar to its departure orbit from Earth a velocity (relative to Mars) of $\Delta v_{HTO,M}$. It then has to decrease its velocity by $\Delta v_{hyp,M}$ to be captured into a circular orbit of radius r (to be determined) and orbital velocity $v_{c,M}$. You are given that for this hyperbolic orbit, $\varepsilon = 3.49 \times 10^6\ \mathrm{J\ kg^{-1}}$, and that the mass of Mars, $M_M = 6.42 \times 10^{23}\ \mathrm{kg}$.

i. Find the value of r that minimises $\Delta v_{hyp,M}$. [Hint: instead of trying to use the same equation as used in b.(ii), rewrite it as $\gamma = f(\alpha)$ where $\gamma = \frac{1}{\sqrt{\varepsilon}}\Delta v_{hyp,M}$ and $\alpha = \frac{\varepsilon r}{GM_M}$ are both dimensionless variables, and find the value of α that minimises γ.]

Recasting our expression for $\Delta v_{hyp,M}$ from b.(ii) in terms of γ and α, as suggested

$$\Delta v_{hyp,M} = \sqrt{2\left(\varepsilon + \frac{GM_M}{r}\right)} - \sqrt{\frac{GM_M}{r}}$$

$$\therefore \gamma\sqrt{\varepsilon} = \sqrt{2\left(\varepsilon + \frac{\varepsilon}{\alpha}\right)} - \sqrt{\frac{\varepsilon}{\alpha}} = \sqrt{\varepsilon}\left(\sqrt{2 + \frac{2}{\alpha}} - \sqrt{\frac{1}{\alpha}}\right)$$

$$\therefore \gamma = \sqrt{2 + \frac{2}{\alpha}} - \sqrt{\frac{1}{\alpha}}$$ [1]

The minimum will correspond to a turning point, so we look where $\frac{\mathrm{d}\gamma}{\mathrm{d}\alpha} = 0$ and make use of the chain rule

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\alpha} = \frac{1}{2}\left(2 + \frac{2}{\alpha}\right)^{-1/2}(-2\alpha^{-2}) - \left(-\frac{1}{2}\alpha^{-3/2}\right)$$ [correct differentiation] [1]

$$\therefore \frac{1}{2}\left(2 + \frac{2}{\alpha}\right)^{-1/2}(-2\alpha^{-2}) - \left(-\frac{1}{2}\alpha^{-3/2}\right) = 0$$

$$\therefore \frac{1}{2}\left(2 + \frac{2}{\alpha}\right)^{-1/2}(2\alpha^{-2}) = \left(\frac{1}{2}\alpha^{-3/2}\right)$$

$$\therefore \left(2 + \frac{2}{\alpha}\right)^{-1/2} = \frac{1}{2}\alpha^{1/2}$$

$$\therefore 2 + 2\alpha^{-1} = 4\alpha^{-1} \quad \therefore 2 = 2\alpha^{-1} \quad \therefore \boxed{\alpha = 1}$$ [1] [3]

Hence the value of r that minimises $\Delta v_{hyp,M}$ is

$$r = \frac{GM_M\alpha}{\varepsilon} = \frac{6.67\times10^{-11}\times6.42\times10^{23}\times1}{3.49\times10^{6}} = \boxed{1.23 \times 10^7\ \mathrm{m}}$$ [1] [1]

[If this answer was achieved without following the suggested hint of recasting the expression in terms of γ and α, the student can still receive 4 marks]

ii. Hence determine the minimum $\Delta v_{hyp,M}$ to capture into a circular orbit at Mars.

Substituting our value of r into the equation for $\Delta v_{hyp,M}$

$$\Delta v_{hyp,M} = \sqrt{2\left(\varepsilon + \frac{GM_M}{r}\right)} - \sqrt{\frac{GM_M}{r}}$$

$$= \sqrt{2\left(3.49 \times 10^6 + \frac{6.67\times10^{-11}\times6.42\times10^{23}}{1.23\times10^{7}}\right)} - \sqrt{\frac{6.67\times10^{-11}\times6.42\times10^{23}}{1.23\times10^{7}}}$$

$$= \boxed{1.87\ \mathrm{km\ s^{-1}}}$$ [1] [1]

[Valid alternative method: since $\gamma = \sqrt{2 + \frac{2}{\alpha}} - \sqrt{\frac{1}{\alpha}}$ then if $\alpha = 1 \Longrightarrow \gamma = 1$, so $\Delta v_{hyp,M} = \gamma\sqrt{\varepsilon} = 1 \times \sqrt{3.49 \times 10^6} = \boxed{1.87\ \mathrm{km\ s^{-1}}}$]

d. From this we can calculate the magnitude of the total change in velocity by the spacecraft that has to be done by its rockets, $\Delta v_{tot} = \Delta v_{hyp,\oplus} + \Delta v_{hyp,M}$, and hence calculate the maximum payload mass that it can carry to Mars.

i. Consider the ejection by the spacecraft's engine of a small mass of fuel δm at an exhaust velocity v_e. Show that the corresponding change in velocity of the spacecraft, δv, is given by $\delta v = \frac{v_e}{m}\delta m$, where m is the current mass of the spacecraft.

By conserving momentum (should be explicitly stated)

$$m\delta v - v_e \delta m = 0 \qquad \therefore \delta v = \frac{v_e}{m}\delta m$$ [1] [1]

[Since it is a 'show that' question, the mark is for clear reasoning rather than the final algebraic answer]

ii. Hence, by integrating, show that the total change in velocity of the spacecraft is given by $\Delta v = v_e \ln\left(\frac{m_1}{m_2}\right)$, where m_0 is the dry mass of the spacecraft (total mass of everything except fuel), and m_1 is the wet mass (total mass including fuel).

