

BAAO
British Astronomy and
Astrophysics Olympiad

British Astronomy and Astrophysics Olympiad 2021-2022

Astronomy & Astrophysics Competition Paper

Monday 24th January 2022

This question paper must not be photographed or taken out of the exam room

Instructions

Time: 3 hours (~ 50 minutes for Q1, ~ 65 minutes for Q2 and ~ 65 minutes for Q3).

Questions: All three questions should be attempted. Each question contains independent parts so that later parts can be attempted even if earlier parts are incomplete.

Solutions: Answers and calculations are to be written on loose paper. Students should ensure their **name** and **school** is clearly written on the **first** answer sheet and that **all** pages are numbered. **EACH QUESTION ANSWERED must be started on a new page.** A standard formula booklet with standard physical constants may be used if desired.

Instructions: To accommodate students sitting the paper at different times, please **do not discuss** any aspect of the paper on the internet until 8 am Saturday 29th January.

Clarity: Solutions must be written legibly, in black pen, and working down the page. Scribble will not be marked and overall clarity is an important aspect of this exam.

Eligibility: The International Olympiad will be held in August 2022; all sixth form students are eligible to participate.

Calculators: Any standard calculator may be used, but calculators cannot be programmable and must not have symbolic algebra capability.

Training Dates and the IOAA (Kyiv, Ukraine, 12th - 22nd August 2022)

*The team will be selected from students taking this paper. The best students that are eligible to represent the UK at the IOAA will be invited to attend the **Training Camp** to be held in Oxford from **Saturday 9th to Wednesday 13th April 2022.** Astronomy material will be covered; problem solving skills and observational skills (telescope and naked eye observations) will be developed. At the Training Camp a Data Analysis exam along with a Round 3 theory paper will be sat. A team of five students (plus one reserve) will be selected for further training, including additional training camps in the summer.*

Important Constants

Constant	Symbol	Value
Speed of light	c	$3.00 \times 10^8 \text{ m s}^{-1}$
Earth's rotation period	1 day	24 hours
Earth's orbital period	1 year	365.25 days
parsec	pc	$3.09 \times 10^{16} \text{ m}$
Astronomical Unit	au	$1.50 \times 10^{11} \text{ m}$
Radius of the Sun	R_{\odot}	$6.96 \times 10^8 \text{ m}$
Radius of the Earth	R_{\oplus}	$6.37 \times 10^6 \text{ m}$
Mass of the Sun	M_{\odot}	$1.99 \times 10^{30} \text{ kg}$
Mass of the Earth	M_{\oplus}	$5.97 \times 10^{24} \text{ kg}$
Luminosity of the Sun	L_{\odot}	$3.85 \times 10^{26} \text{ W}$
Stephan-Boltzmann constant	σ	$5.67 \times 10^{-8} \text{ J m}^{-2} \text{ K}^{-4}$
Gravitational constant	G	$6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Boltzmann constant	k_B	$1.38 \times 10^{-23} \text{ J K}^{-1}$
Permittivity of free space	ϵ_0	$8.85 \times 10^{-12} \text{ F m}^{-1}$
Permeability of free space	μ_0	$4\pi \times 10^{-7} \text{ H m}^{-1}$
Planck's constant	h	$6.63 \times 10^{-34} \text{ J s}$
Elementary charge	e	$1.60 \times 10^{-19} \text{ C}$
Proton rest mass	m_p	$1.67 \times 10^{-27} \text{ kg}$
Electron rest mass	m_e	$9.11 \times 10^{-31} \text{ kg}$
Wien's displacement law	$\lambda_{\text{max}}T$	$2.90 \times 10^{-3} \text{ m K}$
Avagadro's constant	N_A	$6.02 \times 10^{23} \text{ mol}^{-1}$

Basic calculus formulae:

Chain rule $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Product rule $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$

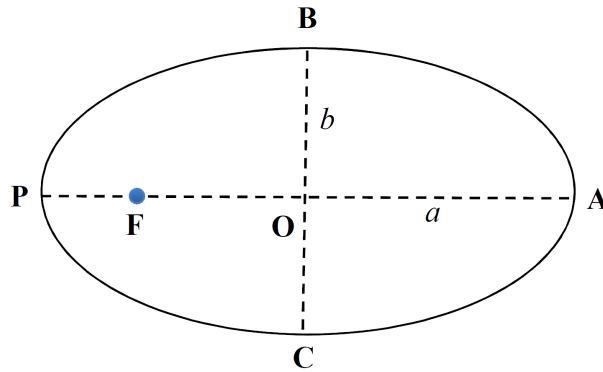
Quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

