BAAO
British Astronomy and Astrophysics Olympiad

British Astronomy and Astrophysics Olympiad 2023-2024

Astro Round 2

Tuesday 27th February 2024

This question paper must not be photographed or taken out of the exam room

Instructions

Time: 3 hours ($\sim$ 60 minutes for Q1, $\sim$ 55 minutes for Q2 and $\sim$ 65 minutes for Q3).

Questions: All three questions should be attempted. Each question contains independent parts so that later parts can be attempted even if earlier parts are incomplete.

Solutions: Answers and calculations are to be written on loose paper. START EACH QUESTION ON A NEW PAGE. Students should ensure their **name** and **school** is clearly written on the **first** answer sheet and that **all** pages are numbered. A standard formula booklet may be used.

Clarity: Solutions must be written legibly, in black pen, and working down the page. Scribble will not be marked and overall clarity is an important aspect of this exam.

Instructions: To accommodate students sitting the paper at different times, please **do not discuss** any aspect of the paper on the internet until 8 am Saturday 2nd March.

Calculators: Any standard calculator may be used, but calculators cannot be programmable and must not have symbolic algebra capability.

Training Dates and the IOAA (Vassouras, Rio de Janeiro, Brazil, 17th - 26th August 2024)

The team will be chosen from students taking this paper. The best students that are eligible to represent the UK at the IOAA will be invited to attend the **Training Camp** *to be held in Oxford from* **Tuesday 2nd to Saturday 6th April 2024**. *Astronomy material will be covered; problem solving skills and observational skills (telescope and naked eye observations) will be developed. At the Training Camp a Data Analysis exam along with a Round 3 theory paper will be sat. A team of five students (plus one reserve) will be selected for further training, including additional training camps in the summer.*

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Important Constants

Constant	Symbol	Value
Speed of light	c	$3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Earth's rotation period	1 day	24 hours
Earth's orbital period	1 year	365.25 days
parsec	pc	$3.09 \times 10^{16}\,\mathrm{m}$
Astronomical Unit	au	$1.50 \times 10^{11}\,\mathrm{m}$
Radius of the Sun	$R_\odot$	$6.96 \times 10^{8}\,\mathrm{m}$
Radius of the Earth	$R_\oplus$	$6.37 \times 10^{6}\,\mathrm{m}$
Mass of the Sun	$M_\odot$	$1.99 \times 10^{30}\,\mathrm{kg}$
Mass of the Earth	$M_\oplus$	$5.97 \times 10^{24}\,\mathrm{kg}$
Luminosity of the Sun	$L_\odot$	3.83×10^{26} W
Absolute magnitude of the Sun	$\mathcal{M}_\odot$	4.74
Hubble constant	H_0	70 km s^{-1} Mpc^{-1}
Stephan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Gravitational constant	G	6.67×10^{-11} m^{3} kg^{-1} s^{-2}
Boltzmann constant	k_B	1.38×10^{-23} J K^{-1}
Permittivity of free space	ε_0	8.85×10^{-12} F m^{-1}
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ H m^{-1}
Planck's constant	h	6.63×10^{-34} J s
Elementary charge	e	1.60×10^{-19} C
Proton rest mass	m_p	1.67×10^{-27} kg
Electron rest mass	m_e	9.11×10^{-31} kg
Wien's displacement law	$\lambda_\mathrm{max}T$	2.90×10^{-3} m K
Avagadro's constant	N_A	6.02×10^{23} mol^{-1}

Basic calculus formulae:

Chain rule $$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x))g'(x)$$

Product rule $$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = \frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

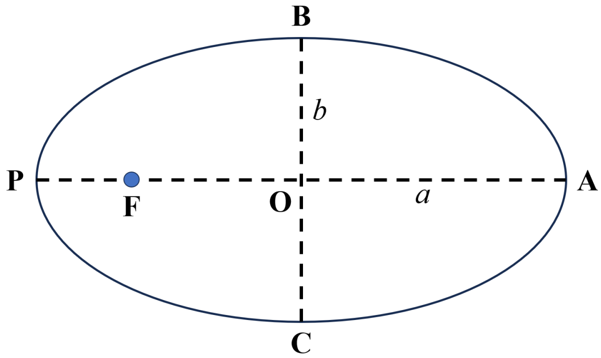
Quotient rule $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{\frac{\mathrm{d}u}{\mathrm{d}x}v - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

Integration by parts $$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Standard integral $$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

Important Formulae

You might find the diagram of an elliptical orbit below useful in solving some of the questions:



Elements of an elliptic orbit:

$a = \mathbf{OA}\ (= \mathbf{OP})$	semi-major axis
$b = \mathbf{OB}\ (= \mathbf{OC})$	semi-minor axis
$e = \sqrt{1 - \frac{b^2}{a^2}}$	eccentricity
$\mathbf{F}$	focus
$\mathbf{PF} = a(1-e)$	periapsis distance (shortest distance from **F**)
$\mathbf{AF} = a(1+e)$	apoapsis distance (longest distance from **F**)

Kepler's Third Law:

$$T^2 = \frac{4\pi^2}{GM}a^3$$

Vis-Viva Equation:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Wien's Displacement Law:

$$\lambda_{\text{max}}T = \text{constant}$$

Stephan-Boltzmann Law:

$$L = 4\pi R^2 \sigma T^4$$

Brightness (Intensity):

$$b = \frac{L}{4\pi d^2}$$

Magnitudes:

$$\frac{b_1}{b_0} = 10^{-0.4(m_1 - m_0)}$$

$$m - \mathcal{M} = 5\log\left(\frac{d}{10}\right)$$

Distance-Parallax Relation:

$$d = \frac{1}{p}$$

Rayleigh Criterion:

$$\theta = \frac{1.22\lambda}{D}$$

Redshift:

$$z = \frac{\Delta\lambda}{\lambda_{\text{emit}}} \approx \frac{v}{c}$$

Hubble's Law:

$$v = H_0 d$$

Qu 1. Maximum Payload to Mars

In order to travel from Earth to Mars, a spacecraft must travel on an elliptical orbit around the Sun. To make the transfer as efficient as possible, this ellipse should be chosen to have its perihelion (distance of closest approach to the sun) equal to the radius of the Earth's orbit, and its aphelion (greatest distance from the sun) equal to that of Mars, as shown in Figure 1 below below. This is known as a Hohmann transfer orbit (HTO), and one can classically calculate the increase in speed between the Earth and a spacecraft necessary to get onto the ellipse. However, this simplistic calculation does not account for the energy needed to escape the Earth's gravitational pull. This means it would only be accurate for a spacecraft orbiting the Sun at 1 au with no planet nearby - in reality, the change in speed of the spacecraft must be larger so it can get onto a hyperbolic orbit around the Earth that will asymptote to the transfer ellipse's perihelion velocity. By considering the increase and decrease in the speed of a spacecraft to move from escaping Earth's gravity and being captured by Mars' gravity, we can calculate the maximum payload we can transfer with current technology.

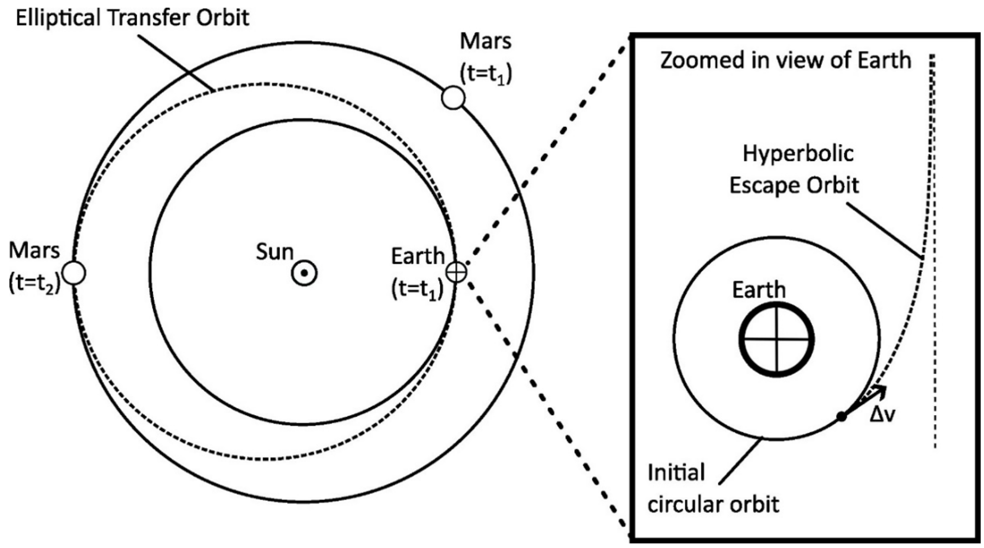


Figure 1: The Hohmann transfer orbit (HTO) between Earth and Mars, as well as the hyperbolic departure orbit from Earth needed to escape the Earth's gravity and move onto the elliptical transfer orbit. Credit: Ben Woodrow.

a. The spacecraft leaves Earth at time t_1 and travels on the elliptical transfer orbit until it reaches its aphelion at time t_2, where it intercepts Mars, which is assumed to have a circular orbit with semi-major axis 1.52 au. Assume the Earth's orbit is also circular and all orbits are coplanar.