Setting up the integral with the bounds the right way round

$$\Delta v = \int_{m_0}^{m_1} \frac{v_e}{m}\,\mathrm{d}m$$ [1]

$$\therefore \Delta v = [v_e \ln m]_{m_0}^{m_1} = v_e \ln m_1 - v_e \ln m_0 = v_e \ln \frac{m_1}{m_0}$$ [1] [2]

[Since it is a 'show that' question, the second mark is for correct evaluation of the integral, clearly showing how they have integrated 1/m]

iii. A future spacecraft for travelling to Mars has a dry mass (not including payload) of 1000 kg, can carry 5000 kg of fuel, and has an effective exhaust velocity from a hydrogen-oxygen rocket of 4.25 km s^{-1}. Calculate the maximum mass of payload it can carry to Mars, assuming that Δv_{tot} is the total change in velocity needed for the mission.

Expanding the equation of the previous part to include the described quantities with correct dry and wet mass expressions [accept any choice of notation]

$$\Delta v_{tot} = v_e \ln \frac{m_{empty} + m_{payload} + m_{fuel}}{m_{empty} + m_{payload}}$$ [1]

Rearranging for $m_{payload}$

$$e^{\Delta v_{tot}/v_e} = \frac{m_e + m_p + m_f}{m_e + m_p}$$

$$\therefore e^{\Delta v_{tot}/v_e}\left(m_e + m_p\right) = m_e + m_p + m_f$$

$$\therefore e^{\Delta v_{tot}/v_e} m_p - m_p = m_e + m_f - e^{\Delta v_{tot}/v_e} m_e$$

$$\therefore m_p = \frac{m_f + (1 - e^{\Delta v_{tot}/v_e})m_e}{(e^{\Delta v_{tot}/v_e} - 1)}$$ [1]

Using our earlier values of $\Delta v_{hyp,\oplus}$ from b.(ii) and $\Delta v_{hyp,M}$ from c.(ii)

$$\Delta v_{tot} = \Delta v_{hyp,\oplus} + \Delta v_{hyp,M} = 3.34 + 1.87 = 5.21 \text{ km s}^{-1}$$

$$\therefore m_p = \frac{5000 + \left(1 - \exp\left(\frac{5.21}{4.25}\right)\right) \times 1000}{\exp\left(\frac{5.21}{4.25}\right) - 1} = \boxed{1077 \text{ kg}}$$ [1] [3]

[Allow ecf from their values of $\Delta v_{hyp,\oplus}$ and $\Delta v_{hyp,M}$. For reference, if using $\Delta v_{hyp,\oplus} = 3.31 \text{ km s}^{-1} \rightarrow m_p = \boxed{1100 \text{ kg}}$, and $\Delta v_{hyp,\oplus} = 3.36 \text{ km s}^{-1} \rightarrow m_p = \boxed{1062 \text{ kg}}$. If they use the given value of $\Delta v_{tot} = 5 \text{ km s}^{-1} \rightarrow m_p = \boxed{1229 \text{ kg}}$]

[In reality, the Orion spacecraft in the NASA Artemis programme (planned to send humans to Mars) has a dry mass 10 times heavier than this so the fuel requirements are enormous!]

Q2 – *Discworld* Astrophysics [30 marks]

a. Figure 3 shows two logarithmic plots of solar intensity versus time over the duration of a year on the disc. In each case, observers recorded the solar intensity every 24 hours. One of the plots shows the curves obtained by observers at the same location each measuring at a different time of day, while the other plot shows the curves obtained by observers measuring at the same time of day as each other but from various locations on the disc.

i. With clear and explicit reasoning, decide which plot is which. Guesses will not be credited.

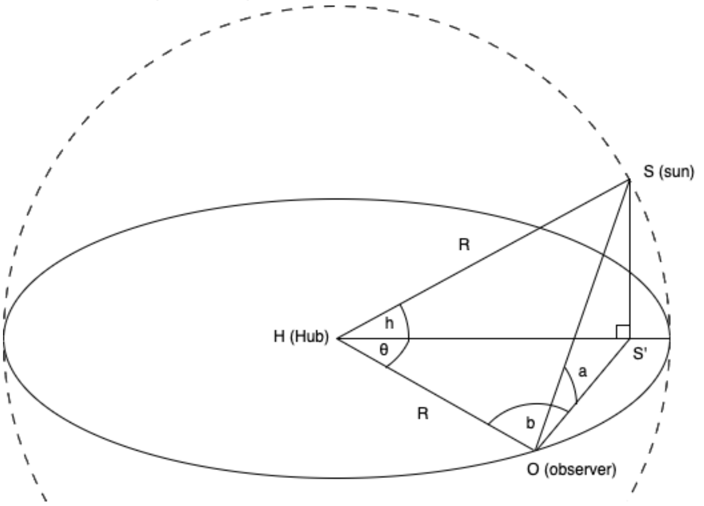
In Plot 1, all 5 observers record the same intensity on the winter solstice (i.e. 200 and 600 days after the first summer solstice) [1]

(Since intensity / distance to sunlet is the same all day on the winter solstice that means)
$\boxed{\text{Plot 1 is taken by observers at the same location}}$ but different times [1] [2]
(and so $\boxed{\text{Plot 2 is taken by observers at the same time}}$ but different locations)

[Do not give the second mark for correct guesses without explanation. If they do not link the shared intensity of the observers in Plot 1 to the winter solstice but get the correct answer, give one mark only. Expect to see both plots allocated a description for the second marking point but do not penalise if the students only write out one in full and leave the second implied]

ii. Derive an equation relating the number of days (n) since the first summer solstice of the year and the altitude of the sunlet (a) as observed at the Rim x hours after sunrise. It should be of the form tan(a) = β sin(πx/12) where β = f(n, x) which should be found.

Relevant diagram e.g.



