BAAO
British Astronomy and Astrophysics Olympiad

British Astronomy and Astrophysics Olympiad 2019-2020

Astronomy & Astrophysics Competition Paper

Monday 20th January 2020

This question paper must not be photographed or taken out of the exam room

Instructions

Time: 3 hours (approximately 60 minutes per question).

Questions: All three questions should be attempted. The last question is worth slightly fewer marks than the rest.

Solutions: Answers and calculations are to be written on loose paper. Students should ensure their **name** and **school** is clearly written on all answer sheets and pages are numbered. **EACH QUESTION ANSWERED must be started on a new page.** A standard formula booklet with standard physical constants should be supplied.

Instructions: To accommodate students sitting the paper at different times, please **do not discuss** any aspect of the paper on the internet until 8 am Saturday 25th January.

Clarity: Solutions must be written legibly, in black pen (the papers are photocopied), and working down the page. Scribble will not be marked and overall clarity is an important aspect of this exam.

Eligibility: The International Olympiad will be held during September 2020; all sixth form students are eligible to participate, even if they will be attending university from October.

Calculators: Any standard calculator may be used, but calculators cannot be programmable and must not have symbolic algebra capability.

Training Dates and the IOAA (Bogota, Columbia, 13th to 22nd September 2020)

The best students taking this paper that are eligible to represent the UK at the IOAA will be invited to attend the **Training Camp** *to be held in the Physics Department at the University of Oxford,* **(Friday 27th March to Tuesday 31st March 2020)**. *Astronomy material will be covered; problem solving skills and observational skills (telescope and naked eye observations) will be developed. At the Training Camp a data analysis exam and a short theory paper will be sat. A team of five students (plus one reserve) will be selected for further training. From April there will be mentoring by email to cover some topics and problems, followed by additional training camps in the summer.*

Important Constants

Constant	**Symbol**	**Value**
Speed of light	c	$3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Earth's rotation period	1 day	24 hours
Earth's orbital period	1 year	365.25 days
parsec	pc	$3.09 \times 10^{16}\,\mathrm{m}$
Astronomical Unit	au	$1.50 \times 10^{11}\,\mathrm{m}$
Semi-major axis of the Earth's orbit		1 au
Radius of the Sun	$R_\odot$	$6.96 \times 10^{8}\,\mathrm{m}$
Radius of the Earth	$R_\oplus$	$6.37 \times 10^{6}\,\mathrm{m}$
Mass of the Sun	$M_\odot$	$1.99 \times 10^{30}\,\mathrm{kg}$
Mass of the Earth	$M_\oplus$	$5.97 \times 10^{24}\,\mathrm{kg}$
Luminosity of the Sun	$L_\odot$	$3.85 \times 10^{26}\,\mathrm{W}$
Stephan-Boltzmann constant	σ	5.67×10^{-8} J m^{-2} K^{-4}
Gravitational constant	G	6.67×10^{-11} m^{3} kg^{-1} s^{-2}
Boltzmann constant	k_B	1.38×10^{-23} J K^{-1}
Permittivity of free space	ε_0	8.85×10^{-12} F m^{-1}
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ H m^{-1}
Planck's constant	h	6.63×10^{-34} J s
Elementary charge	e	1.60×10^{-19} C
Proton rest mass	m_p	1.67×10^{-27} kg
Electron rest mass	m_e	9.11×10^{-31} kg
Wien's displacement law	$\lambda_\mathrm{max} T$	2.90×10^{-3} m K

Basic calculus formulae:

Chain rule $$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x))g'(x)$$

Product rule $$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = \frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

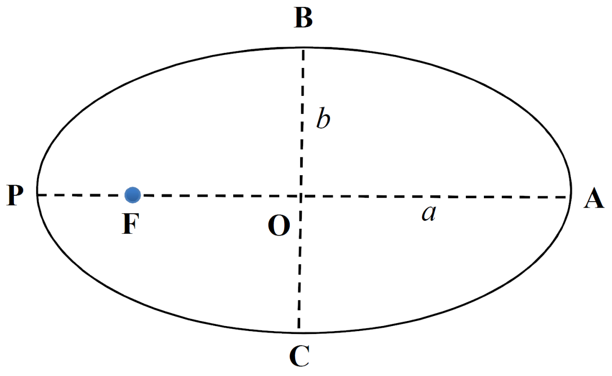
Quotient rule $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{\frac{\mathrm{d}u}{\mathrm{d}x}v - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

Integration by parts $$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Standard integral $$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

Important Formulae

You might find the diagram of an elliptical orbit below useful in solving some of the questions:



Elements of an elliptic orbit:

$a = \mathbf{OA}\ (= \mathbf{OP})$	semi-major axis
$b = \mathbf{OB}\ (= \mathbf{OC})$	semi-minor axis
$e = \sqrt{1 - \frac{b^2}{a^2}}$	eccentricity
F	focus
P	periapsis (point nearest to F)
A	apoapsis (point furthest from F)

Kepler's Third Law: For an elliptical orbit, the square of the period, T, of an object about the focus is proportional to the cube of the semi-major axis, a (as defined above), such that

$$T^2 = \frac{4\pi^2}{GM}a^3 \ ,$$

where M is the total mass of the system (typically dominated by the central object) and G is the universal gravitational constant.