(i) Calculate the transfer time $\Delta t = t_2 - t_1$ in years, and hence calculate the Earth-Sun-Mars angle, φ, at $t = t_1$ required for a successful transfer.

(ii) Calculate the time between consecutive occurrences of this phase angle, known as the synodic period T_{syn}.

(iii) Calculate the perihelion velocity, v_p, of the transfer orbit and the orbital velocity of the Earth, $v_\oplus$ (these are both in the Sun's rest frame) and hence show that the required increase in the velocity of the spacecraft, $\Delta v_{HTO,\oplus}$ (parallel to the Earth's orbit) in the Earth's rest frame is $\Delta v_{HTO,\oplus} = v_p - v_\oplus \approx 3$ km s^{-1}.

(iv) What is the minimum distance from the Earth (in Earth radii) necessary for the spacecraft to be travelling at $\Delta v_{HTO,\oplus}$ (as measured by observers on Earth) and NOT still be in an elliptical orbit around the Earth?

To put your answer to the previous question into context, geostationary satellites have an orbital radius of $\sim 6.6R_\oplus$, and given all spacecraft orbits prior to interplanetary transfer are in low Earth orbit (which is much closer) it should be clear that in order to enter the HTO, the spacecraft must first escape the gravity of the Earth. Whilst this could be achieved with a parabolic orbit, we need the asymptotic speed to be equal to the $\Delta v_{HTO,\oplus}$ calculated and so only a hyperbolic orbit can achieve this.

b. The spacecraft is initially in a circular orbit around the Earth with a radius of $r_0 = 8.00 \times 10^6$ m and orbital velocity $v_{c,\oplus}$ (all velocities in this part of the question are in the Earth's rest frame). It then makes a single, instantaneous change in velocity $\Delta v_{hyp,\oplus}$, parallel to its direction of travel, to accelerate into a hyperbolic orbit, with a perigee velocity of $v_{p,\oplus} = v_{c,\oplus} + \Delta v_{hyp,\oplus}$ at a perigee distance of r_0, and an asymptotic velocity $v_{\infty,\oplus}$ (the value that the spacecraft's velocity approaches as it gets far away from the Earth). To join the HTO we will assume $v_{\infty,\oplus} = \Delta v_{HTO,\oplus}$.

 (i) Calculate the total specific energy, $\varepsilon = \frac{E_{\text{tot}}}{m}$, of the desired hyperbolic orbit, where E_{tot} is the total energy of the orbit and m is the mass of the spacecraft.

 (ii) Find an expression for $\Delta v_{hyp,\oplus}$ in terms of ε, r_0, G and $M_\oplus$ and hence calculate its value. [Hint: you should find $\Delta v_{hyp,\oplus} > \Delta v_{HTO,\oplus}$.]

 (iii) Hence calculate the specific angular momentum $h = rv_\perp$ of the hyperbolic orbit, where r is the spacecraft's distance from the focus of the hyperbolic orbit (in this case the centre of the Earth) and $v_\perp$ is the component of its velocity perpendicular to that distance. Note that h is conserved over the orbit.

 (iv) Let θ be the perigee-Earth-spacecraft angle, which increases with time t as the spacecraft moves along the hyperbolic orbit. Show that $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h}{r^2}$, derive an expression for $\frac{\mathrm{d}r}{\mathrm{d}t}$ (in terms of h, r, ε, and any relevant constants), and hence find an expression for $\frac{\mathrm{d}r}{\mathrm{d}\theta}$.

 (v) It can be shown that solving your expression for $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ gives the result $r = \frac{p}{1+e\cos\theta}$ where p and e are constants given by $p = \frac{h^2}{GM_\oplus}$ and $e = \sqrt{1 + \frac{h^2\varepsilon}{G^2 M_\oplus^2}}$. Use this to determine the required Sun-Earth-Spacecraft angle at the time when the spacecraft accelerates into its hyperbolic orbit.

c. Once the spacecraft arrives at Mars, it will approach Mars on a hyperbolic orbit similar to its departure orbit from Earth a velocity (relative to Mars) of $\Delta v_{HTO,M}$. It then has to decrease its velocity by $\Delta v_{hyp,M}$ to be captured into a circular orbit of radius r (to be determined) and orbital velocity $v_{c,M}$. You are given that for this hyperbolic orbit, $\varepsilon = 3.49 \times 10^6$ J kg^{-1}, and that the mass of Mars, $M_M = 6.42 \times 10^{23}$ kg.