[1]

[**Individual notation will vary** and could be done as separate diagrams, but should ideally have 4 points [H, S, S', O] and 4 angles (a, b, h, θ). Angle b may only be clearly indicated in a diagram in part iii) of this question, so look at these parts together when deciding diagram mark. No penalty if only two triangles drawn]

Altitude of sunlet $= h = \frac{2\pi x}{24} = \frac{\pi x}{12}$ (so $SS' = R\sin\frac{\pi x}{12}$ and $HS' = R\cos\frac{\pi x}{12}$) [1]

[This mark can be given from implied working for SS' and / or HS']

Angle around the Rim the sunrise has moved since summer solstice $= \theta = \frac{\pi n}{400}$ [1]

Applying the cosine rule to triangle HOS'

$$(OS')^2 = (HO)^2 + (HS')^2 - 2(HO)(HS')\cos\theta$$

$$\therefore (OS')^2 = R^2 + R^2\cos^2\frac{\pi x}{12} - 2R^2\cos\frac{\pi x}{12}\cos\frac{\pi n}{400}$$

$$\therefore OS' = R\sqrt{1 - 2\cos\frac{\pi x}{12}\cos\frac{\pi n}{400} + \cos^2\frac{\pi x}{12}}$$ [1]

Hence in triangle OSS'

$$\tan(a) = \frac{SS'}{OS'} = \frac{R\sin\frac{\pi x}{12}}{R\sqrt{1-2\cos\frac{\pi x}{12}\cos\frac{\pi n}{400}+\cos^2\frac{\pi x}{12}}} = \frac{\sin\frac{\pi x}{12}}{\sqrt{1-2\cos\frac{\pi x}{12}\cos\frac{\pi n}{400}+\cos^2\frac{\pi x}{12}}}$$

$$\therefore \beta = \boxed{\left(1 - 2\cos\frac{\pi x}{12}\cos\frac{\pi n}{400} + \cos^2\frac{\pi x}{12}\right)^{-1/2}}$$ [1] [5]

[Accept algebraically equivalent forms e.g. $\boxed{\left(\left(\cos\frac{\pi x}{12} - \cos\frac{\pi n}{400}\right)^2 + \sin^2\frac{\pi n}{400}\right)^{-1/2}}$ from other valid methods, such as vector algebra. Simplification is not required. Expect students to state β explicitly but still give credit if they get the correct final version of tan (a) and leave β implied]

iii. Derive an equation relating n and the angle subtended at the observer by the Hub and the projection of the sunlet onto the disc (b). It should be of the form sin(b) = βγ where γ = g(n, x) which should be found.

Applying the sine rule to triangle HOS'

$$\frac{\sin(b)}{HS'} = \frac{\sin\frac{\pi n}{400}}{OS'}$$ [1]

$$\therefore \sin(b) = \frac{R\cos\frac{\pi x}{12}\sin\frac{\pi n}{400}}{R\sqrt{1-2\cos\frac{\pi x}{12}\cos\frac{\pi n}{400}+\cos^2\frac{\pi x}{12}}} = \frac{\cos\frac{\pi x}{12}\sin\frac{\pi n}{400}}{\sqrt{1-2\cos\frac{\pi x}{12}\cos\frac{\pi n}{400}+\cos^2\frac{\pi x}{12}}}$$

$$\therefore \gamma = \boxed{\cos\frac{\pi x}{12}\sin\frac{\pi n}{400}}$$ [1] [2]

[Accept algebraically equivalent forms from other valid methods. Expect students to state γ explicitly but still give credit if they get the correct final version of sin (b) and leave γ implied]

b. The warmest part of the day at the Hub is at midday, and at this peak temperature the top layer of the ground is at a temperature of 0 °C.

i. Assuming the Hub is thermally isolated from the rest of the disc and that the icy surface reflects 80% of the incident radiation but emits like a perfect black body, show that the luminosity of the sunlet is ~ 3×10^{-9} $L_\odot$. [Hint: Consider that at this temperature the surface is in thermal equilibrium and so the power absorbed by a 1 m^2 patch of ground equals the power emitted by that same patch.]

Incident power on a patch of ground of area A at a distance R from the sunlet

$$P_i = \frac{L}{4\pi R^2} \times A \times (1 - 0.8)$$ [1]

Using the hint given, in thermal equilibrium this must be equal to the emitted power, so

$$\frac{L}{4\pi R^2} \times A \times (1 - 0.8) = \sigma A T^4$$ [1]

Given that T = 0 °C = 273 K and that the areas cancel

$$L = \frac{4\pi R^2 \sigma T^4}{1-0.8} = \frac{4\pi \times (8000\times10^3)^2 \times 5.67\times10^{-8} \times 273^4}{0.2}$$

$$= 1.27 \times 10^{18}\ W = \boxed{3.31 \times 10^{-9}\ L_\odot} \quad (\sim 3 \times 10^{-9}\ L_\odot)$$ [1] [3]

[Must be in units of $L_\odot$ and at least 2 s.f. for the final mark since it was a 'show that'. If the student applies the concept of 80% reflected incorrectly then can still get second mark through ecf, although will not be able to get third mark]

ii. The people of Discworld see the same spectrum of colours as we do (plus a magical eighth colour known as octarine), so we assume that its sunlet is a blackbody radiator with a peak wavelength of 500 nm similar to our own Sun. Calculate the sunlet's radius, $r_\star$.

Using Wien's displacement law

$$T_\star = \frac{2.90\times10^{-3}}{\lambda_{max}} = \frac{2.90\times10^{-3}}{500\times10^{-9}} = 5800\text{ K} \qquad [1]$$

Using the Stephan-Boltzmann law

$$L = 4\pi r_\star^2 \sigma T_\star^4 \therefore r_\star = \sqrt{\frac{L}{4\pi\sigma T_\star^4}} = \sqrt{\frac{1.27\times10^{18}}{4\pi\times5.67\times10^{-8}\times5800^4}} = \boxed{3.96\times10^4\text{ m}} \quad [1] \qquad [2]$$

[If using $L = 3\times10^{-9}\,L_\odot$ then this leads to $r_\star = \boxed{3.77\times10^4\text{ m}}$ which also gets full credit]

[The luminosity and size of this sunlet emphasise how small it really is – it is actually entirely unphysical to have a star this small, but *Discworld* is a piece of fiction!]

c. A spherical object of negligible mass and radius r is a distance x from a mass m. Find the strength of the gravitational field of m at the location of the centre of the spherical object, g_{centre}, and at a point on the surface of the far side of the spherical object, g_{far}. Given that $r \ll x$, derive an expression for the tidal acceleration, a_{tidal}, on the spherical object due to mass m where $a_{\text{tidal}} = g_{\text{centre}} - g_{\text{far}}$.