Integration by parts $\int u\frac{dv}{dx} dx = uv - \int v\frac{du}{dx} dx$

Standard integral $\int \frac{1}{x} dx = \ln|x| + C$

Important Formulae

You might find the diagram of an elliptical orbit below useful in solving some of the questions:



Elements of an elliptic orbit:

- $a = \text{OA} (= \text{OP})$ semi-major axis
- $b = \text{OB} (= \text{OC})$ semi-minor axis
- $e = \sqrt{1 - \frac{b^2}{a^2}}$ eccentricity
- F** focus
- $\text{PF} = a(1 - e)$ periapsis distance (shortest distance from **F**)
- $\text{AF} = a(1 + e)$ apoapsis distance (longest distance from **F**)

Kepler's Third Law: For an elliptical orbit, the square of the period, T , of an object about the focus is proportional to the cube of the semi-major axis, a (as defined above), such that

$$T^2 = \frac{4\pi^2}{GM} a^3,$$

where M is the total mass of the system (typically dominated by the central object) and G is the universal gravitational constant.

Vis-Viva Equation: For an elliptical orbit, the speed v of an object at a distance r from the focus is related to the semi-major axis, a , total mass of the system, M , and universal gravitational constant, G , (as defined above), such that

$$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right).$$

Magnitudes: The apparent magnitudes of two objects, m_1 and m_0 , are related to their apparent brightnesses, b_1 and b_0 , via the formula

$$\frac{b_1}{b_0} = 10^{-0.4(m_1 - m_0)}.$$

The absolute magnitude of an object, \mathcal{M} , is the same as its apparent magnitude when viewed from 10 pc, hence the relationship between apparent and absolute magnitude and distance is

$$m - \mathcal{M} = 5 \log \left(\frac{d}{10} \right),$$

where d is measured in parsecs.

Qu 1. Sunrise in Oxford

It is often said that the Sun rises in the East and sets in the West, however this is only true twice a year - at the equinoxes. In the Northern hemisphere, the Sun will rise northwards of East on the June solstice, and southwards of East on the December solstice; this is directly tied in with the varying length of day too, since the Sun either has a greater or shorter distance to travel across the sky (see Figure 1).

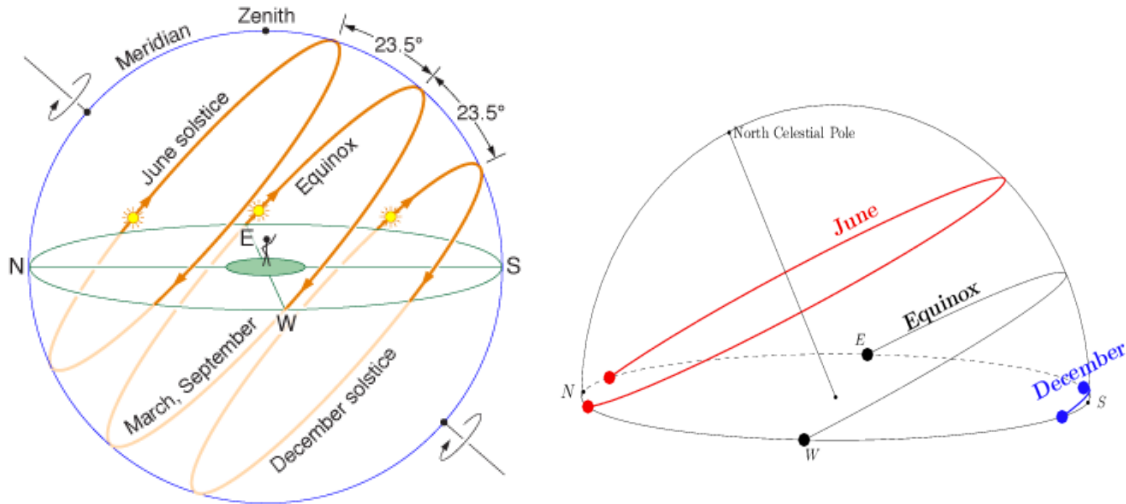


Figure 1: *Left:* The path of the Sun across the sky during the equinoxes and solstices, as viewed by an observer in the Northern hemisphere at a latitude of $\sim 40^\circ$. Credit: Daniel V. Schroeder / Weber State University. *Right:* The same idea but viewed from Iceland at a latitude of 65° , where by being so close to the Arctic circle the day length can get close to 24 hours in June and almost no daylight in December. Credit: Kristín Bjarnadóttir / University of Iceland.

During the equinox, the Sun travels along the projection of the Earth's equator. In this question, we will assume a circular orbit for the Earth, and all angles will be calculated in degrees.