Vis-Viva Equation: For an elliptical orbit, the speed v of an object at a distance r from the focus is related to the semi-major axis, a, total mass of the system, M, and universal gravitational constant, G, (as defined above), such that

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \ .$$

Magnitudes: The apparent magnitudes of two objects, m_1 and m_0, are related to their apparent brightnesses, b_1 and b_0, via the formula

$$\frac{b_1}{b_0} = 10^{-0.4(m_1 - m_0)} \ .$$

The absolute magnitude of an object, $\mathcal{M}$, is the same as its apparent magnitude when viewed from 10 pc, hence the relationship between apparent and absolute magnitude and distance is

$$m - \mathcal{M} = 5\log\left(\frac{d}{10}\right) ,$$

where d is measured in parsecs.

Qu 1. Apollo 11

In July 1969 the mission Apollo 11 was the first to successfully allow humans to walk on the Moon. This was an incredible achievement as the engineering necessary to make it a possibility was an order of magnitude more complex than anything that had come before. The Apollo 11 spacecraft was launched atop the Saturn V rocket, which still stands as the most powerful rocket ever made.

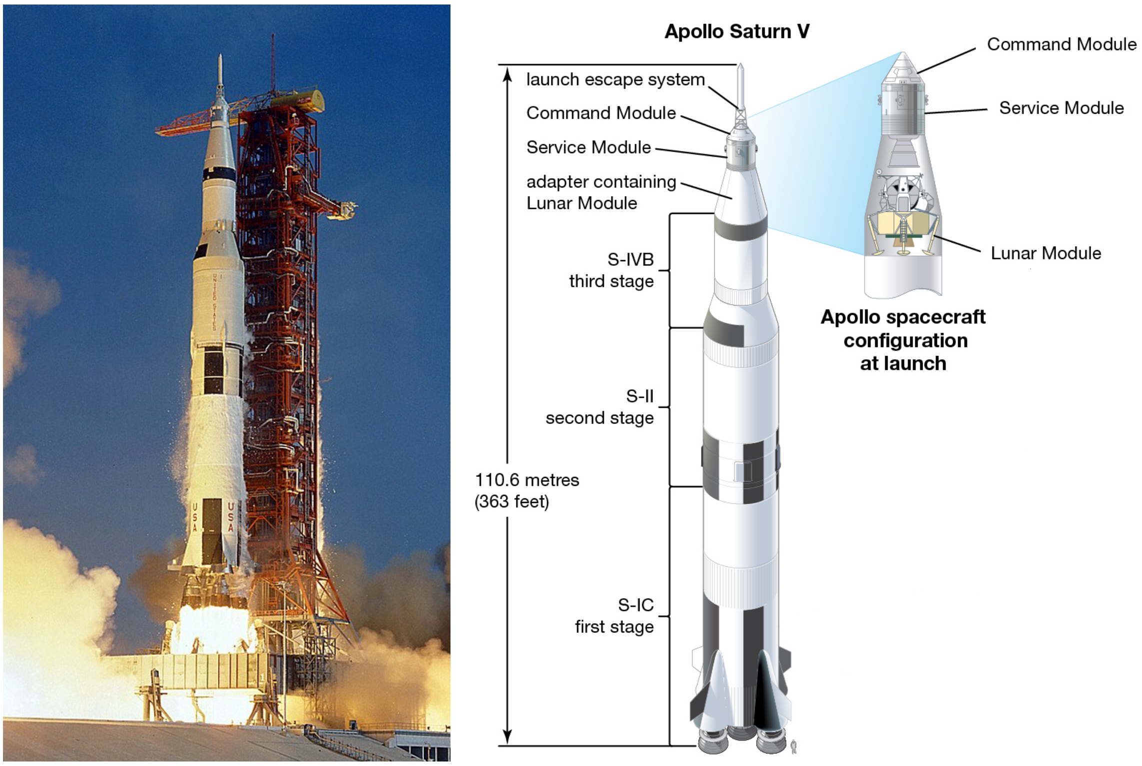


Figure 1: *Left*: The launch of Apollo 11 upon the Saturn V rocket. Credit: NASA.
Right: Showing the three stages of the Saturn V rocket (each detached once its fuel was expended), plus the Apollo spacecraft on top (containing three astronauts) which was delivered into a translunar orbit. At the base of the rocket is a person to scale, emphasising the enormous size of the rocket. Credit: Encyclopaedia Britannica.