Starting with the gravitational field strengths

$$g_{centre} = \frac{Gm}{x^2} \quad \text{and} \quad g_{far} = \frac{Gm}{(x+r)^2} \qquad \text{[0.5 marks for each]} \qquad [1]$$

Hence the tidal acceleration is

$$a_{tidal} = g_{centre} - g_{far} = \frac{Gm}{x^2} - \frac{Gm}{(x+r)^2} = \frac{Gm}{x^2} - \frac{Gm}{x^2\left(1+\frac{r}{x}\right)^2}$$

$$= \frac{Gm}{x^2}\left(1 - \left(1+\frac{r}{x}\right)^{-2}\right) \approx \frac{Gm}{x^2}\left(1 - \left(1 - \frac{2r}{x}\right)\right) \qquad [1]$$

$$= \frac{Gm}{x^2} \times \frac{2r}{x} = \boxed{\frac{2Gmr}{x^3}} \qquad [1] \qquad [3]$$

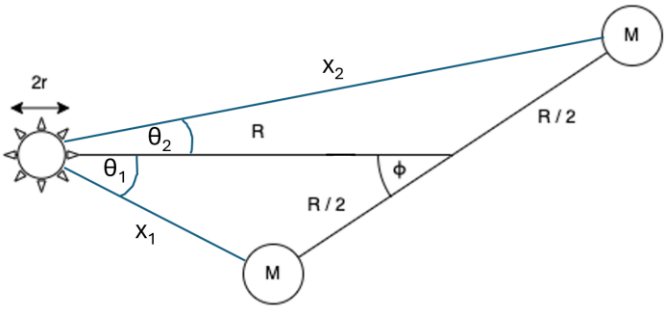
[The second mark is for correct application of the first order expansion suggested by the hint]

[We have used the centre and far side of the small spherical object in our derivation here, but you would get the exact same answer if you used the centre and near side – this is why we can use this expression in the next part of the question]

d. Model the disc as two equal and diametrically opposed point masses each of mass M separated by R symmetrically around the Hub, and neglect the gravitational effects of the elephants and the turtle on the sunlet. Take the disc's thickness as being R/4.

i. Derive an expression for the tidal acceleration on the surface of the sunlet due to the disc at sunrise on some day of the year. Give your answer in terms of M, R, r⋆ and ϕ where ϕ is the Sunlet-Hub-Supercontinent angle.

Relevant diagram e.g.



[1]

[No penalty here if they treat x₁ and x₂ as meeting at the same point on the surface of the sunlet and / or if they do not indicate the angle θ (notations may vary)]

Using the cosine rule

$$x_1^2 = R^2 + \left(\frac{R}{2}\right)^2 - 2R\left(\frac{R}{2}\right)\cos\phi = R^2\left(\frac{5}{4} - \cos\phi\right) \qquad [1]$$

$$x_2^2 = R^2 + \left(\frac{R}{2}\right)^2 - 2R\left(\frac{R}{2}\right)\cos(\pi - \phi) = R^2\left(\frac{5}{4} + \cos\phi\right) \qquad [1]$$

The total tidal acceleration at the surface of the sunlet in the direction of the centre of mass of the system is going to be equal to the sum of the components of the magnitudes of the tidal force caused by each (point) mass M

$$a_{tot} = \frac{2GMr_\star}{x_1^3}\cos\theta_1 + \frac{2GMr_\star}{x_2^3}\cos\theta_2 \qquad \text{[for realising use of cos θ]} \qquad [1]$$

$$= \boxed{\frac{2GMr_\star}{R^3}\left[\left(\frac{5}{4} - \cos\phi\right)^{-\frac{3}{2}}\cos\theta_1 + \left(\frac{5}{4} + \cos\phi\right)^{-\frac{3}{2}}\cos\theta_2\right]} \qquad [1] \qquad [5]$$

[If the student has a formula that ignores the θ factor (or their equivalent notation) max 4 marks. This is one route – there are many other (much longer) ways and so accept any valid approach that arrives at an algebraically equivalent expression e.g. using vectors with $\vec{\imath}$ the unit vector horizontally (choosing it to be positive to the left in our diagram) and $\vec{\jmath}$ the unit vector vertically (choosing it to be positive downwards) then you can derive an expression that avoids θ and is purely dependent on ϕ. Same first mark (diagram) then the remaining 4:

$$\overrightarrow{x_1} = \frac{R}{2}(2 - \cos\phi)\vec{\imath} + \frac{R}{2}(\sin\phi)\vec{\jmath} \quad \text{and} \quad |\overrightarrow{x_1}| = x_1 = \frac{R}{2}\sqrt{5 - 4\cos\phi} \qquad [1]$$

$$\overrightarrow{x_2} = \frac{R}{2}(2 + \cos\phi)\vec{\imath} - \frac{R}{2}(\sin\phi)\vec{\jmath} \quad \text{and} \quad |\overrightarrow{x_2}| = x_2 = \frac{R}{2}\sqrt{5 + 4\cos\phi} \qquad [1]$$

$$\therefore \overrightarrow{a_{tot}} = 2GMr_\star\left(\frac{\overrightarrow{x_1}}{x_1^4} + \frac{\overrightarrow{x_2}}{x_2^4}\right) = \frac{16GMr_\star}{R^3}\left(\frac{(2-\cos\phi)\vec{\imath} + (\sin\phi)\vec{\jmath}}{(5-4\cos\phi)^2} + \frac{(2+\cos\phi)\vec{\imath} - (\sin\phi)\vec{\jmath}}{(5+4\cos\phi)^2}\right)$$

$$= \frac{16GMr_\star}{R^3(25-16(\cos\phi)^2)^2}[((2 - \cos\phi)(5 + 4\cos\phi)^2 + (2 + \cos\phi)(5 - 4\cos\phi)^2)\vec{\imath}$$

$$+ \sin\phi((5 + 4\cos\phi)^2 - (5 - 4\cos\phi)^2)\vec{\jmath}]$$

$$= \frac{64GMr_\star}{R^3(17-8\cos 2\phi)^2}((23 - 2\cos 2\phi)\vec{\imath} + 10(\sin 2\phi)\vec{\jmath}) \qquad [1]$$

$$\therefore |\overrightarrow{a_{tot}}| = \boxed{\frac{64GMr_\star}{R^3(17-8\cos 2\phi)}\sqrt{529 + 4\cos^2 2\phi - 92\cos 2\phi + 100\sin^2 2\phi}} \qquad [1]$$