 (i) Find the value of r that minimises $\Delta v_{hyp,M}$. [Hint: instead of trying to use the same equation as used in b.(ii), rewrite it as $\gamma = f(\alpha)$ where $\gamma = \frac{1}{\sqrt{\varepsilon}}\Delta v_{hyp,M}$ and $\alpha = \frac{\varepsilon r}{GM_M}$ are both dimensionless variables, and find the value of α that minimises γ.]

 (ii) Hence determine the minimum $\Delta v_{hyp,M}$ to capture into a circular orbit at Mars.

d. From this we can calculate the magnitude of the total change in velocity by the spacecraft that has to be done by its rockets, $\Delta v_{tot} = \Delta v_{hyp,\oplus} + \Delta v_{hyp,M}$, and hence calculate the maximum payload mass that it can carry to Mars.

 (i) Consider the ejection by the spacecraft's engine of a small mass of fuel δm at an exhaust velocity v_e. Show that the corresponding change in velocity of the spacecraft, δv, is given by $\delta v = \frac{v_e}{m}\delta m$, where m is the current mass of the spacecraft.

 (ii) Hence, by integrating, show that the total change in velocity of the spacecraft is given by $\Delta v = v_e \ln \frac{m_1}{m_0}$, where m_0 is the dry mass of the spacecraft (total mass of everything except fuel), and m_1 is the wet mass (total mass including fuel).

 (iii) A future spacecraft for travelling to Mars has a dry mass (not including payload) of 1000 kg, can carry 5000 kg of fuel, and has an effective exhaust velocity from a hydrogen-oxygen rocket of 4.25 km s^{-1}. Calculate the maximum mass of payload it can carry to Mars, assuming that Δv_{tot} is the total change in velocity needed for the mission. *Use $\Delta v_{tot} = 5$ km s^{-1} if you did not make progress with previous parts.*

Qu 2. *Discworld* Astrophysics

The inhabitants of Terry Pratchett's fantasy *Discworld* series reside on a disc-shaped world carried through the universe by four elephants standing on a turtle. This disc is orbited by a "tiny sunlet" in what we shall assume to be 24-hour days with a uniform angular velocity and the geometry shown in Figure 2. The disc itself rotates about a perpendicular axis through its centre with a period of 800 days, resulting in eight seasons throughout the year, although they have a different definition of a 'solstice'. For them, a summer solstice (corresponding to the day with the highest peak temperature) occurs when the sunlet rises at your point on the Rim, and a second one happens 400 days later when the sunlet is setting at your point on the Rim, hence you get two summers (and indeed two of each season). Similarly, a winter solstice occurs 200 days after the summer solstice (when the disc has rotated a further 90°). The Hub (the centre of the disc) has no seasons and is covered in permafrost.

The radius of the disc, R, is 8000 km; assume that this is also approximately equal to the sunlet's orbital radius and that the disc's average density is equal to that of the Earth.

