BAAO
British Astronomy and Astrophysics Olympiad

British Astronomy and Astrophysics Olympiad 2018-2019

Astronomy & Astrophysics Competition Paper

Monday 21st January 2019

This question paper must not be taken out of the exam room

Instructions

Time: 3 hours plus 15 minutes reading time (no writing permitted). Approx 45 minutes per question.

Questions: All four questions should be attempted.

Marks: The questions carry similar marks.

Solutions: Answers and calculations are to be written on loose paper or in examination booklets. Students should ensure their name and school is clearly written on all answer sheets and pages are numbered. A standard formula booklet with standard physical constants should be supplied.

Instructions: To accommodate students sitting the paper at different times, please **do not discuss** any aspect of the paper on the internet until 8 am Saturday 26th January.

Clarity: Solutions must be written legibly, in black pen (the papers are photocopied), and working down the page. Scribble will not be marked and overall clarity is an important aspect of this exam paper.

Eligibility: The International Olympiad will be held during August 2019; all sixth form students are eligible to participate, even if they will be attending university from October.

Training Dates and the International Astronomy and Astrophysics Olympiad (IOAA)

The IOAA this year will be held in Zanka, Hungary, from 2nd to 11th August 2019.

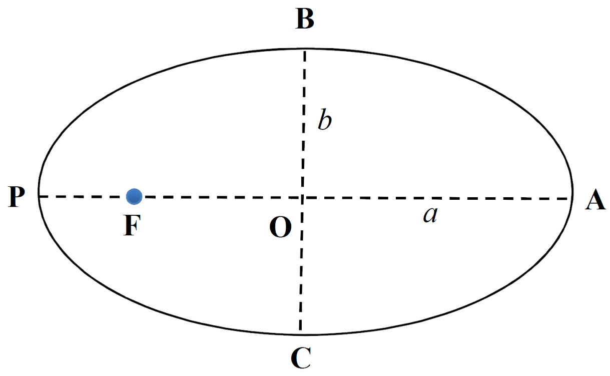
The best students taking this paper that are eligible to represent the UK at the IOAA will be invited to attend the **Training Camp** *to be held in the Physics Department at the University of Oxford,* **(Saturday 13th April to Wednesday 17th April 2019)**. *Astronomy material will be covered; problem solving skills and observational skills (telescope and naked eye observations) will be developed. At the Training Camp a data analysis exam and a short theory paper will be sat. A team of five students (plus one reserve) will be selected for further training. From May there will be mentoring by email to cover some topics and problems, followed by additional training camps in the summer.*

Important Constants

Constant	Symbol	Value
Speed of light	c	$3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Earth's rotation period	1 day	24 hours
Earth's orbital period	1 year	365.25 days
parsec	pc	$3.09 \times 10^{16}\,\mathrm{m}$
Astronomical Unit	au	$1.50 \times 10^{11}\,\mathrm{m}$
Semi-major axis of the Earth's orbit		1 au
Radius of the Sun	$R_{\odot}$	$6.96 \times 10^{8}\,\mathrm{m}$
Radius of the Earth	$R_{\oplus}$	$6.37 \times 10^{6}\,\mathrm{m}$
Mass of the Sun	$M_{\odot}$	$1.99 \times 10^{30}\,\mathrm{kg}$
Mass of the Earth	$M_{\oplus}$	$5.97 \times 10^{24}\,\mathrm{kg}$
Luminosity of the Sun	$L_{\odot}$	$3.85 \times 10^{26}\,\mathrm{W}$
Stephan-Boltzmann constant	σ	5.67×10^{-8} J m^{-2} K^{-4}
Gravitational constant	G	6.67×10^{-11} m^{3} kg^{-1} s^{-2}
Boltzmann constant	k_{B}	1.38×10^{-23} J K^{-1}
Permittivity of free space	ε_0	8.85×10^{-12} F m^{-1}
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ H m^{-1}
Planck's constant	h	6.63×10^{-34} J s
Elementary charge	e	1.60×10^{-19} C
Proton rest mass	m_{p}	1.67×10^{-27} kg
Electron rest mass	m_{e}	9.11×10^{-31} kg

Important Formulae

You might find the diagram of an elliptical orbit below useful in solving some of the questions:



Elements of an elliptic orbit:	$a = \mathbf{OA}\ (= \mathbf{OP})$	semi-major axis
	$b = \mathbf{OB}\ (= \mathbf{OC})$	semi-minor axis
	$e = \sqrt{1 - \frac{b^2}{a^2}}$	eccentricity
	F	focus
	P	periapsis (point nearest to F)
	A	apoapsis (point furthest from F)

Kepler's Third Law: For an elliptical orbit, the square of the period, T, of an object about the focus is proportional to the cube of the semi-major axis, a (as defined above), such that

$$T^2 = \frac{4\pi^2}{GM} a^3 ,$$

where M is the total mass of the system (typically dominated by the central object) and G is the universal gravitational constant.

Magnitudes: The apparent magnitudes of two objects, m_1 and m_0, are related to their apparent brightnesses, b_1 and b_0, via the formula

$$\frac{b_1}{b_0} = 10^{-0.4(m_1 - m_0)} .$$

The absolute magnitude of an object, $\mathcal{M}$, is the same as its apparent magnitude when viewed from 10 pc, hence the relationship between apparent and absolute magnitude and distance is

$$m - \mathcal{M} = 5 \log \left(\frac{d}{10} \right) ,$$

where d is measured in parsecs.

Qu 1. Parker Solar Probe

The Parker Solar Probe (PSP) is part of a mission to learn more about the Sun, named after the scientist that first proposed the existence of the solar wind, and was launched on 12$^{\text{th}}$ August 2018. Over the course of the 7 year mission it will orbit the Sun 24 times, and through 7 flybys of Venus it will lose some energy in order to get into an ever tighter orbit (see Figure 1). In its final 3 orbits it will have a perihelion (closest approach to the Sun) of only $r_{\text{peri}} = 9.86\,R_{\odot}$, about 7 times closer than any previous probe, the first of which is due on 24$^{\text{th}}$ December 2024. In this extreme environment the probe will not only face extreme brightness and temperatures but also will break the record for the fastest ever spacecraft.

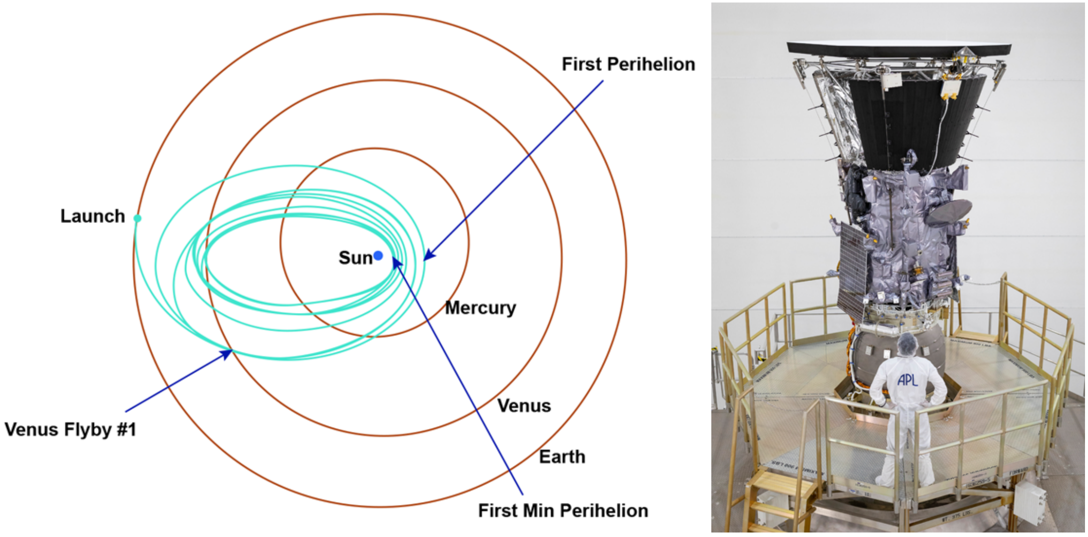


Figure 1: *Left*: The journey PSP will take to get from the Earth to the final orbit around the Sun. *Right*: The probe just after assembly in the John Hopkins University Applied Physics Laboratory. Credit: NASA / John Hopkins APL / Ed Whitman.

a. When the probe is at its closest perihelion:

(i) Calculate the apparent magnitude of the Sun, given that from Earth $m_{\odot} = -26.74$.