A simple model for the vertical angle between the Sun and the horizon (known as the altitude), h , as a function of the bearing on the horizon, A (measured clockwise from North, also called the azimuth), the latitude of the observer, ϕ (positive in Northern hemisphere, negative in Southern hemisphere), and the vertical angle of the Sun relative to the celestial equator (known as the solar declination), δ , is given as:

$$h = -(90^\circ - \phi) \cos(A) + \delta$$

The solar declination can be considered to vary sinusoidally over the year, going from a maximum of $\delta = +23.44^\circ$ at the June solstice (roughly 21st June) to a minimum of $\delta = -23.44^\circ$ on the December solstice (roughly 21st December).

- a. Consider an observer in Oxford ($\phi = +51.8^\circ$) on the June solstice.
 - (i) Calculate the bearing of sunrise. Take the Sun to be a point source and ignore any atmospheric effects.
 - (ii) By calculating the angle the solar path makes with the horizon, η , at sunrise for both the solstice and the equinox, estimate the duration of sunrise on the solstice if sunrise takes 3 mins 26 secs on the equinox. Assume the same solar angular velocity in both cases. [Hint: $\frac{d}{dx} \cos(\alpha x) = -\alpha \frac{\pi}{180} \sin(\alpha x)$ if x is in degrees (this will give an answer in degrees per degree, rather than degrees per radian as it would be without the $\frac{\pi}{180}$ factor).]

It can be shown using spherical trigonometry that the precise model connecting δ , h , ϕ and A is:

$$\sin(\delta) = \sin(h) \sin(\phi) + \cos(h) \cos(\phi) \cos(A) .$$

- b. Considering just the bearing of sunrise, suggest (with qualitative justification only) which of the following situations the simple model will be the best approximation for the precise model:
 A) a pole at solstice; B) a pole at equinox; C) the equator at solstice; or D) the equator at equinox.

Using the precise model, the path of the Sun across the sky forms a shape that is not quite the cosine shape of the simple model, and is shown in Figure 2.

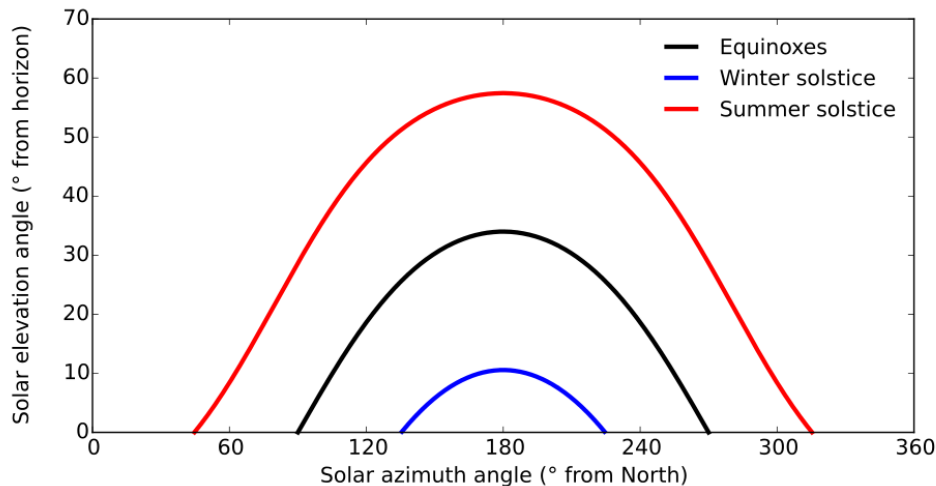


Figure 2: The altitude of the Sun as a function of bearing during the equinoxes and solstices, as viewed by an observer at a latitude of $+56^\circ$. Whilst it resembles the cosine shape of the simple model well at this latitude, there are small deviations. Credit: Wikipedia.

By using further spherical trigonometry, we can derive a second helpful equation in the precise model:

$$\sin(h) = \sin(\phi) \sin(\delta) + \cos(\phi) \cos(\delta) \cos(H) .$$

Here, H is the solar hour angle, which measures the angle between the Sun and solar noon as measured along the projection of the Earth's equator on the sky. Conventionally, $H = 0^\circ$ at solar noon, is negative before solar noon, and is positive afterwards. Since the sun's hour angle increases at an approximately constant rate due to the rotation of the Earth, we can convert this angle into a time using the conversion $360^\circ = 24^h$.

- c. Reconsider the Oxford observer at the June solstice, but this time use the two equations of the precise model. Ignore any atmospheric effects.
- (i) Calculate the bearing of sunrise and the duration of the day (in hours and minutes), taking the Sun to be a point source.
 - (ii) Calculate the duration of sunrise (in minutes and seconds), assuming a solar angular diameter of 0.525° .
- d. This exam is being taken on 24th January and is 3 hours long.
- (i) Estimate the solar declination on this date.
 - (ii) Hence, calculate the latitude where today's day length is equal to the exam length, taking the Sun to be a point source.
 - (iii) What is the latitude with the longest sunrise today? Give its duration in minutes and seconds.