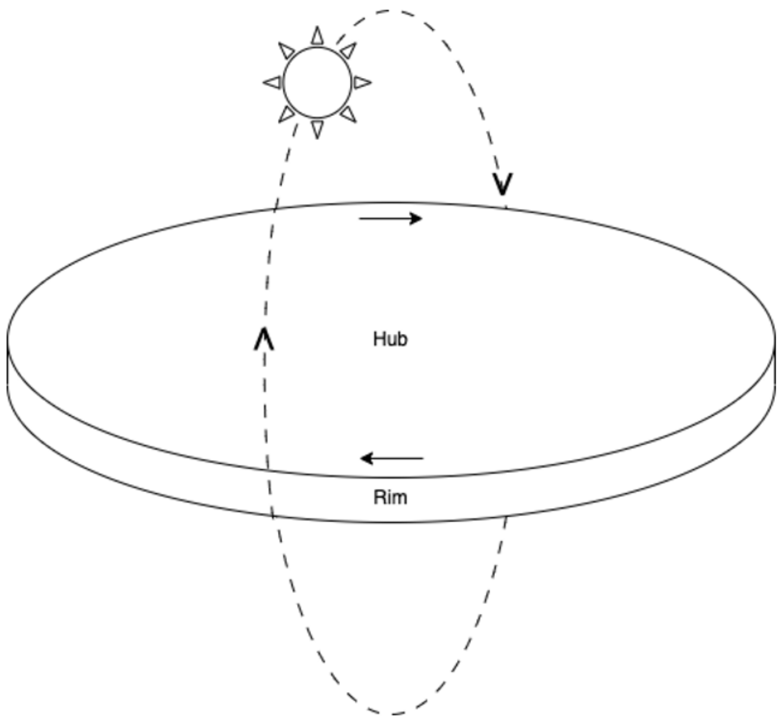


Figure 2: The overall geography of *Discworld*, showing the vast flat world being orbited by the tiny sunlet. The days are 24 hours in length and a year lasts 800 days. The edge of the disc is known as the Rim and the centre is known as the Hub. Credit: Francesca Di Cecio.

a. Figure 3 shows two logarithmic plots of solar intensity versus time over the duration of a year on the disc. In each case, observers recorded the solar intensity every 24 hours. One of the plots shows the curves obtained by observers at the same location each measuring at a different time of day, while the other plot shows the curves obtained by observers measuring at the same time of day as each other but from various locations on the disc.

   (i) With clear and explicit reasoning, decide which plot is which. Guesses will not be credited.

   (ii) Derive an equation relating the number of days (n) since the first summer solstice of the year and the altitude of the sunlet (a) as observed at the Rim x hours after sunrise. It should be of the form $\tan(a) = \beta \sin(\pi x/12)$ where $\beta = f(n, x)$ which should be found.

   (iii) Derive an equation relating n and the angle subtended at the observer by the Hub and the projection of the sunlet onto the disc (b). It should be of the form $\sin(b) = \beta\gamma$ where $\gamma = g(n, x)$ which should be found.

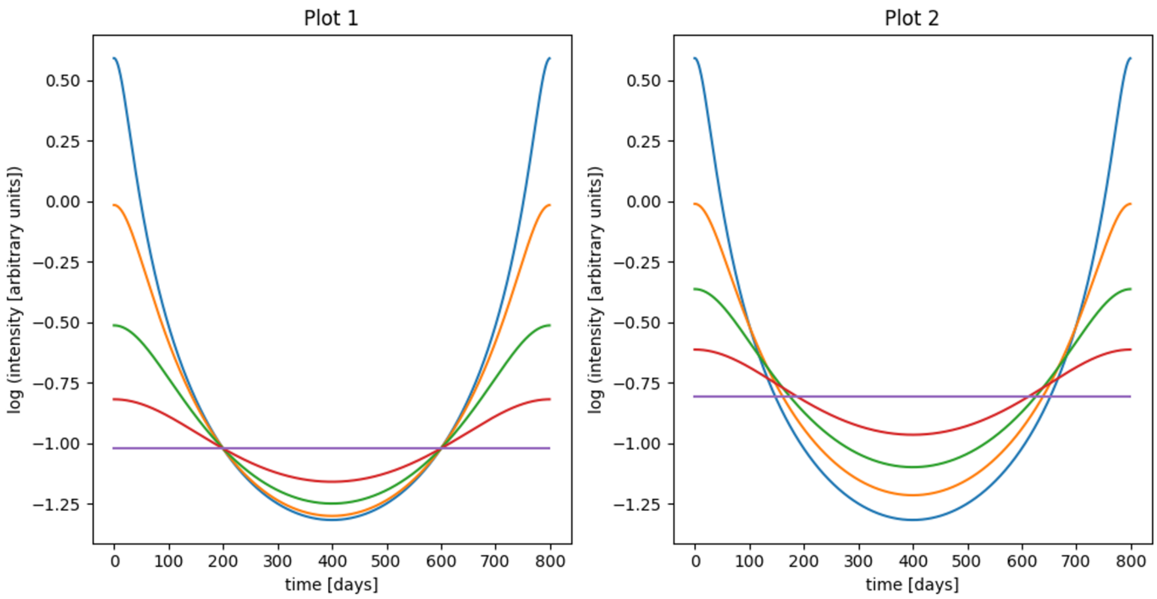


Figure 3: Two logarithmic plots of solar intensity versus time over the duration of a year on the disc. One corresponds to 5 observers taking readings once every 24 hours for a year from the same location but at different times of day. The other corresponds to 5 observers taking readings once every 24 hours for a year from different locations but at the same time of day. Credit: Francesca Di Cecio.

b. The warmest part of the day at the Hub is at midday, and at this peak temperature the top layer of the ground is at a temperature of 0 °C.