(ii) Calculate the temperature the heat shield must be able to survive. Assume that the heat shield of the probe absorbs all of the incident radiation, radiates as a perfect black body, and that only one side of the probe ever faces the Sun (to protect the instruments) such that the emitting (surface) area is double the absorbing (cross-sectional) area.

b. The speed, v, of an object in an elliptical orbit of semi-major axis a around an object of mass M when a distance r away can be calculated as

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) .$$

Given that in its final orbit PSP has a orbital period of 88 days, calculate the speed of the probe as it passes through the minimum perihelion. Give your answer in km s^{-1}.

c. After the first flyby of Venus on 3$^{\text{rd}}$ October 2018 it was moved into an orbit with a 150 day period, and the subsequent first perihelion on 6$^{\text{th}}$ November 2018 was at a distance of $35.7\,R_{\odot}$. Given its mass at launch was 685 kg, calculate the total amount of energy that had to be lost by the probe to get from this first orbit (ignoring the orbital properties prior to the Venus flyby) to the final orbit. Ignore any change in the mass of the probe due to burning fuel.

Close to the Sun the communications equipment is very sensitive to the extreme environment, so the mission is planned for the probe to take all of its primary science measurements whilst within 0.25 au of the Sun, and then to spend the rest of the orbit beaming that data back to Earth, as shown in Figure 2.

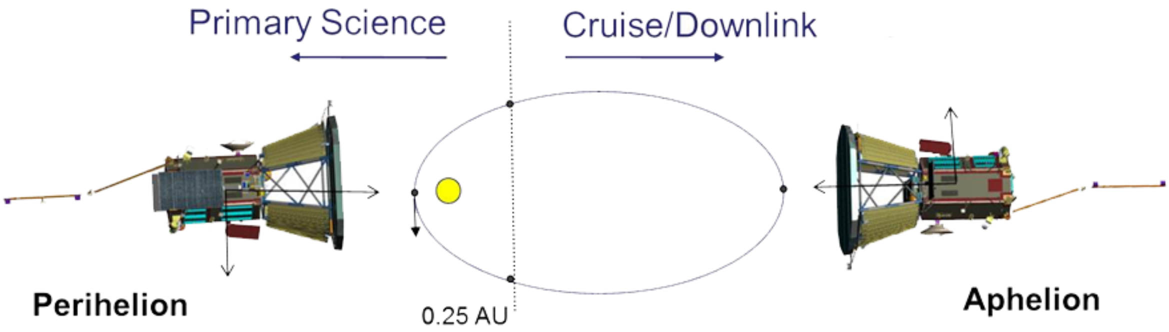


Figure 2: The way PSP is planned to split each orbit into taking measurements and sending data back. Credit: NASA / Johns Hopkins APL.

When considering the position of an object in an elliptical orbit as a function of time, there are two important angles (called 'anomalies') necessary to do the calculation, and they are defined in Figure 3. By constructing a circular orbit centred on the same point as the ellipse and with the same orbital period, the eccentric anomaly, E, is then the angle between the major axis and the perpendicular projection of the object (some time t after perihelion) onto the circle as measured from the centre of the ellipse ($\angle xcz$ in the figure). The mean anomaly, M, is the angle between the major axis and where the object would have been at time t if it was indeed on the circular orbit ($\angle ycz$ in the figure, such that the shaded areas are the same).

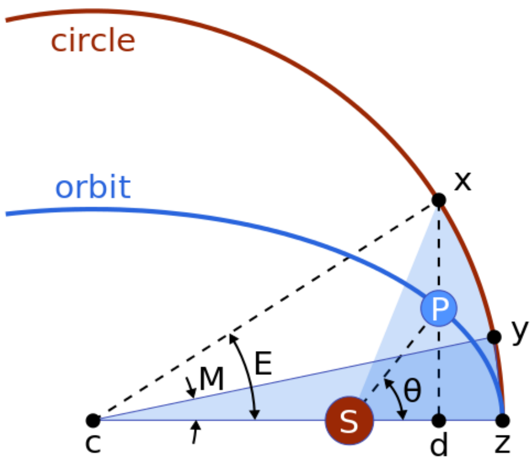


Figure 3: The definitions of the anomalies needed to get the position of an object in an ellipse as a function of time. The Sun (located at the focus) is labeled S and the probe P. M and E are the mean and eccentric anomalies respectively. The angle θ is called the true anomaly and is not needed for this question. Credit: Wikipedia.

d. Derive a formula for the distance from the focus for an elliptical orbit, r (SP in the figure) in terms of the semi-major axis a, the eccentricity e, and the eccentric anomaly E.