Qu 2. Stellar Structure

The Sun is our closest star so it is arguably the most studied. Results from detailed observations of how sound waves propagate through the plasma of the Sun allow us to get a sense of the general structure of the Sun, with an upper layer of convecting plasma (leading to the ‘bubbling’ we see with granulation on the surface) and radiative heat transfer below, including a central core region where the pressure and temperature are large enough for nuclear fusion to occur (see Figure 3).

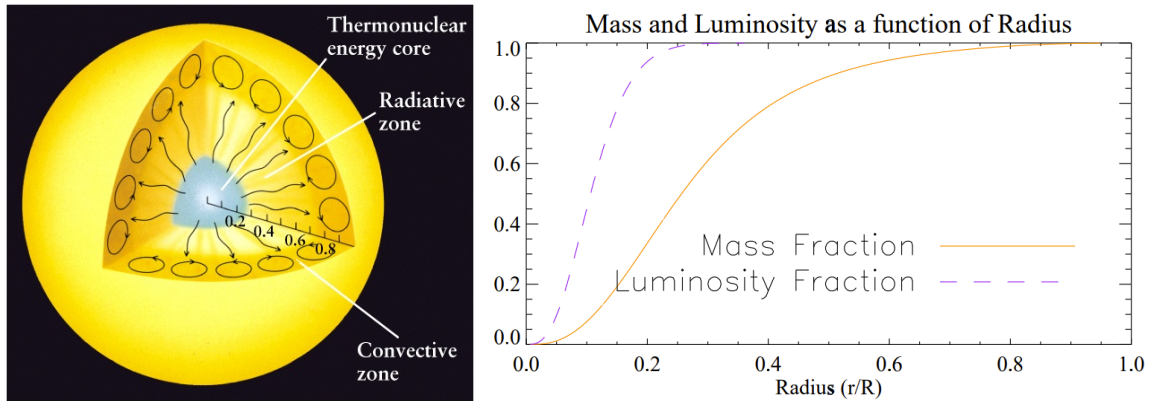


Figure 3: *Left:* The general structure of the Sun, with a core in which all the nuclear reactions take place, and convective cells in an outer layer. Credit: Dmitri Pogosyan / University of Alberta.

Right: The fraction of the Sun’s mass and the contribution to the Sun’s luminosity as a function of solar radius as determined from detailed computer simulations. Essentially all of the nuclear reactions creating the photons for the Sun’s luminosity take place in a core with a radius of $0.20 R_{\odot}$ and a mass of $0.35 M_{\odot}$. Credit: Kevin France / University of Colorado.

Estimating the conditions in the cores of stars is an important aspect of constructing stellar models. This question explores some of the equations governing stellar structure and estimates the central temperature and pressure of the Sun.

- a. Let r denote distance from the centre of a star. We define the variables $\rho(r)$, $p(r)$ and $T(r)$ to be the density, pressure and temperature at radius r respectively, and $m(r)$ to be the mass enclosed within radius r . We will now try and derive an estimate for the pressure at the centre of the Sun.

- (i) By considering forces on a box of height dr at radius r , show that

$$\frac{dp}{dr} = -\rho \frac{Gm}{r^2}.$$

[This is known as the equation of hydrostatic equilibrium.]

- (ii) We can get a good estimate of the central pressure if we use m as our independent variable rather than r . Derive an expression for $\frac{dm}{dr}$ in terms of r and ρ , and hence express $\frac{dp}{dm}$ in terms of m and r .
- (iii) Assuming that the pressure at the surface, p_s , is negligible compared to the pressure at the centre of the Sun, p_c , the edge of the core is at $r = 0.20 R_{\odot}$ and encloses a mass of $m = 0.35 M_{\odot}$, and that $\frac{dp}{dm}$ is constant throughout the star and equal to the value at the edge of the core, calculate a value for p_c .

- b. The Sun is composed predominantly of ionized hydrogen and helium, with approximate *mass* fractions $X = 0.70$ and $Y = 0.30$ respectively (taken to be constant throughout the Sun). These ionized nuclei and electrons may be treated as an ideal monatomic gas, obeying the equation of state

$$pV = Nk_B T ,$$

where N is the number of particles contained within volume V and k_B is the Boltzmann constant. They also have an average kinetic energy per particle

$$E_K = \frac{3}{2}k_B T .$$

- (i) Show that the kinetic energy per unit mass of the solar plasma is given by:

$$u = \frac{3}{2} \frac{k_B T}{m_p} \left(2X + \frac{3}{4}Y \right) ,$$

where m_p is the mass of a proton (treated to be the same as the mass of a neutron).