 (i) Assuming the Hub is thermally isolated from the rest of the disc and that the icy surface reflects 80% of the incident radiation but emits like a perfect black body, show that the luminosity of the sunlet is $\sim 3 \times 10^{-9}\, L_{\odot}$. [Hint: Consider that at this temperature the surface is in thermal equilibrium and so the power absorbed by a 1 m^2 patch of ground equals the power emitted by that same patch.]

 (ii) The people of *Discworld* see the same spectrum of colours as we do (plus a magical eighth colour known as octarine), so we assume that its sunlet is a blackbody radiator with a peak wavelength of 500 nm similar to our own Sun. Calculate the sunlet's radius, $r_{\star}$.

c. A spherical object of negligible mass and radius r is a distance x from a mass m. Find the strength of the gravitational field of m at the location of the centre of the spherical object, g_{centre}, and at a point on the surface of the far side of the spherical object, g_{far}. Given that $r \ll x$, derive an expression for the tidal acceleration, a_{tidal}, on the spherical object due to mass m where $a_{tidal} = g_{centre} - g_{far}$. [Hint: You may use the result $(1+y)^n \approx 1 + ny$ when $y \ll 1$.]

d. The disc's mass is dominated by the Supercontinent, on which most of the 41 novels are set, and a smaller landmass known as the Counterweight Continent, which is very dense due to its abundance of gold. For this question, model the disc as two equal and diametrically opposed point masses each of mass M separated by R symmetrically around the Hub, and neglect the gravitational effects of the elephants and the turtle on the sunlet. Take the disc's thickness as being $R/4$.

 (i) Derive an expression for the tidal acceleration on the surface of the sunlet due to the disc at sunrise on some day of the year. Give your answer in terms of M, R, $r_{\star}$ and ϕ where ϕ is the Sunlet-Hub-Supercontinent angle.

 (ii) Use your equation to calculate the value of the maximum and minimum tidal acceleration experienced by the sunlet at sunrise. [Hint: Consider the value of ϕ needed in each case.]

 (iii) Compare these to the surface gravitational field strength of the sunlet, assuming it has the same average density as the Sun, and comment on whether or not the sunlet will noticeably change shape at sunrise in *Discworld*.

Qu 3. Heat Transfer in the Sun

There are three main methods of heat transfer: conduction, convection and radiation. Since stars are made of plasma, only convection and radiation are significant, with conduction only important in white dwarfs. Convection is dominant in regions of the star that have a steep temperature gradient and / or high opacity (meaning the material is effectively opaque to EM radiation), whilst radiation is dominant in regions with a low temperature gradient and low opacity (meaning the material is much more transparent). This means that the heat transfer method that dominates will vary with distance from the core and also stellar mass (see Figure 4).

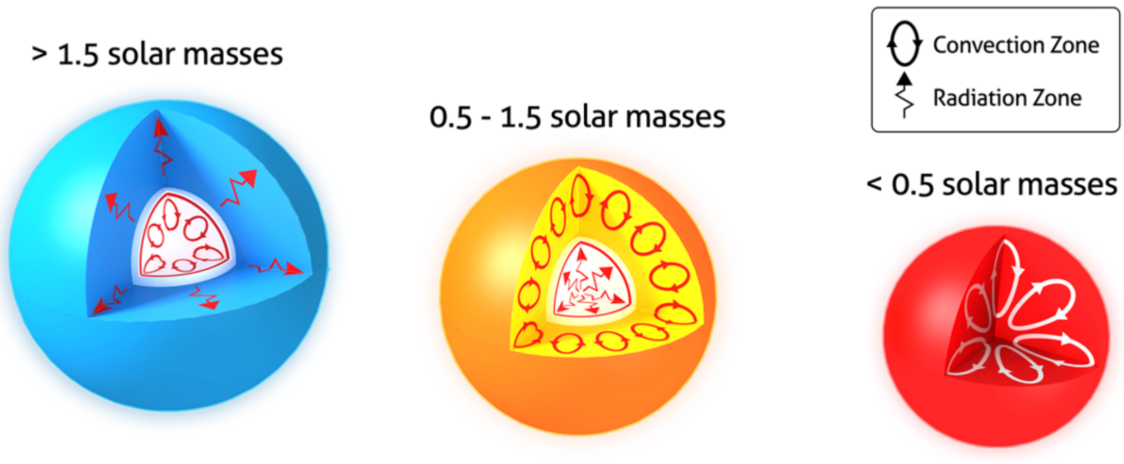


Figure 4: Types of heat transfer in different stars, showing the dependence on stellar mass on what type dominates at different distances from the core. Credit: Sun.org - www.sun.org.