The eccentric anomaly can be related to the mean anomaly through Kepler's Equation,

$$M = E - e \sin E \, .$$

e. Calculate how long PSP spends doing primary science in its final orbit. Give your answer in days.

Qu 2. 360 Days in a Year

A day on Earth can be defined in two ways: relative to the Sun (called solar or synodic time) or relative to the background stars (called sidereal time). The mean solar day is 24 hours (within a few milliseconds), whilst the mean sidereal day is shorter at 23 hours 56 minutes 4 seconds (to the nearest second). The solar day is longer as over the course of a sidereal day the Earth has moved slightly in its orbit around the Sun and so has to rotate slightly further for the Sun to be back in the same direction (see Figure 4).

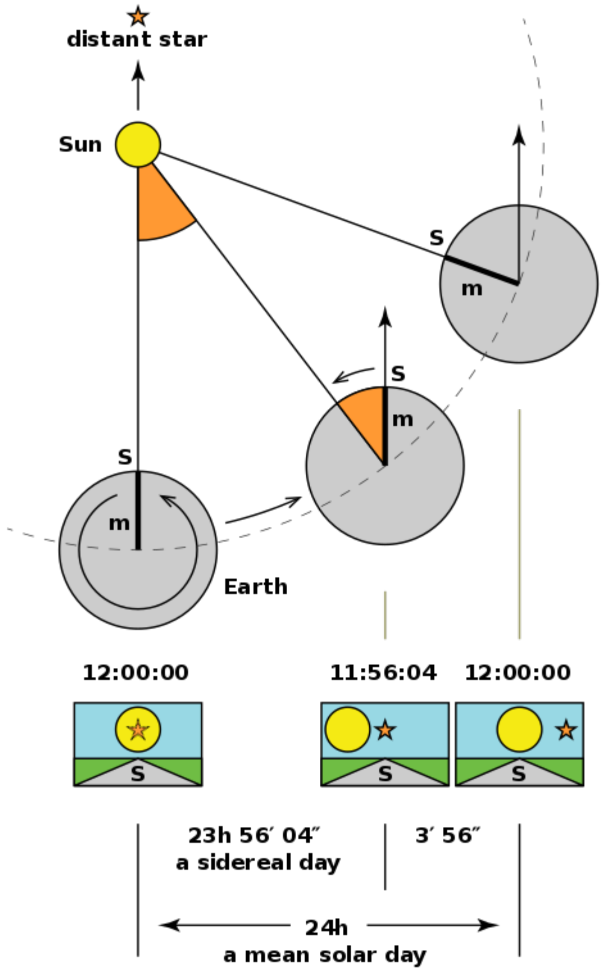


Figure 4: A solar day is defined as the time between two consecutive passages of the Sun through the meridian, corresponding to local midday (which in the Northern hemisphere is in the South), whilst a sidereal day is the time for a distant star to do the same. The difference between the two is due to the Earth having moved slightly in its orbit around the Sun.
Credit: Wikipedia.

The length of a year on Earth is 365.25 solar days (to 2 d.p.), however some ancient civilizations used to believe that there were once exactly 360 solar days in a year, with various myths explaining where the extra days came from. In this question you will look at how to return the Earth to this time.

[Note that this question is very sensitive to the precision of the fundamental constants used, so throughout please take $G = 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, $R_\oplus = 6371$ km, $M_\oplus = 5.972 \times 10^{24}$ kg, $M_\odot = 1.989 \times 10^{30}$ kg and 1 au $= 1.496 \times 10^{11}$ m.]

a. How many sidereal days elapse during a year? Give your answer to 2 d.p.

b. Without further calculation, suggest how many sidereal days there would be if a year was in fact only 360 solar days.

c. What reduction in the Earth's semi-major axis would be required for the year to be shortened down to 360 solar days?

d. Imagine creating an incredibly powerful rocket, positioned on the Earth's equator, that when fired once can apply a huge force to the Earth in a very short time period, delivering a total impulse of Δp. Assuming the Earth's orbit is initially circular, calculate:

   (i) The total impulse required to slow the Earth's rotation to give a year of 360 solar days, but with no change in the orbit.