Stage	Initial Mass (t)	Final mass (t)	I_{sp} (s)	Burn duration (s)
S-IC	2283.9	135.6	263	168
S-II	483.7	39.9	421	384
S-IV (Burn 1)	121.0	–	421	147
S-IV (Burn 2)	–	13.2	421	347
Apollo Spacecraft	49.7	–	–	–

Table 1: Data about each stage of the rocket used to launch the Apollo 11 spacecraft into a translunar orbit. Masses are given in tonnes (1 t = 1000 kg) and for convenience include the interstage parts of the rocket too. The specific impulse, I_{sp}, of the stage is given at sea level atmospheric pressure for S-IC and for a vacuum for S-II and S-IVB.

The Saturn V rocket consisted of three stages (see Fig 1), since this was the only practical way to get the Apollo spacecraft up to the speed necessary to make the transfer to the Moon. When fully fueled the mass of the total rocket was immense, and lots of that fuel was necessary to simply lift the fuel of the later stages into high altitude - in total about 3000 t (1 tonne, t = 1000 kg) of rocket on the launchpad was required to send about 50 t on a mission to the Moon. The first stage (called S-IC) was the heaviest, the second (called S-II) was considerably lighter, and the third stage (called S-IVB) was fired twice - the first to get the spacecraft into a circular 'parking' orbit around the Earth where various safety checks were made, whilst the second burn was to get the spacecraft on its way to the Moon. Once each rocket stage was fully spent it was detached from the rest of the rocket before the next stage ignited. Data about each stage is given in Table 1.

The thrust of the rocket is given as

$$F = -I_{\rm sp} g_0 \dot{m}$$

where the specific impulse, $I_{\rm sp}$, of each stage is a constant related to the type of fuel used and the shape of the rocket nozzle, g_0 is the gravitational field strength of the Earth at sea level (i.e. $g_0 = 9.81$ m s^{-2}) and $\dot{m} \equiv {\rm d}m/{\rm d}t$ is the rate of change of mass of the rocket with time.

The thrust generated by the first two stages (S-IC and S-II) can be taken to be constant. However, the thrust generated by the third stage (S-IVB) varied in order to give a constant acceleration (taken to be the same throughout both burns of the rocket).

a. Ignoring the effects of air resistance, the weight of the rocket, and assuming 1-D motion only:

   (i) Show that the thrust generated by the S-IC is about 3.3×10^7 N and hence calculate the acceleration experienced by the astronauts firstly at lift-off and secondly when the S-IC finishes its burn (ignore that the S-IC ignites a few seconds before lift off). Give your answer in units of g_0.

   (ii) Determine the constant acceleration produced by the third stage (S-IVB) of the Saturn V rocket and hence calculate the total mass carried into the parking orbit at the end of the first burn.

   (iii) Sketch an acceleration-time graph of the journey from lift-off to reaching the parking orbit. Give accelerations in units of g_0. Assume the time between one stage finishing, detaching, and ignition of the next stage is negligible (i.e. you will have discontinuities in the graph at the end of each stage).

   (iv) By using your graph or otherwise, work out the speed of the rocket when reaching the parking orbit.

b. In reality, the effects of air resistance and the weight of the rocket are substantial. Once in the parking orbit it is travelling at 7.79 km s^{-1}.

   (i) What is its height above the Earth's surface (measured from sea level)? Give your answer in km.

   (ii) The Apollo 11 spacecraft was in the parking orbit for 2 hours 32 mins 27 secs. How many revolutions of the Earth did it do?

By the end of the second burn the Apollo spacecraft, apart from a few short burns to give mid-course corrections, coasted all the way to the far side of the Moon where the engines were then fired again to circularise the orbit. All of the early Apollo missions were on a orbit known as a 'free-return trajectory', meaning that if there was a problem then they were already on an orbit that would take them back to Earth after passing around the Moon. The real shape of such a trajectory (in a rotating frame of reference) is like a stretched figure of 8 and is shown in the top panel of Fig 2. To calculate this precisely is non-trivial and required substantial computing power in the 1960s. However, we can have two simplified models that can be used to estimate the duration of the translunar coast, and they are shown in the bottom panel of the Fig 2.

The first is a Hohmann transfer orbit (dashed line), which is a single ellipse with the Earth at one focus. In this model the gravitational effect of the Moon is ignored, so the spacecraft travels from A (the perigee) to B (the apogee). The second (solid line) takes advantage of a 'patched conics' approach by having two ellipses whose apoapsides coincide at point C where the gravitational force on the spacecraft is equal from both the Earth and the Moon. The first ellipse has a periapsis at A and ignores the gravitational effect of the Moon, whilst the second ellipse has a periapsis at B and ignores the gravitational effect of the Earth. If the spacecraft trajectory and lunar orbit are coplanar and the Moon is in a circular orbit around the Earth then the time to travel from A to B via C is **double** the value attained if taking into account the gravitational forces of the Earth and Moon together throughout the journey, which is a much better estimate of the time of a real translunar coast.