This can be further simplified to $\frac{64GMr_\star}{R^3(17-8u)^2}\sqrt{629 - 92u - 96u^2} = \boxed{\frac{64GMr_\star}{R^3(17-8u)^{3/2}}\sqrt{37 + 12u}}$ where $u = \cos 2\phi$, although this simplification is not needed for the marks]

ii. Use your equation to calculate the value of the maximum and minimum tidal acceleration experienced by the sunlet at sunrise. [Hint: Consider the value of ϕ needed in each case.]

First we calculate M given the density of the disc is the same as the Earth's average density

$$\rho_\oplus = \frac{M_\oplus}{\frac{4}{3}\pi R_\oplus^3} = \frac{5.97\times 10^{24}}{\frac{4}{3}\pi(6.37\times 10^6)^3} = 5514\ \mathrm{kg\ m^{-3}} \qquad [1]$$

$$\therefore M = \frac{1}{2}M_{tot} = \frac{1}{2}\rho_\oplus V_{disc} = \frac{1}{2}\rho_\oplus \pi R^2\left(\frac{R}{4}\right) = \frac{1}{8}\rho_\oplus \pi R^3$$

$$= \frac{1}{8}\times 5514 \times \pi \times (8000\times 10^3)^3 = 1.11\times 10^{24}\ \mathrm{kg} \qquad [1]$$

For the maximum tidal acceleration

$$\phi = 0 \quad (\text{or } \pi) \qquad (\text{so } \theta_1 = \theta_2 = 0) \qquad [0.5]$$

$$\therefore a_{tot,max} = \frac{2GMr_\star}{R^3}\left[\left(\frac{5}{4}-1\right)^{-\frac{3}{2}} + \left(\frac{5}{4}+1\right)^{-\frac{3}{2}}\right] = \frac{448GMr_\star}{27R^3}$$

$$= \frac{448\times 6.67\times 10^{-11}\times 1.11\times 10^{24}\times 3.96\times 10^4}{27\times(8000\times 10^3)^3} = \boxed{0.0950\ \mathrm{m\ s^{-2}}} \qquad [1] \qquad [3.5]$$

For the minimum tidal acceleration

$$\phi = \frac{\pi}{2} \qquad \left(\text{so } \theta_1 = \theta_2 = \tan^{-1}\left(\frac{R/2}{R}\right) = \tan^{-1}\left(\frac{1}{2}\right)\right) \qquad [0.5]$$

$$\therefore a_{tot,min} = \frac{2GMr_\star}{R^3}\left[\left(\frac{5}{4}-0\right)^{-\frac{3}{2}}\cos\left(\tan^{-1}\left(\frac{1}{2}\right)\right) + \left(\frac{5}{4}+0\right)^{-\frac{3}{2}}\cos\left(\tan^{-1}\left(\frac{1}{2}\right)\right)\right]$$

$$= \frac{64GMr_\star}{25R^3} = \frac{64\times 6.67\times 10^{-11}\times 1.11\times 10^{24}\times 3.96\times 10^4}{25\times(8000\times 10^3)^3} = \boxed{0.0147\ \mathrm{m\ s^{-2}}} \qquad [1] \qquad [1.5]$$

[If the minimum tidal acceleration is calculated correctly first, then it is worth 3.5 marks and the maximum tidal acceleration is 1.5 marks i.e. they swap. The mass may not be calculated explicitly so award those marks if the working is clear in subsequent stages. If their formula in the previous part does not include angle θ (or equivalent) allow full ecf here – they will get the same value for the maximum tidal acceleration, but a slightly overestimated value for the minimum tidal acceleration of $a_{tot,min} = \frac{32GMr_\star}{5\sqrt{5}R^3} = \boxed{0.0164\ \mathrm{m\ s^{-2}}}$. Allow full ecf using $r_\star = 37.7\ \mathrm{km}$ which gives $a_{tot,max} = \boxed{0.905\ \mathrm{m\ s^{-2}}}$ and $a_{tot,min} = \boxed{0.0140\ \mathrm{m\ s^{-2}}}$ (or, if leaving out θ, $a_{tot,min} = \boxed{0.0156\ \mathrm{m\ s^{-2}}}$)]

iii. Compare these to the surface gravitational field strength of the sunlet, assuming it has the same average density as the Sun, and comment on whether or not the sunlet will noticeably change shape at sunrise in Discworld.

First we calculate the mass of the sunlet, given it has the same average density as the Sun

$$M_\star = \frac{4}{3}\pi r_\star^3 \rho_\odot = \frac{4}{3}\pi r_\star^3 \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3} = \frac{r_\star^3}{r_\odot^3}M_\odot = \left(\frac{3.96\times 10^4}{6.96\times 10^8}\right)^3 \times 1.99\times 10^{30}$$

$$= 3.67\times 10^{17}\ \mathrm{kg} \qquad [1]$$

Hence the surface gravitational field strength is

$$g_\star = \frac{GM_\star}{r_\star^2} = \boxed{0.0156\ \mathrm{m\ s^{-2}}} \qquad [1] \qquad [2]$$

[Allow ecf using $r_\star = 37.7\ \mathrm{km}$ which gives $M_\star = 3.18\times 10^{17}\ \mathrm{kg}$ and $g_\star = \boxed{0.0149\ \mathrm{m\ s^{-2}}}$]

As the tidal acceleration is comparable to (or several times greater than) the surface gravitational field strength, $\boxed{\text{the sunlet WILL noticeably change shape at sunrise}}$ [1] [1]

[Needs to have a clear reason based upon comparison between $g_\star$ and a_{tot} to get this mark]

[The sunlet is actually within the Roche limit of the disc and so will form at least a teardrop shape (known as a Roche lobe) at sunrise since $0.94g_\star \leq a_{tot} \leq 6.09g_\star$. In fact, over the course of a year the sunlet should completely disintegrate – lucky it's fiction!]