You are given that opacity, κ, is related to the density and temperature of the plasma following Kramers' opacity law $\kappa \propto \rho T^{-7/2}$, and that in small stars the proton-proton chain for turning hydrogen into helium dominates in the core and has a relatively low temperature gradient whilst in large stars the CNO cycle dominates in the core, with a much steeper temperature gradient, and a shell of fusion via the proton-proton chain around that.

a. Use the above information plus your own knowledge about the key differences between small and large stars to qualitatively explain the pattern seen in Figure 4.

In the Sun, the radiative zone extends out to a radius $r \approx 0.7\ R_{\odot}$, with the convective zone at larger radii. In this question we will investigate the physics of the two zones, starting with the convective zone.

b. Consider a parcel of gas at a distance r from a star's centre. It has pressure p_{gas} and density ρ_{gas}, with initial values p_0 and ρ_0 respectively. The pressure and density of the surrounding stellar material will be denoted by $p_\star$ and $\rho_\star$ respectively. All of these variables are dependent on r.

   (i) If the parcel behaves adiabatically (i.e. no heat energy is exchanged between the parcel and its surroundings, leading to $p_{gas}(r)V_{gas}(r)^\gamma = \text{constant}$ where $V_{gas}(r)$ is the volume of the parcel and γ is a constant), derive an equation relating $p'_{gas} \equiv \frac{\mathrm{d}p_{gas}}{\mathrm{d}r}$ and $\rho'_{gas} \equiv \frac{\mathrm{d}\rho_{gas}}{\mathrm{d}r}$ of the form $p'_{gas} = B \times \rho'_{gas}$ where B is an expression to be determined.

   (ii) State the condition for the parcel to rise by convection.

   (iii) The parcel begins at equilibrium with its surroundings so $p_\star = p_0$ and $\rho_\star = \rho_0$. Use the ideal gas law to relate the pressure $p_\star$, density $\rho_\star$, and temperature $T_\star$ gradients (with respect to r) in the star, and hence show that the condition for convection can be expressed as

$$\frac{p'_{gas}}{p_{gas}} < \gamma \frac{\rho_\star}{\rho_{gas}} \left(\frac{p'_\star}{p_\star} - \frac{T'_\star}{T_\star} \right) .$$

Radiative flux refers to the total energy passing through a surface (in our case, a spherical layer of star) per unit time (i.e. the luminosity). It is proportional in magnitude to the temperature gradient across the surface and to the area of the layer, with a constant of proportionality λ known as thermal conductivity, as shown mathematically below,

$$L(r) = -4\pi r^2 \lambda \frac{\mathrm{d}T}{\mathrm{d}r} \, .$$

The kinetic theory of gases gives us that

$$\lambda = \frac{1}{3} C_V \ell c \, ,$$

where C_V is the heat capacity per unit volume of the photon gas, ℓ is the mean free path of photons in the gas, and c is the speed of light. In the following parts of the question, you will find expressions for C_V and ℓ to derive a more useful expression for radiative flux L as a function of distance from the centre of the star r.

c. Given that the photon gas energy density (i.e. energy per unit volume), $u = \alpha \times f(T, \sigma, c)$ where σ is the Stephan-Boltzmann constant and $\alpha = 4$ (a dimensionless constant), use dimensional analysis to find an expression for u in terms of these variables, and hence C_V.

d. Consider a beam of light of intensity I propagating across a short distance $\mathrm{d}x$ through the star. The change in the intensity of the beam across this distance $\mathrm{d}I$ is proportional to the number of atoms encountered as well as the number of photons doing the encountering.

(i) Write an expression relating $\mathrm{d}I$ to the density $\rho_\star$ of the star as well as other parameters provided. There should be a constant of proportionality κ in your answer with dimensions of length squared divided by mass; this is the opacity of the gas. Integrate this result to find an expression for $I(x)$, setting $I(0) = I_0$.

(ii) Using your result, evaluate the photon mean free path ℓ, and hence write an expression for the radiative flux $L(r)$ that includes temperature, density, and opacity.