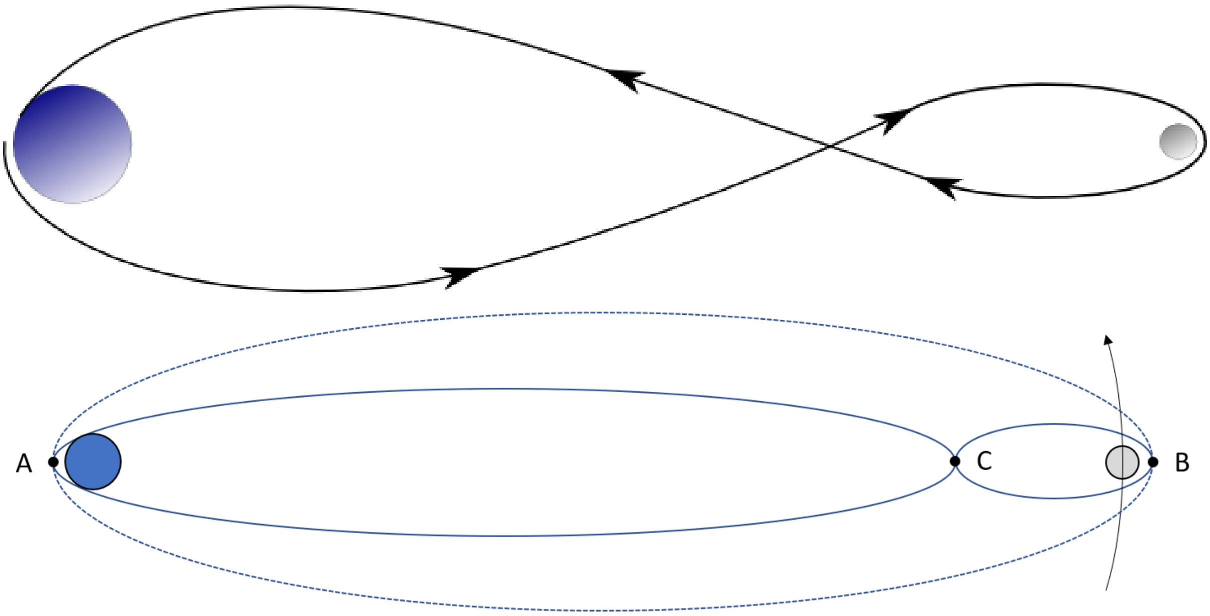


Figure 2: *Top*: The real shape of a translunar free-return trajectory, with the Earth on the left and the Moon on the right (orbiting around the Earth in an anti-clockwise direction). This diagram (and the one below) is shown in a co-ordinate system co-rotating with the Earth and is not to scale. Credit: NASA.
Bottom: Two simplified ways of modelling the translunar trajectory. The simplest is a Hohmann transfer orbit (dashed line, outer ellipse), which is an ellipse that has the Earth at one focus and ignores the gravitational effect of the Moon. A better model (solid line, inner ellipses) of the Apollo trajectory is the use of two ellipses that meet at point C where the gravitational forces of the Earth and Moon on the spacecraft are equal.

For the Apollo 11 journey, the end of the second burn of the S-IVB rocket (point A) was 334 km above the surface of the Earth, and the end of the translunar coast (point B) was 161 km above the surface of the Moon. The distance between the centres of mass of the Earth and the Moon at the end of the translunar coast was 3.94×10^8 m. Take the radius of the Earth to be 6370 km, the radius of the Moon to be 1740 km, and the mass of the Moon to be 7.35×10^{22} kg.

c. For the Hohmann transfer orbit (dashed line), find its semi-major axis and hence the duration of a translunar coast from A to B (expressed in hours and minutes).

d. For the patched conics approach (solid lines):

   (i) Find the distance from the centre of the Earth to point C, and hence the semi-major axes of both ellipses.

   (ii) Calculate the speed of the spacecraft at point A and point B. Give your answer in km s^{-1} and as a percentage of the escape speed of the spacecraft at that distance from the relevant closest gravitational body. Comment on what this implies for the eccentricity of the orbits.

   (iii) Determine the best estimate of the duration of the real Apollo 11 translunar coast. Give your answer in hours and minutes.

Qu 2. Event Horizon Telescope and Super Massive Black Holes

The Event Horizon Telescope (EHT) is a project to use many widely-spaced radio telescopes as a Very Long Baseline Interferometer (VBLI) to create a virtual telescope as big as the Earth. This extraordinary size allows sufficient angular resolution to be able to image the space close to the event horizon of a super massive black hole (SMBH), and provide an opportunity to test the predictions of Einstein's theory of General Relativity (GR) in a very strong gravitational field. In April 2017 the EHT collaboration managed to co-ordinate time on all of the telescopes in the array so that they could observe the SMBH (called M87*) at the centre of the Virgo galaxy, M87, and they plan to also image the SMBH at the centre of our galaxy (called Sgr A*).