Q3 – Heat Transfer in the Sun [35 marks]

a. Use the above information plus your own knowledge about the key differences between small and large stars to qualitatively explain the pattern seen in Figure 4.

In this section the marks are for some attempt at an explanation for the five distinct environments, so it is one mark for each row

Feature	**Explanation**
Fully convective in small stars	High opacity due to low average star temperature (and relatively high average density)
Radiative core in medium stars	Temperature is high enough for low opacity (and temperature dependence of p-p chain is not steep enough)
Convective outer layers in medium stars	Temperature drops with increasing radius until opacity too high for radiative transfer
Convective core in large stars	Hot enough for low opacity BUT CNO cycle is dominant and has extremely steep temperature dependence in the core (so convection favoured)
Radiative outer layers in large stars	Still very hot for low opacity, and since now in p-p chain burning shell the temperature dependence is no longer steep enough to generate convective mixing

[5]

[Do not have to mention any changes in density for the marks – the dependence of opacity on temperature is far more significant]

b. Consider a parcel of gas at a distance r from a star's centre. It has pressure p_{gas} and density ρ_{gas}, with initial values p_0 and ρ_0 respectively. The pressure and density of the surrounding stellar material will be denoted by $p_\star$ and $\rho_\star$ respectively. All of these variables are dependent on r.

i. If the parcel behaves adiabatically (i.e. no heat energy is exchanged between the parcel and its surroundings, leading to $p_{gas}(r)V_{gas}(r)^\gamma = \text{constant}$ where $V_{gas}(r)$ is the volume of the parcel and γ is a constant), derive an equation relating $p'_{gas} \equiv \frac{\mathrm{d}p_{gas}}{\mathrm{d}r}$ and $\rho'_{gas} \equiv \frac{\mathrm{d}\rho_{gas}}{\mathrm{d}r}$ of the form $p'_{gas} = B \times \rho'_{gas}$ where B is an expression to be determined.

First we need to use the given relations to connect the pressure and density of the gas

$$V_{gas} = \frac{M_{gas}}{\rho_{gas}} \propto \frac{1}{\rho_{gas}} \qquad \therefore \frac{p_{gas}}{\rho_{gas}^\gamma} = \text{constant} \qquad [1]$$

Differentiating with respect to r

$$\frac{d}{dr}\left(\frac{p_{gas}}{\rho_{gas}^\gamma}\right) = \frac{p'_{gas}}{\rho_{gas}^\gamma} - \gamma\frac{p_{gas}\rho'_{gas}}{\rho_{gas}^{\gamma+1}} = 0 \qquad [1]$$

This can be simplified to

$$\frac{p'_{gas}}{p_{gas}} - \gamma\frac{\rho'_{gas}}{\rho_{gas}} = 0 \qquad [1]$$

$$\therefore p'_{gas} = \gamma\frac{p_{gas}}{\rho_{gas}}\rho'_{gas} \qquad \therefore B = \boxed{\gamma\frac{p_{gas}}{\rho_{gas}}} \qquad [1] \qquad [4]$$

[Expect students to state B explicitly but still give credit if they get the correct final version of p'_{gas} and leave B implied]

ii. State the condition for the parcel to rise by convection.

$$\boxed{\rho_{gas} < \rho_\star} \qquad [1] \qquad [1]$$

[Accept $\boxed{\rho'_{gas} < \rho'_\star}$ too since parcel of gas starts in equilibrium, also award MP3 in part (iii)]

iii. The parcel begins at equilibrium with its surroundings so $p_\star = p_0$ and $\rho_\star = \rho_0$. Use the ideal gas law to relate the pressure $p_\star$, density $\rho_\star$, and temperature $T_\star$ gradients (with respect to r) in the star, and hence show that the condition for convection can be expressed as

$$\frac{p'_{gas}}{p_{gas}} < \gamma \frac{\rho_\star}{\rho_{gas}}\left(\frac{p'_\star}{p_\star} - \frac{T'_\star}{T_\star}\right)$$

From the ideal gas law

$$\frac{pV}{T} = \text{constant} \;\therefore \frac{\rho_\star T_\star}{p_\star} = \text{constant}$$ (rearranged with one eye on what we want)

$$\therefore \frac{d}{dr}\left(\frac{\rho_\star T_\star}{p_\star}\right) = \frac{\rho'_\star T_\star}{p_\star} + \frac{\rho_\star T'_\star}{p_\star} - \frac{\rho_\star T_\star p'_\star}{p_\star^2} = 0$$ [1]

$$\therefore \frac{\rho'_\star}{\rho_\star} = \frac{p'_\star}{p_\star} - \frac{T'_\star}{T_\star}$$ [1] [2]

Since the parcel of gas begins in equilibrium, our convection condition from b.(ii) becomes

$$\rho'_{gas} < \rho'_\star$$ [Award this mark if student already wrote this in b.(ii)] [1]

Using our result from b.(i)

$$\therefore \frac{\rho_{gas}}{\gamma p_{gas}} p'_{gas} < \rho'_\star$$

$$\therefore \frac{\rho_{gas}}{\gamma p_{gas}} p'_{gas} < \rho_\star\left(\frac{p'_\star}{p_\star} - \frac{T'_\star}{T_\star}\right)$$ [1] [2]

(this can then be rearranged to get the expression we are after)

[Since this is a ‘show that’, the marks are for the route rather than the final expression]

We can assume that as the parcel rises, it expands sufficiently quickly that it is always in pressure equilibrium so that $p_{gas} \approx p_\star$. Also, $\rho_\star \approx \rho_{gas}$ to a rough approximation, so the inequality in this question can be rewritten in a more useful form as:

$$\frac{p'_\star}{p_\star}\left(\frac{1}{\gamma} - 1\right) < -\frac{T'_\star}{T_\star}$$

c. Given that the photon gas energy density (i.e. energy per unit volume), $u = \alpha \times f(T, \sigma, c)$ where σ is the Stephan-Boltzmann constant and $\alpha = 4$ (a dimensionless constant), use dimensional analysis to find an expression for u in terms of these variables, and hence C_V.