- (ii) The Virial Theorem states that for a stable, gravitationally bound system of particles the total kinetic energy in the system, E_K and the total gravitational potential energy in the system, E_G , are related as

$$E_K = -\frac{1}{2}E_G .$$

In the context of the plasma in the Sun, this becomes

$$\int_0^{M_\odot} u \, dm = -\frac{1}{2} \int_0^{M_\odot} -\frac{Gm}{r} \, dm .$$

Using the Virial Theorem, and given $E_G \approx G M_\odot^2 / R_\odot$, estimate the Sun's mean temperature

$$\langle T_\odot \rangle = \frac{1}{M_\odot} \int_0^{M_\odot} T \, dm .$$

[This is our initial lower bound for the central temperature]

The primary nuclear fusion pathway responsible for much of the Sun's luminous output is called the proton-proton chain (p-p chain). All of the most common steps are shown in Figure 4.

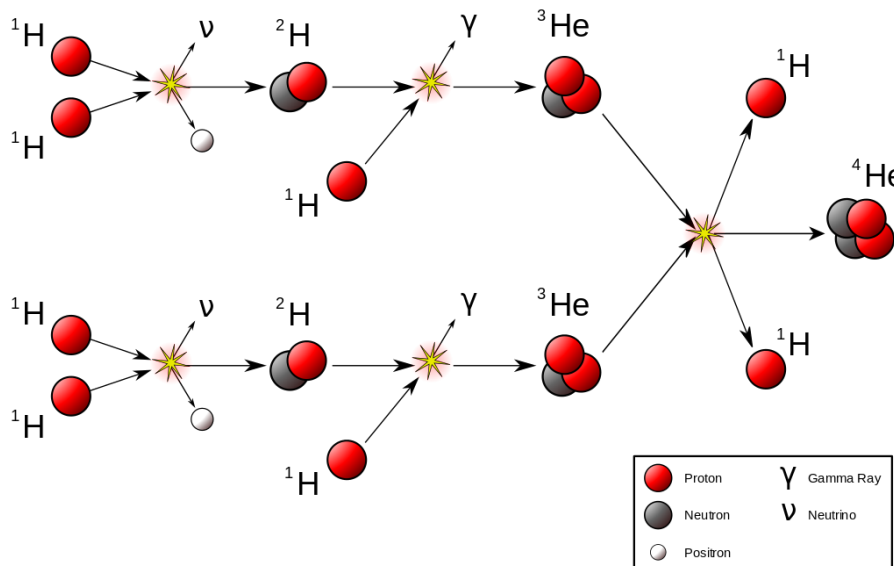
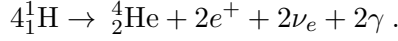


Figure 4: An overview of the all the steps in the most common form of the p-p chain, turning four protons into one helium-4 nucleus plus some light particles and photons. Credit: Wikipedia.

The net reaction of the p-p chain is



The rate-limiting step is the first one (${}^1_1\text{H} + {}^1_1\text{H}$) as the probability of interaction is so low, since it requires the decay of a proton into a neutron and so involves the weak nuclear force.

Nuclear physics allow us to calculate the reaction rate coefficient, R , energy produced per reaction, Q , and hence the energy generation rate, q . The reaction rate is related to the number of particles that have enough energy to undergo quantum tunnelling, and the distribution as a function of energy is known as the Gamow peak, with the top of the curve at energy E_0 . For two nuclei, ${}^{A_i}_{Z_i}\text{C}_i$ and ${}^{A_j}_{Z_j}\text{C}_j$, with mass fractions X_i and X_j , then to first order and ignoring electron screening,

$$R = \frac{4}{3^{2.5}\pi^2} \frac{h}{\mu_r m_p} \frac{4\pi\epsilon_0}{Z_i Z_j e^2} S(E_0) \tau^2 e^{-\tau}, \quad \text{where } \mu_r = \frac{A_i A_j}{A_i + A_j},$$

and $S(E_0)$ measures the probability of interaction at the maximum of the Gamow peak whilst τ is a characteristic width of the Gamow peak,

$$\tau = \frac{3E_0}{k_B T} \quad \text{where } E_0 = \left(\frac{bk_B T}{2} \right)^{2/3} \quad \text{given } b = \sqrt{\frac{\mu_r m_p}{2}} \frac{\pi Z_i Z_j e^2}{h\epsilon_0}.$$

Here h is Planck's constant, ϵ_0 is the permittivity of free space and e is the elementary charge (the charge on a proton). Finally, the energy generation rate per unit mass is

$$q = \frac{\rho}{m_p^2} \left(\frac{1}{1 + \delta_{ij}} \right) \frac{X_i X_j}{A_i A_j} R Q,$$

where δ_{ij} is the Kronecker delta, so equals 1 when $i = j$ and 0 otherwise.