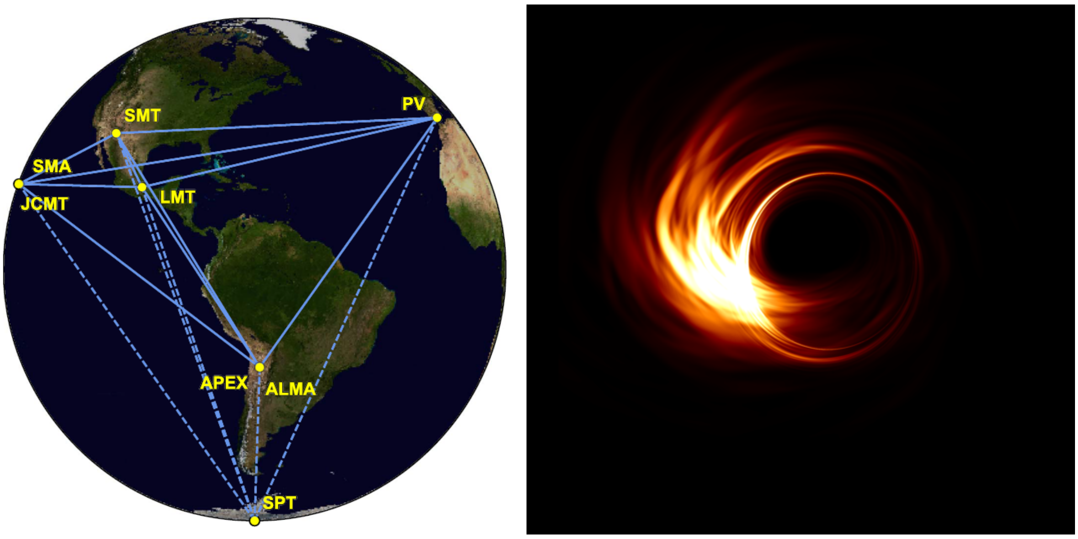


Figure 3: *Left*: The locations of all the telescopes used during the April 2017 observing run. The solid lines correspond to baselines used for observing M87, whilst the dashed lines were the baselines used for the calibration source. Credit: EHT Collaboration.
Right: A simulated model of what the region near an SMBH could look like, modelled at much higher resolution than the EHT can achieve. The light comes from the accretion disc, but the paths of the photons are bent into a characteristic shape by the extreme gravity, leading to a 'shadow' in middle of the disc - this is what the EHT is trying to image. The left side of the image is brighter than the right side as light emitted from a substance moving towards an observer is brighter than that of one moving away. Credit: Hotaka Shiokawa.

Some data about the locations of the eight telescopes in the array are given below in 3-D cartesian geocentric coordinates with X pointing to the Greenwich meridian, Y pointing 90° away in the equatorial plane (eastern longitudes have positive Y), and positive Z pointing in the direction of the North Pole. This is a left-handed coordinate system.

Facility	Location	X (m)	Y (m)	Z (m)
ALMA	Chile	2225061.3	-5440061.7	-2481681.2
APEX	Chile	2225039.5	-5441197.6	-2479303.4
JCMT	Hawaii, USA	-5464584.7	-2493001.2	2150654.0
LMT	Mexico	-768715.6	-5988507.1	2063354.9
PV	Spain	5088967.8	-301681.2	3825012.2
SMA	Hawaii, USA	-5464555.5	-2492928.0	2150797.2
SMT	Arizona, USA	-1828796.2	-5054406.8	3427865.2
SPT	Antarctica	809.8	-816.9	-6359568.7

The minimum angle, θ_{min} (in radians) that can be resolved by a VLBI array is given by the equation

$$\theta_{\text{min}} = \frac{\lambda_{\text{obs}}}{d_{\text{max}}} \, ,$$

where λ_{obs} is the observing wavelength and d_{max} is the longest straight line distance between two telescopes used (called the baseline), assumed perpendicular to the line of sight during the observation.

a. Determine d_{max} for the observations of M87 (i.e. the solid lines in Fig 3) and hence θ_{min} if the EHT uses radio waves of frequency 230 GHz. Give your answer for d_{max} in km and θ_{min} in microarcseconds (1 degree = 3600 arcseconds).

An important length scale when discussing black holes is the gravitational radius, $r_g = \frac{GM}{c^2}$, where G is the gravitational constant, M is the mass of the black hole and c is the speed of light. The familiar event horizon of a non-rotating black hole is called the Schwartzschild radius, $r_S \equiv 2r_g$, however this is not what the EHT is able to observe - instead the closest it can see to a black hole is called the photon sphere, where photons orbit in the black hole in unstable circular orbits. On top of this the image of the black hole is gravitationally lensed by the black hole itself magnifying the apparent radius of the photon sphere to be between $(2\sqrt{3+2\sqrt{2}})r_g$ and $(3\sqrt{3})r_g$, determined by spin and inclination; the latter corresponds to a perfectly spherical non-spinning black hole. The area within this lensed image will appear almost black and is the 'shadow' the EHT is looking for.

b. Assuming Sgr A* is a non-spinning black hole with mass $4.1 \times 10^6 \, M_\odot$ and a distance from us of 8.34 kpc:
   (i) Derive the (unlensed) radius of the photon sphere, r_{ph}, in units of r_g, by considering a balance between the centripetal and (Newtonian) gravitational forces, but with the relativistic correction $v' = v\sqrt{1 - 2r_g/r}$ where v' is the classical velocity and $r = r_{ph}$ when $v = c$.
   (ii) Determine the angular diameter (in microarcseconds) of the lensed photon sphere of Sgr A* and hence verify that the EHT can resolve it.

c. The angular diameter of M87* as determined from the images gained by the EHT (shown in Fig 4) is 42 microarcseconds, and the galaxy is 16.8 Mpc away from us. Determine the minimum and maximum possible masses of the SMBH in units of $M_\odot$.