   (ii) The total impulse required to change the orbit to give a year of 360 solar days, but with no change in the length of a solar day, also explaining how the rocket needs to be fired.

e. Tidal interactions between the Moon and the Earth mean that the Earth's rotation rate is slowing down, such that a solar day has lengthened over the last 2800 years by an average of 2.3 ms per century. Similar interactions between the Earth and the Sun, as well as mass loss by the Sun due to nuclear fusion and the solar wind, mean that the distance between them is increasing by about 1.5 cm per year. Assuming these rates have stayed constant over time and that the Earth's orbit has remained circular throughout, is there any time in either the Earth's past or future when it had or when it will have a year with 360 solar days? Give your answer in Myr (where 1 Myr = 10^6 years). [For reference, the age of the Earth is 4543 Myr.]

Helpful equations:

The moment of inertia, I, of a sphere of mass M and radius R is $I = \frac{2}{5}MR^2$.

The angular momentum, L, of a spinning object with an angular velocity of ω is $L = I\omega = r \times p$, where p is the linear momentum of a point particle a distance r from the axis of rotation.

The speed, v, of an object in an elliptical orbit of semi-major axis a around an object of mass M when a distance r away can be calculated as

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) .$$

Qu 3. Stellar Mass Limits

In the heart of every star, nuclear fusion is taking place. For most stars that involves hydrogen being turned into helium, a process that starts by bringing two protons close enough that the strong nuclear force can act upon them. The smallest stars are the ones that have a core that is only just hot enough for fusion to occur, whilst in the biggest ones the radiation pressure of the photons given out by the fusion reaction pushing on the stellar material can overcome the gravitational forces holding it together.

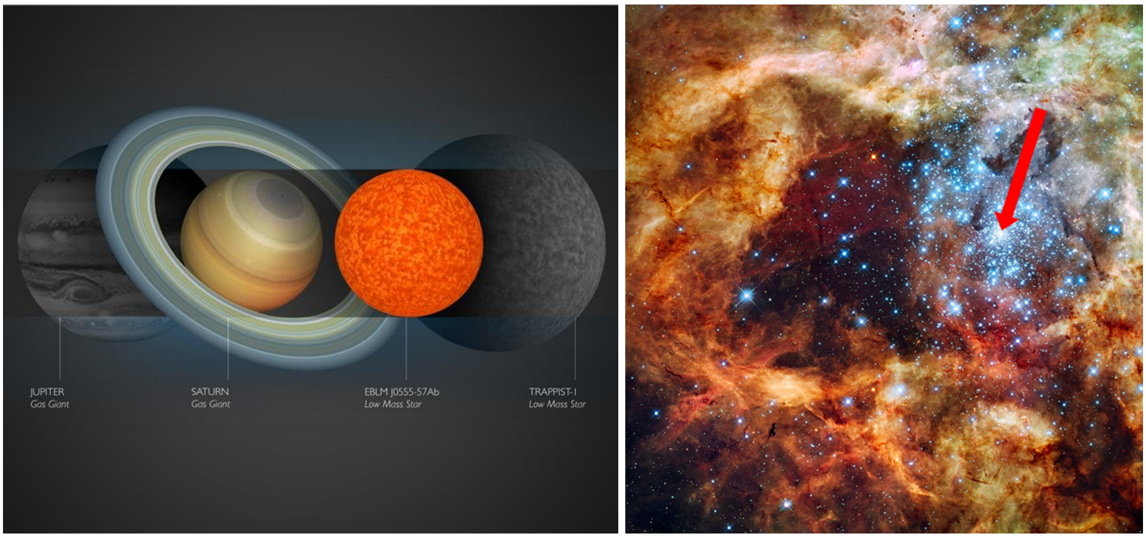


Figure 5: *Left*: The lowest mass star we know of, EBLM J0555-57Ab, was found by von Boetticher *et al.* (2017) and is about the size of Saturn with a mass of $0.081\,M_\odot$. Credit: Amanda Smith, University of Cambridge. *Right*: The highest mass star we know of, R136a1, is in the centre of the clump of stars on the right of this HST image of the Tarantula Nebula. Schneider *et al.* (2014) suggest it has a mass of $315\,M_\odot$, which is above what stellar evolution models allow. Despite its large mass, other stars have far bigger radii. Credit: NASA & ESA.