First we need to express all values in SI base units

$$[u] = \mathrm{J\,m^{-3}} = \mathrm{kg\,m^{-1}\,s^{-2}} \qquad [T] = \mathrm{K}$$

$$[\sigma] = \mathrm{W\,m^{-2}\,K^{-4}} = \mathrm{kg\,s^{-3}\,K^{-4}} \qquad [c] = \mathrm{m\,s^{-1}}$$ [1]

Setting up the equations to allow for dimensional consistency

$$u = T^p \sigma^q c^r$$

$$\therefore [\mathrm{kg}]: 1 = q \quad [\mathrm{m}]: -1 = r \quad [\mathrm{s}]: -2 = -3q - r \quad [\mathrm{K}]: 0 = p - 4q$$ [1]

Solving these gives

$$p = 4, q = 1, r = -1 \quad \therefore u = \frac{\alpha\sigma T^4}{c} = \boxed{\frac{4\sigma T^4}{c}}$$ [1] [3]

[Allow all the marks if they set up their equations in terms of J, m, s, K instead of all base units]

We can now work out C_V by differentiating u with respect to T

$$C_V = \frac{\mathrm{d}u}{\mathrm{d}T} = \boxed{\frac{16\sigma T^3}{c}}$$ [1] [1]

d. Consider a beam of light of intensity I propagating across a short distance $\mathrm{d}x$ through the star. The change in the intensity of the beam across this distance $\mathrm{d}I$ is proportional to the number of atoms encountered as well as the number of photons doing the encountering.

i. Write an expression relating $\mathrm{d}I$ to the density $\rho_\star$ of the star as well as other parameters provided. There should be a constant of proportionality κ in your answer with dimensions of length squared divided by mass; this is the opacity of the gas. Integrate this result to find an expression for $I(x)$, setting $I(0) = I_0$.

From the given information (plus some dimensional analysis) it is clear that

$$\mathrm{d}I \propto \rho_\star I\,\mathrm{d}x$$

Introducing κ and recognising the intensity is decreasing

$$\mathrm{d}I = -\kappa\rho_\star I\,\mathrm{d}x$$

[1]

[This mark can be gained by either this line or the one before if the student makes no further progress]

Carrying out the integration by separating variables and using I_0 as the boundary condition

$$\int \frac{1}{I}\mathrm{d}I = -\kappa\rho_\star \int \mathrm{d}x \qquad \therefore \boxed{I(x) = I_0 e^{-\kappa\rho_\star x}}$$

[1] [2]

ii. Using your result, evaluate the photon mean free path ℓ, and hence write an expression for the radiative flux $L(r)$ that includes temperature, density, and opacity.

We know that $dI \propto$ probability of particle interacting between x and $x + dx$, so the expected value can be calculated using the (normalised) standard result from statistics

$$\ell = \frac{\int_0^\infty xI\mathrm{d}x}{\int_0^\infty I\mathrm{d}x}$$

[1]

Evaluating the numerator using integration by parts

$$\begin{aligned}\int_0^\infty xI\,\mathrm{d}x &= \int_0^\infty xI_0 e^{-\kappa\rho_\star x}\,\mathrm{d}x \\ &= \left[\frac{I_0 x e^{-\kappa\rho_\star x}}{-\kappa\rho_\star}\right]_0^\infty - \int_0^\infty -\frac{I_0}{\kappa\rho_\star} e^{-\kappa\rho_\star x}\,\mathrm{d}x \\ &= 0 + \frac{1}{\kappa\rho_\star}\int_0^\infty I\,\mathrm{d}x\end{aligned}$$

[1]

$$\therefore \ell = \frac{\int_0^\infty xI\mathrm{d}x}{\int_0^\infty I\mathrm{d}x} = \frac{\frac{1}{\kappa\rho_\star}\int_0^\infty I\mathrm{d}x}{\int_0^\infty I\mathrm{d}x} = \boxed{\frac{1}{\kappa\rho_\star}}$$

[1] [3]

Hence, using our results for C_V from c. in the given expression for λ, the radiative flux is

$$\begin{aligned}L(r) = -4\pi r^2\lambda\frac{\mathrm{d}T}{\mathrm{d}r} &= -4\pi r^2\left(\frac{1}{3}C_V\ell c\right)\frac{\mathrm{d}T}{\mathrm{d}r} \\ &= -4\pi r^2\left(\frac{1}{3}\times\frac{16\sigma T^3}{c}\times\frac{1}{\kappa\rho_\star}\times c\right)\frac{\mathrm{d}T}{\mathrm{d}r} \\ &= \boxed{-\frac{64\pi r^2\sigma T^3}{3\kappa\rho_\star}\frac{\mathrm{d}T}{\mathrm{d}r}}\end{aligned}$$

[1] [1]

[The second marking point can be awarded for recognising $\left[\frac{I_0 x e^{-\kappa\rho_\star x}}{-\kappa\rho_\star}\right]_0^\infty = 0$ if their second integral is not correct meaning they make no further progress. Accept ecf on their expression for C_V]

e. Derive an expression for the rate at which energy is dissipated due to shear stresses in the tachocline, in terms of h, μ, R and Δ where R is the outer radius of the radiative zone and Δ is the difference between the angular velocity of the convective zone at the equator and at the poles. Hence evaluate this energy loss (to 1 s.f. in W) given $h = 0.03\ R_\odot$, $\mu = 3\ \mathrm{cm^2\ s^{-1}}$, $R = 0.7\ R_\odot$ and $\Delta = 1 \times 10^{-6}\ \mathrm{rad\ s^{-1}}$.