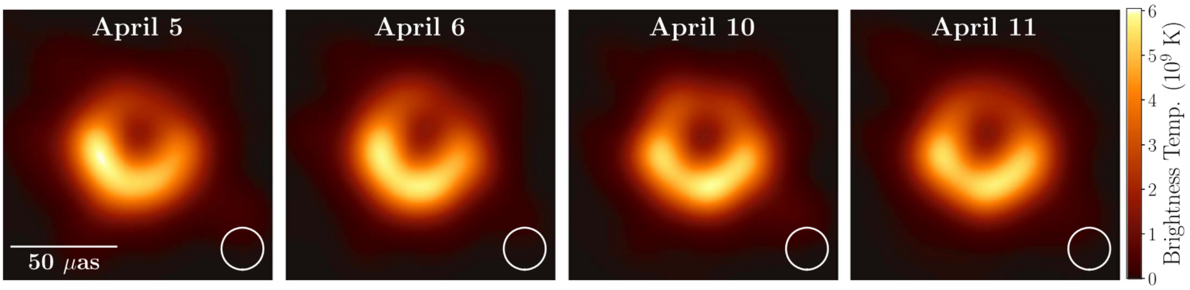


Figure 4: Four nights of data were taken for M87* during the observing window of the EHT, and whilst the diameter of the disk stayed relatively constant the location of bright spots moved, possibly indicating gas that is orbiting the black hole. Credit: EHT Collaboration.

The EHT observed M87* on four separate occasions during the observing window (see Fig 4), and the team saw that some of the bright spots changed in that time, suggesting they may be associated with orbiting gas close to the black hole. The Innermost Stable Circular Orbit (ISCO) is the equivalent of the photon sphere but for particles with mass (and is also stable). The total conserved energy of a circular orbit close to a non-spinning black hole is given by

$$E = mc^2 \left(\frac{1 - \frac{2r_g}{r}}{\sqrt{1 - \frac{3r_g}{r}}} \right) \, ,$$

and the radius of the ISCO, r_{ISCO}, is the value of r for which E is minimised.

d. Determine r_{ISCO} for a non-spinning black hole. Give your answer in units of r_g.

We expect that most black holes are in fact spinning (since most stars are spinning) and the spin of a black hole is quantified with the spin parameter $a \equiv J/J_{\max}$ where J is the angular momentum of the black hole and $J_{\max} = GM^2/c$ is the maximum possible angular momentum it can have. The value of a varies from $-1 \leq a \leq 1$, where negative spins correspond to the black hole rotating in the opposite direction to its accretion disk, and positive spins in the same direction. If $a = 1$ then $r_{\text{ISCO}} = r_g$, whilst if $a = -1$ then $r_{\text{ISCO}} = 9r_g$. The angular velocity of a particle in the ISCO is given by

$$\omega^2 = \frac{GM}{\left(r_{\text{ISCO}}^{3/2} + a r_g^{3/2}\right)^2} \, .$$

e. Taking the mass of M87* as $6.5 \times 10^9 \, M_\odot$:

(i) Determine the period of a particle in the ISCO of M87* for the $a = 1$, $a = -1$ and $a = 0$ (i.e. non-spinning) cases. Give your answer in days.

(ii) One of the bright patches in Fig 4 seemed to move a quarter of the way around the ring between April 5 and 10 (from the left hand side to the bottom). Could it be attributable to gas moving in the ISCO? If so, is the spin likely to be positive or negative?

(iii) Determine the minimum and maximum ISCO periods for Sgr A* and hence suggest a possible reason why M87 has been imaged first, even though Sgr A* has a larger angular diameter, given that each 'exposure' with the EHT was 7 mins long (with multiple exposures from each observing run added together for the final image from each night).

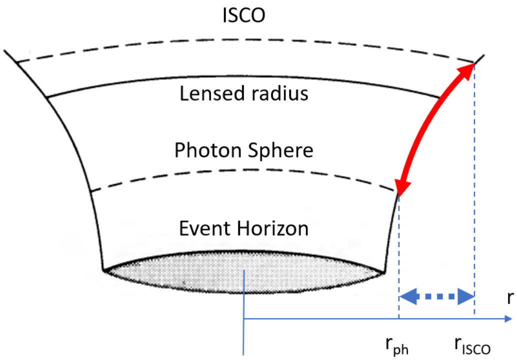


Figure 5: Due to the curvature of spacetime, the real distance travelled by a particle moving from the ISCO to the photon sphere (indicated with the solid red arrow) is longer than you would get purely from subtracting the radial co-ordinates of those orbits (indicated with the dashed blue arrow), which would be valid for a flat spacetime. Relations between these distances are not to scale in this diagram. Credit: Modified from Bardeen et al. (1972).