Evaluating the fundamental constants and defining $T_6 \equiv \frac{T}{10^6 \text{ K}}$ gives

$$\tau = 42.59 [Z_i^2 Z_j^2 \mu_r T_6^{-1}]^{1/3},$$

whilst for the proton-proton interaction $Q = 13.366 \text{ MeV}$ (half the overall energy of the p-p chain), $Z_i = Z_j = A_i = A_j = \delta_{ij} = 1$, and $S(E_0)$ is $4.01 \times 10^{-50} \text{ keV m}^2$ (Adelberger et al. 2011), so

$$R = 6.55 \times 10^{-43} T_6^{-2/3} e^{-33.80 T_6^{-1/3}} \text{ m}^3 \text{ s}^{-1} \quad \text{and} \quad q = 0.251 \rho X^2 T_6^{-2/3} e^{-33.80 T_6^{-1/3}} \text{ W kg}^{-1}.$$

- c. Considering the evaluated equations for τ , R , and q we can use this with the measured luminosity of the Sun to get a new estimate for the central temperature.
 - (i) Considering the simplified equation for q and assuming that the core has a mass of $0.35 M_\odot$, throughout which T and ρ are constant, and that the Sun's luminosity is equal to the power produced by the p-p chain fusion processes occurring within its core, estimate the central temperature. [Hint: you are also given that $u = \frac{3}{2} \frac{p_c}{\rho_c}$ where ρ_c is the central density, and you may need to use a numerical method with $\langle T_\odot \rangle$ as an initial guess.]
 - (ii) Using this new central temperature, and considering R and the central number density of protons, n_p , estimate the typical amount of time a proton needs to wait to undergo fusion, giving your answer in years.
[This is an estimate for the length of time the Sun spends on the main sequence.]
 - (iii) Since $q \propto \tau^2 e^{-\tau}$ and $\tau \propto T^{-1/3}$, it can be approximated at a given temperature as $q \propto T^\alpha$, quantifying the sensitivity of the fusion reaction to temperature. By considering $\frac{d(\ln q)}{d(\ln T)}$ give an expression for α as a function of τ and calculate it at your central temperature.
 - (iv) The carbon-nitrogen-oxygen (CNO) cycle is an alternative pathway that becomes important at higher temperatures, where heavier elements catalyse the process of turning hydrogen into helium. There the rate limiting step is between ${}^1_7\text{N}$ and ${}^1_1\text{H}$. Compare the temperature dependence of the CNO cycle to the p-p chain at the Sun's central temperature.

Qu 3. James Webb Space Telescope

The James Webb Space Telescope (JWST) is an incredibly exciting next generation telescope that was successfully launched on 25th December 2021. Its mirror is approximately 6.5 m in diameter, much larger than the 2.4 m mirror of the Hubble Space Telescope (HST), and so it has far greater resolution and sensitivity. Whilst HST largely imaged in the visible, JWST will do most of its work in the near- and mid-infrared (NIR and MIR respectively). This will allow it to pick up heavily redshifted light, such as that from the first generation of stars in the very first galaxies.



Figure 5: *Left:* A full-scale model of JWST next to some of the scientists and engineers involved in its development at the Goddard Space Flight Center. Credit: NASA / Goddard Space Flight Center / Pat Izzo. *Right:* The position of the second Lagrangian point, L_2 , relative to the Earth. Credit: ESA.

- a. The telescope will spend its expected 10-20 year mission in a halo orbit about the second Lagrangian point, L_2 (see Figure 5). This is one of five special points in the Sun-Earth system where the gravitational forces from the two bodies provide the centripetal force required to have a (small mass) object there have an orbital period identical to the Earth. At the L_2 point, this means it orbits quicker than you would expect for an object that distance from the Sun.
 - (i) Taking 1 year as 365.25 days and 1 au as 1.496×10^{11} m, using numerical methods show that the distance between the Earth and L_2 is $\sim 1.5 \times 10^6$ km. Give your answer to 4 s.f.
 - (ii) Rather than go to L_2 directly, the JWST is on an orbit that will take it to within 200 000 km of L_2 , where it will then do a final large burn of the rockets to insert it into the halo orbit around L_2 . Assuming it is on a simple elliptical transfer orbit ignoring the influence of the Sun, and had a perigee at an altitude of 2100 km above the surface of the Earth, how long will it take JWST to get to the L_2 orbital insertion phase of its mission? Give your answer in days.