For a spherical main sequence star made of a plasma (a fully ionized gas of electrons and nuclei) that is acting like an ideal gas, the temperature at the core can be approximately calculated as

$$T_{\text{int}} \simeq \frac{GM\bar{\mu}}{k_{\text{B}}R} \quad \text{where} \quad \bar{\mu} = \frac{m_{\text{p}}}{2X + 3Y/4 + Z/2} \, .$$

In this equation, M is the mass of the star, R is its radius, k_{B} is the Boltzmann constant, and $\bar{\mu}$ is the mean mass of the plasma particles (i.e nuclei and electrons) with m_{p} the mass of a proton.

a. Given the Sun's composition has hydrogen fraction, $X = 0.72$, helium fraction $Y = 0.26$ and 'metals' (i.e. any element lithium and heavier) fraction $Z = 0.02$, estimate the temperature at the centre of the Sun.

b. Classically, two protons need to have enough energy to overcome their electrostatic repulsion in order to fuse. Assuming the kinetic energy of each proton, E_K, and the electric potential energy between two protons, E_P, are

$$E_K = \frac{3}{2}k_{\text{B}}T \quad \text{and} \quad E_P = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{b} \, ,$$

where e is the fundamental charge and b is the distance between the protons, calculate the value of $T_{\text{classical}}$ necessary to allow fusion to occur, given that at that point $b = 1$ fm ($= 10^{-15}$ m). [You should find that it's much larger than your answer to part a.]

Classically, the core of the Sun is not hot enough for fusion, and yet fusion is clearly happening. The key is that it is a fundamentally quantum process, and so protons are able to 'quantum tunnel' through the Coloumb barrier (see Figure 6), allowing fusion to occur at lower temperatures. In quantum mechanics, fusion will happen when $b = \lambda$ where λ is the de Broglie wavelength of the proton, related to the momentum of the proton by $\lambda = h/p$.

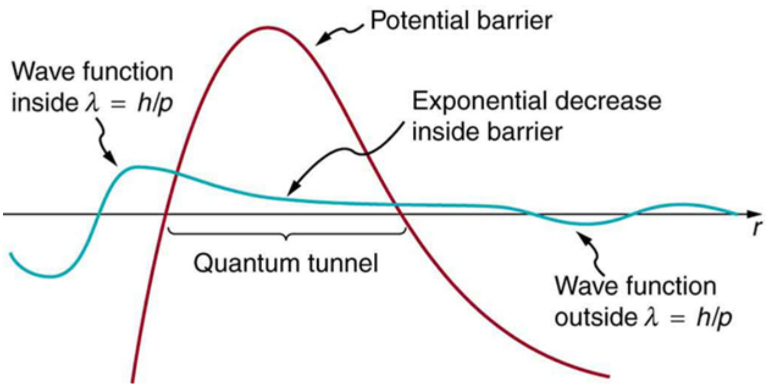


Figure 6: A diagram showing the way a particle can pass through a classically impenetrable potential barrier due to its wave-like properties on the quantum scale.
Credit: Brooks/Cole - Thomson Learning.

c. Given that the proton momentum is related to the average kinetic energy of a particle in the plasma by $E_K = p^2/2m_{\rm p}$ calculate the value of λ and hence calculate $T_{\rm quantum}$. [You should find that it's below your answer to part a.]

In the smallest stars, electron degeneracy prevents them from compressing in radius and thus stops the core reaching $T_{\rm int} \gtrsim T_{\rm quantum}$. At the limit of electron degeneracy, the number density of electrons $n_{\rm e} = 1/\lambda_{\rm e}^3$ where $\lambda_{\rm e}$ is the de Broglie wavelength of the electrons.

d. Assuming the star to be of uniform density at this limit with $\rho = m_{\rm p} n_{\rm e}$ and the electrons to be in thermal equilibrium with the plasma, show that the minimum mass of a star for which $T_{\rm int} = T_{\rm quantum}$ is $\sim 0.1\, M_\odot$.