The spacetime near a black hole is curved, as described by the equations of GR. This means that the distance between two points can be substantially different to the distance you would expect if spacetime was flat. GR tells us that the proper distance travelled by a particle moving from radius r_1 to radius r_2 around a black hole of mass M (with $r_1 > r_2$) is given by

$$\Delta l = \int_{r_2}^{r_1} \left(1 - \frac{2r_g}{r}\right)^{-1/2} \mathrm{d}r \, .$$

f. A particle close to M87* moves directly from r_{ISCO} to r_{ph} (and subsequently into the black hole). What is the **extra** distance travelled by it due to the curvature of spacetime, as described in Fig 5? Give your answer in au, and assume M87* is non-spinning.

Qu 3. Inspirals of Exoplanets Near Tidal Destruction

Some of the very first exoplanets to be discovered in large surveys were dubbed ‘hot Jupiters’ as they were similar in mass to Jupiter (i.e. a gas giant) but were much closer to their star than Mercury is to the Sun (and hence are in a very hot environment). Planetary formation models suggest that they were unlikely to have formed there, but instead formed much further out from the star and migrated inwards, due to gravitational interactions with other planets in the system. Studies of ‘hot Jupiters’ show that there is an overabundance of them with periods of $\sim 3-4$ days, and very few with periods shorter than that. Since large, close-in planets should be the easiest to detect in all of the main methods of finding exoplanets, this scarcity is likely to be a real effect and suggests that exoplanets which are that close to their star are in a relatively rapid (by astronomical standards) inspiral towards destruction by their star.

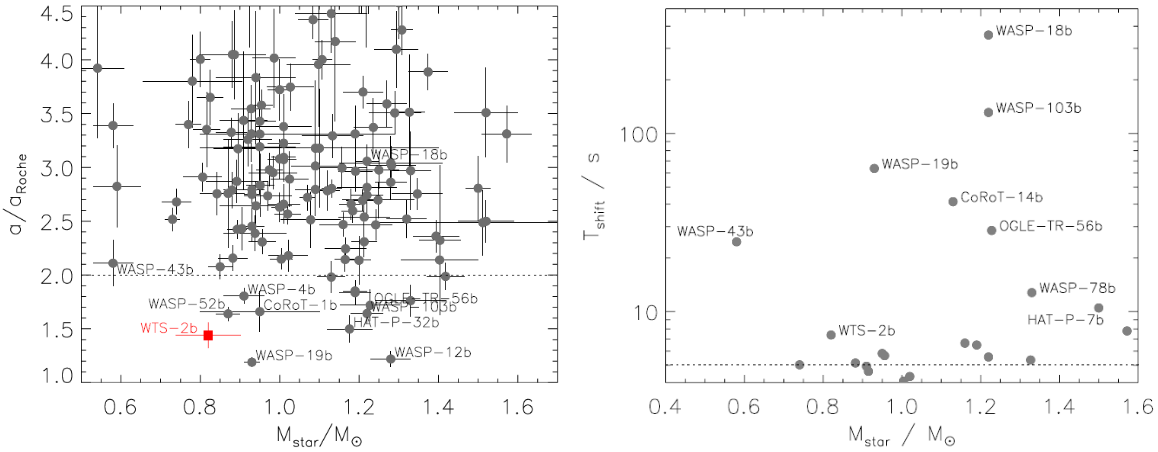


Figure 6: *Left*: The orbital radius of several ‘hot Jupiters’ scaled by the Roche radius of the system (where tidal forces would destroy the planet). There is an expected pile up close to radii double the Roche radius (dotted line), and very few with radii smaller than that - those that are will inevitably spiral into the star and be destroyed by the tidal forces when they get too close. Credit: Birkby et al. (2014).
Right: As the planets inspiral we should see a shift in when their transits occur. This figure shows the predicted size of the shift after a period of 10 years if the tidal dissipation quality factor $Q'_\star = 10^6$, as well as the current detection limit of 5 seconds (dotted line). Therefore measuring if there is any shift in the transit times over the course of a decade of observations can put stringent limits on the value of $Q'_\star$. Credit: Birkby et al. (2014).