The resolution limit of a telescope is set by the amount of diffraction light rays experience as they enter the system, and is related to the diameter of a telescope, D , and the wavelength being observed, λ . The resolution limit of a CCD is set by the size of the pixels.

Three of the imaging cameras on JWST are tabulated with some properties below:

Instrument	Wavelength range (μm)	CCD plate scale (arcseconds / pixel)
NIRCam (short wave)	0.6 – 2.3	0.031
NIRCam (long wave)	2.4 – 5.0	0.065
MIRI	5.6 – 25.5	0.11

An arcsecond is a measure of angle where $1^\circ = 3600$ arcseconds.

The familiar variation in intensity on a screen, I_{slit} , due to diffraction through an infinitely tall single slit is given as

$$I_{\text{slit}} = I_0 \left(\frac{\sin(x)}{x} \right)^2, \quad \text{where } x = \frac{\pi D \theta}{\lambda},$$

and I_0 is the initial intensity. For a circular aperture, the formula is slightly different and is given as

$$I_{\text{circ}} = I_0 \left(\frac{2J_1(x)}{x} \right)^2.$$

Here $J_1(x)$ is the Bessel function of the first kind and is calculated as

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!} \left(\frac{x}{2} \right)^{n+2r} \quad \text{so } J_1(x) = \frac{x}{2} \left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \dots \right).$$

The x -axis intercepts and shape of the maxima are quite different, as shown in Figure 6. The position of the first minimum of I_{slit} is at $x_{\text{min}} = \pi$ meaning that $\theta_{\text{min,slit}} = \lambda/D$, whilst for I_{circ} it is at $x_{\text{min}} = 3.8317\dots$ so $\theta_{\text{min,circ}} \approx 1.22\lambda/D$. This is one way of defining the minimum angular resolution, although since the flux drops off so steeply away from the central maximum a more convenient one for use with CCDs is the angle corresponding to the full width half maximum (FWHM).

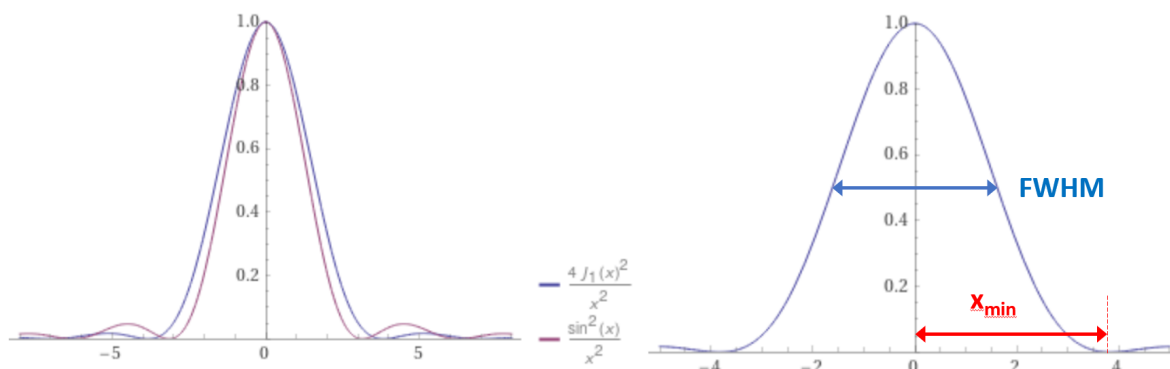


Figure 6: Left: The I_{slit} (purple) and I_{circ} (blue - the wider central maximum) functions, normalised so that $I_0 = 1$. You can see the shapes and x -intercepts are different.

Right: How x_{min} and the full width half maximum (FWHM) are defined. Here it is shown for I_{circ} .

- b. To achieve suitable sampling, an image will be considered diffraction limited when it has ≥ 2 pixels per θ_{FWHM} . The diameter of the JWST primary mirror is 6.5 m, however since it is composed of hexagons and hexagonal in shape, it is not straightforward to work out the equivalent circular mirror diameter. To a good approximation it can be taken to be 6.0 m.

- (i) Given $\theta_{\text{FWHM}} = \alpha \frac{\lambda}{D}$, find α for I_{circ} , giving your answer to 3 s.f.
- (ii) Hence, determine which of the three imaging instruments is diffraction limited for the greatest fraction of its wavelength range.