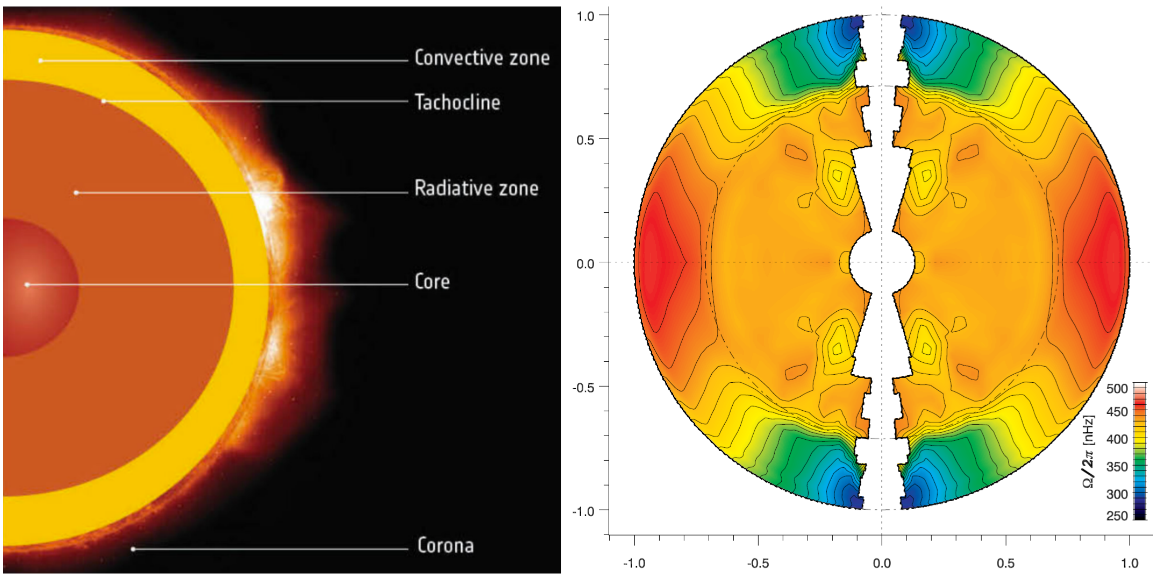


Figure 5: *Left:* A simplified cross-section through the Sun, showing the tachocline at the bottom of the convective zone. Credit: ESA.
Right: Rotation rate, as a function of depth and latitudes, inferred from 12.6 yr of helioseismology observations. The radiative zone clearly moves at a constant angular velocity, whilst the convective zone demonstrates that it has a higher angular velocity at the equator than at the poles. Credit: Korzennik and Eff-Darwich (2011).

Using helioseismology we can determine that the radiative zone of the Sun rotates like a solid body at a constant angular velocity, whilst the convective zone undergoes differential rotation (so it rotates faster at the equator than at the poles) as shown in Figure 5. The thin layer between the radiative and convective zones is called the tachocline and since the change in angular velocity is rapid as you move through it, it experiences strong shear forces. The rapid acceleration (or deceleration) of the plasma is part of the solar magnetic dynamo responsible for creating the magnetic field that leads to sunspots.

For the final part of this question you may assume the following:

- The angular velocity of the convective zone varies linearly with latitude.
- For mathematical convenience, the angular velocity of the radiative zone is equal to the angular velocity of the convective zone at a latitude of $45°$.
- The tachocline is a spherical shell of thickness h.
- The angular velocity of the tachocline varies linearly between the boundaries with the radiative zone and the convective zone.
- The parallel force per unit area due to a velocity gradient $\frac{\mathrm{d}u}{\mathrm{d}r}$ in a fluid (shear stress) is given by $\tau = \mu \frac{\mathrm{d}u}{\mathrm{d}r}$ where μ is the dynamic viscosity of the fluid.

e. Derive an expression for the rate at which energy is dissipated due to shear stresses in the tachocline, in terms of h, μ, R and Δ where R is the outer radius of the radiative zone and Δ is the difference between the angular velocity of the convective zone at the equator and at the poles. Hence evaluate this energy loss (to 1 s.f. in W) given $h = 0.03\ R_\odot$, $\mu = 3$ cm^2 s^{-1}, $R = 0.7\ R_\odot$ and $\Delta = 1 \times 10^{-6}$ rad s^{-1}.
Advice: To simplify your algebra, you should consider working in the rotating frame of reference of the radiative zone and note that the fictitious forces introduced due to this change of frame are entirely radial. You may also use the following result:

$$\int x^2 \cos x \, \mathrm{d}x = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

where C is an arbitrary integration constant.

END OF PAPER

Questions proposed by:
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Ben Woodrow (University of Oxford)
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