The Roche radius, where a planet will be torn apart due to the tidal forces acting on it, is defined as

$$a_{\mathrm{Roche}} \approx 2.16 R_P \left(\frac{M_\star}{M_P}\right)^{1/3} ,$$

where R_P is the radius of the planet, M_P is the mass of the planet and $M_\star$ is the mass of the star. If a gas giant is knocked into a highly elliptical orbit (i.e. $e \approx 1$) that has a periapsis $r_{\mathrm{peri}} < a_{\mathrm{Roche}}$ then it will not survive. However, if the periapsis just grazes the Roche radius ($r_{\mathrm{peri}} \approx a_{\mathrm{Roche}}$) then the orbit will rapidly circularise. By conserving angular momentum, it can be shown that the circular orbit will have a radius $a = 2a_{\mathrm{Roche}}$ (see the left panel of Fig 6). Exoplanets observed to be in an orbit with a radius less than that will be unstable and angular momentum will be transferred from the planet to the star, causing the star to spin more rapidly and the planet’s orbital radius to decrease. Eventually this will result in the planet’s orbit crossing the Roche radius and being destroyed by the tidal forces.

The duration of this inspiral will be dependent on how well the star can dissipate the orbital energy through frictional processes within the star, and can be parameterised by the tidal dissipation quality factor, $Q'_\star$. By looking for changes in the orbital period of the planet, detectable by shifts in the timing of transits by the planet in front of the star, we can determine an estimate of $Q'_\star$, which hence tells us about the internal structure of stars. These ‘hot Jupiters’ are the best laboratory we have for this, as they are the most likely to produce a measurable shift (i.e. ~ 5 s) in transit times within only ~ 10 years (see the right panel of Fig 6). We will try and reproduce these results in this question.

The WTS-2 system consists of a star of mass $M_\star = 0.820\,M_\odot$, peak in its black-body spectrum at $\lambda_{\text{max}} = 580$ nm, and distance from us of 1.03 kpc, with an orbiting planet (called WTS-2b) with a period $P = 1.0187$ days, mass of 1.12 M_J and radius 1.36 R_J. The mass and radius of Jupiter are $M_J = 1.90 \times 10^{27}$ kg and $R_J = 7.15 \times 10^7$ m respectively.

a. Show that WTS-2b has an orbital radius of $\sim 1.4\,a_{\text{Roche}}$.

b. Given the apparent magnitude of WTS-2 in the visible is $m = 16.14$ and the absolute magnitude of the Sun in the same part of the EM spectrum is $\mathcal{M}_\odot = 4.83$:

(i) Calculate the luminosity of the star, $L_\star$. Give your answer in units of $L_\odot$.

(ii) Hence work out the radius of the star, $R_\star$. Give your answer in units of $R_\odot$.

(iii) Show that the incident flux (in W m^{-2}) on the planet is $\sim 10^3$ times larger than what we receive on Earth from the Sun.

The change in the semi-major axis of the planet, a, due to tidal forces is given by

$$\left|\frac{\dot{a}}{a}\right| = 6k_2\Delta t\frac{M_P}{M_\star}\left(\frac{R_\star}{a}\right)^5 n^2$$

where the dot notation is used to indicate the differential with respect to time (i.e. $\dot{a} \equiv \mathrm{d}a/\mathrm{d}t$), k_2 is a constant related to the density structure of the star, Δt is the (assumed constant) time lag between where the planet is in its orbit and the location of the tidal bulge on the star, and $n = 2\pi/P$. By separating variables and integrating this equation, an expression can be derived for the time it takes for a to decrease to zero. This is known as the inspiral time, τ, and even though the planet will be destroyed when $a = a_{\text{Roche}}$ the time to go from $a = a_{\text{Roche}}$ to $a = 0$ is negligible in comparison to the time to get to $a = a_{\text{Roche}}$, so τ is a good estimate of the remaining lifetime of the planet.

c. Derive an equation for τ, showing $\tau \propto a^8$. Hint: n is a function of a due to Kepler's third law.

The tidal dissipation quality factor is defined as

$$Q'_\star = \frac{3}{4k_2\Delta t n}$$

and so absorbs some of our unknown parameters about the stellar structure. Based upon studies of binary stars it seems a value of $Q'_\star \sim 10^6 - 10^7$ is reasonable, although it is poorly constrained. The size of the shift in transit timings, T_{shift}, over an observation period of T seconds is

$$T_{\text{shift}} = \frac{1}{2n}T^2\left(\frac{\mathrm{d}n}{\mathrm{d}t}\right) .$$

d. For the planet WTS-2b, assuming $Q'_\star = 10^6$:

(i) Show that $\tau \propto \dfrac{Q'_\star}{n}\dfrac{M_\star}{M_P}\left(\dfrac{a}{R_\star}\right)^5$, finding the constant of proportionality (as a fraction), and hence the remaining lifetime of the planet. Give your answer in Myr. [1 Myr = 10^6 years]

(ii) Show that $T_{\text{shift}} \propto \dfrac{T^2}{\tau}$, finding the constant of proportionality, and hence verify that T_{shift} is measurable (i.e. > 5 s) if $T = 10$ years. Hint: Using the chain rule, $\dfrac{\mathrm{d}n}{\mathrm{d}t} = \dfrac{\mathrm{d}n}{\mathrm{d}a}\dot{a}$.

END OF PAPER

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