As well as having the largest mirror of any space telescope ever launched, it is also one of the most sensitive, with its greatest sensitivity in the NIRCcam F200W filter (centred on a wavelength of $1.989 \mu\text{m}$) where after 10^4 seconds it can detect a flux of 9.1 nJy ($1 \text{ Jy} = 10^{-26} \text{ W m}^{-2} \text{ Hz}^{-1}$) with a signal-to-noise ratio (S/N) of 10, corresponding to an apparent magnitude of $m = 29.0$. This extraordinary sensitivity can be used to pick up light from the earliest galaxies in the Universe.

The scale factor, a , parameterises the expansion of the Universe since the Big Bang, and is related to the redshift, z , as

$$a = (1 + z)^{-1} \quad \text{where } z \equiv \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}},$$

with λ_{obs} the observed wavelength and λ_{emit} the rest frame wavelength. The current rate of expansion of the Universe is given by the Hubble constant, H_0 , and this is related to the current Hubble time, t_{H_0} , and current Hubble distance, D_{H_0} , as

$$t_{H_0} \equiv H_0^{-1} \quad \text{and} \quad D_{H_0} \equiv ct_{H_0} .$$

Here the subscript 0 indicates the values are measured today. The Hubble constant is more appropriately known as the Hubble parameter as it is a function of time, and the evolution of H as a function of z is

$$E(z) = \frac{H}{H_0} \equiv [\Omega_{0,m}(1+z)^3 + \Omega_{0,\Lambda} + \Omega_{0,r}(1+z)^4]^{1/2} ,$$

where Ω is the normalised density parameter, and the subscript m , r , and Λ indicate the contribution to Ω from matter, radiation, and dark energy, respectively. The proper age of the Universe $t(z)$ at redshift z is best evaluated in terms of a as

$$t = t_{H_0} \int_0^{(1+z)^{-1}} \frac{a}{(\Omega_{0,m}a + \Omega_{0,\Lambda}a^4 + \Omega_{0,r})^{1/2}} da .$$

If $\Omega_{0,r} = 0$ and $\Omega_{0,m} + \Omega_{0,\Lambda} = 1$ (corresponding to what it known as a flat Universe), then via the standard integral $\int (b^2 + x^2)^{-1/2} dx = \ln(x + \sqrt{b^2 + x^2}) + C$ this integral can be evaluated analytically to give

$$t = t_{H_0} \frac{2}{3\Omega_{0,\Lambda}^{1/2}} \ln \left[\left(\frac{\Omega_{0,\Lambda}}{\Omega_{0,m}} \right)^{1/2} (1+z)^{-3/2} + \left(\frac{\Omega_{0,\Lambda}}{\Omega_{0,m}(1+z)^3} + 1 \right)^{1/2} \right] .$$

Finally, the luminosity distance, $D_L(z)$, corresponding to the distance away that an object appears to be due to its measured flux given its intrinsic luminosity (i.e. $f \equiv L/4\pi D_L^2$) is given as

$$D_L = (1+z_i)D_{H_0} \int_0^{z_i} \frac{1}{E(z)} dz = (1+z_i)D_{H_0} \int_{a_i}^1 \frac{1}{(\Omega_{0,m}a + \Omega_{0,\Lambda}a^4 + \Omega_{0,r})^{1/2}} da$$

where z_i is the redshift of interest and a_i is the equivalent scale factor. Even for the flat Universe case with $\Omega_{0,r} = 0$ this integral cannot be done analytically so must be evaluated numerically.

- c. Computer models suggest the first galaxies formed around $z \sim 10 - 20$. One of the best ways to look for high-redshift galaxies is to try and detect the emission from the Lyman alpha ($\text{Ly}\alpha$) emission line at $\lambda_{\text{emit}} = 121.6 \text{ nm}$ as it is a relatively bright line. Some of the brightest galaxies in that initial era of galaxy formation would have an absolute magnitude of $\mathcal{M} \sim -20$. In this question, you are given that $\Omega_{0,m} = 0.3$, $\Omega_{0,\Lambda} = 0.7$, $\Omega_{0,r} = 0$ and $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$.
 - (i) Calculate the redshift at which the $\text{Ly}\alpha$ line is detected in the centre of the F200W filter.
 - (ii) How long after the Big Bang does this correspond to? Give your answer in years.
 - (iii) Calculate the luminosity distance to the galaxy and hence its apparent magnitude. Assume all emitted flux is picked up by the telescope.
 - (iv) If the minimum flux detectable decreases proportionally to $t_{\text{exp}}^{1/2}$ where t_{exp} is the length of the exposure, estimate the minimum exposure time necessary for JWST to image this galaxy with $S/N = 10$. Give your answer in hours.

END OF PAPER

Questions proposed by:
Dr Alex Calverley (Royal Grammar School, Guildford)
Josh Brown (University of Cambridge)



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