Since the fictitious forces are entirely radial, they can be ignored for the purposes of this question. To further simplify matters, we can parametrise the problem in terms of $\theta = \varphi - \pi\ /\ 4$ where φ is latitude, so that the relative velocity u is zero when θ is zero. Let $\Omega(\theta)$ be the angular velocity of the convective zone and $u(r,\theta)$ be the speed of rotation of the tachocline where r is the distance from the centre of the sun.
(We will use partial differentiation here rather than implicit differentiation for simplicity, but this does not change the final answer compared to if done more rigorously)

From the first two bullet points given

$$\Omega(\theta) = \frac{\theta}{\pi/2}\Delta = \frac{2\theta}{\pi}\Delta \qquad [1]$$

From the third and fourth bullet points given

$$u(r,\theta) = \Omega(\theta)\frac{r-R}{h}r = \frac{2\theta r(r-R)}{\pi h}\Delta \qquad [1]$$

Hence, using the relation given in the fifth bullet point

$$\tau(r,\theta) = \mu\frac{\partial u}{\partial r} = \frac{2\theta}{\pi h}\mu\Delta(2r-R) \qquad [1]$$

The power dissipated across a volume element of area dA and height dr is given by

$$P = \text{force} \times \text{(relative) speed} \therefore \mathrm{d}P = \tau \mathrm{d}A \times \frac{\partial u}{\partial r}\mathrm{d}r \qquad [1]$$

The area element is

$$\mathrm{d}A = 2\pi(r\cos\varphi) \times r\,\mathrm{d}\theta = 2\pi r\cos\left(\theta + \frac{\pi}{4}\right) r\,\mathrm{d}\theta \qquad [1]$$

Hence our expression becomes

$$\mathrm{d}P = \frac{2\theta}{\pi h}\mu\Delta(2r-R) \times 2\pi r\cos\left(\theta+\frac{\pi}{4}\right) r\,\mathrm{d}\theta \times \frac{2\theta}{\pi h}\Delta(2r-R)\,\mathrm{d}r$$

$$= \frac{8}{\pi h^2}\mu\Delta^2\theta^2\cos\left(\theta+\frac{\pi}{4}\right)\mathrm{d}\theta\, r^2(2r-R)^2\mathrm{d}r \qquad [1]$$

For the total power we need to integrate over both θ and r

$$P = \iint \mathrm{d}P = 2\int_{-\pi/4}^{\pi/4}\mathrm{d}P\,\mathrm{d}\theta \times \int_R^{R+h}\mathrm{d}P\,\mathrm{d}r$$

$$= \frac{8}{\pi h^2}\mu\Delta^2\left[2\int_{-\pi/4}^{\pi/4}\theta^2\cos\left(\theta+\frac{\pi}{4}\right)\mathrm{d}\theta \times \int_R^{R+h} r^2(2r-R)^2\,\mathrm{d}r\right]$$

Evaluating the first integral using the provided integration result (note: these limits mean it is just for one hemisphere, corresponding to latitudes from $\varphi = 0$ to $\varphi = \frac{\pi}{2}$, hence the factor of 2 above)

$$\int_{-\pi/4}^{\pi/4}\theta^2\cos\left(\theta+\frac{\pi}{4}\right)\mathrm{d}\theta = \left(\frac{\pi}{4}\right)^2\sin\frac{\pi}{2} + 2\frac{\pi}{4}\cos 0 - 2\sin\frac{\pi}{2} = \frac{\pi^2}{16} + \frac{\pi}{2} - 2 \qquad [1]$$

Evaluating the second integral

$$\int_R^{R+h} r^2(2r-R)^2\,\mathrm{d}r = \int_R^{R+h} 4r^4 - 4Rr^3 + R^2r^2\,\mathrm{d}r$$

$$= \left[\frac{4}{5}r^5 - Rr^4 + \frac{1}{3}R^2r^3\right]_R^{R+h}$$

$$= \frac{4}{5}(R+h)^5 - R(R+h)^4 + \frac{1}{3}R^2(R+h)^3 - \frac{4}{5}R^5 + R^5 - \frac{1}{3}R^5 \qquad [1]$$

Taking out a common factor of R^5 and then expanding the brackets to get the first order terms only (since we know we only want the final result to 1 s.f and $h \ll R$)

$$= R^5\left[\frac{4}{5}\left(1+\frac{h}{R}\right)^5 - \left(1+\frac{h}{R}\right)^4 + \frac{1}{3}\left(1+\frac{h}{R}\right)^3 - \frac{4}{5} + 1 - \frac{1}{3}\right]$$

$$\approx R^5\left[\frac{4}{5}\left(1+5\frac{h}{R}\right) - \left(1+4\frac{h}{R}\right) + \frac{1}{3}\left(1+3\frac{h}{R}\right) - \frac{4}{5} + 1 - \frac{1}{3}\right]$$

$$= R^5\left[4\frac{h}{R} - 4\frac{h}{R} + \frac{h}{R}\right] = hR^4 \qquad [1]$$

Putting this all together for the final expression (and given there are two hemispheres)

$P = \frac{8}{\pi h^2}\mu\Delta^2\left[2\times\left(\frac{\pi^2}{16}+\frac{\pi}{2}-2\right)\times hR^4\right] = \boxed{\frac{\mu\Delta^2R^4}{\pi h}(\pi^2+8\pi-32)}$ [1] [10]

Substituting the given values

$P = \frac{3\times10^{-4}\times\left(1\times10^{-6}\right)^2\times\left(0.7\times6.96\times10^8\right)^4}{\pi\times0.03\times6.96\times10^8}(\pi^2+8\pi-32)$

$= 7.736\times10^{11}\ \mathrm{W} \sim \boxed{8\times10^{11}\ \mathrm{W}}$ (must be 1 s.f. for this mark) [1] [1]

[This is only small compared to the luminosity of the Sun which is why the radiative and convective zones have not yet reached a common rotational velocity and are unlikely to do so (due to this resistive force) throughout the Sun's life – it would need much greater dissipation for that]