In the largest stars, radiation pressure pushes on the outer layers of the star stronger than gravity pulls them in. The brightest luminosity for a star is known as the Eddington luminosity, $L_{\rm Edd}$. The acceleration due to radiation pressure can be calculated as

$$g_{\rm rad} = \frac{\kappa_{\rm e} I}{c} \quad \text{where} \quad \kappa_{\rm e} = \frac{\sigma_{\rm T}}{2m_{\rm p}}(1 + X)$$

and $\kappa_{\rm e}$ is the electron opacity of the stellar material, $\sigma_{\rm T}$ is the Thomson scattering cross-section for electrons ($= 66.5$ fm^2), X is the hydrogen fraction, and I is the intensity of radiation (in W m^{-2}). Assuming main-sequence stars follow a mass-luminosity relation of $L \propto M^3$, the maximum mass of a star can be found by considering one that is radiating at $L_{\rm Edd}$.

e. By balancing the radiative acceleration with the gravitational acceleration at the surface of a star, derive a formula for $L_{\rm Edd}$ in terms of M, and hence calculate the maximum mass of a star with a hydrogen fraction like the Sun. Give your answer in $M_\odot$.

Qu 4. Superluminal Jets

The speed of light is considered to be the speed limit of the Universe, however knots of plasma in the jets from active galactic nuclei (AGN) have been observed to be moving with apparent transverse speeds in excess of this, called superluminal speeds. Some of the more extreme examples can be appearing to move at up to 6 times the speed of light (see Figure 7).

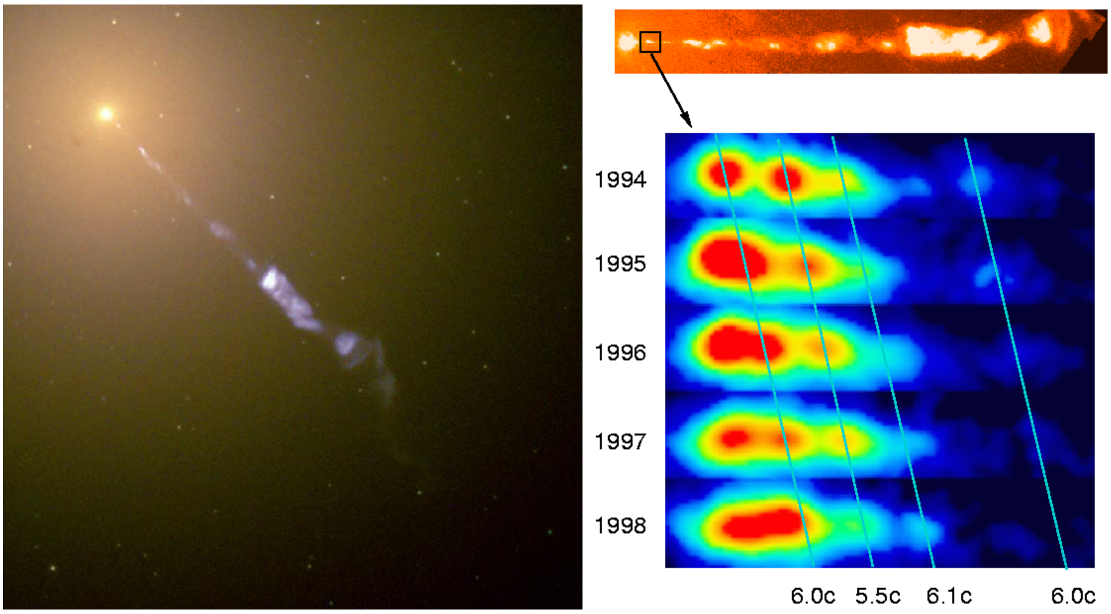


Figure 7: *Left*: The jet coming from the elliptical galaxy M87 as viewed by the Hubble Space Telescope (HST). *Right*: Sequence of HST images showing motion at six times the speed of light. The slanting lines track the moving features, and the speeds are given in units of the velocity of light, c.
Credit: NASA / Space Telescope Science Institute / John Biretta.

This can be explained by understanding that the jet is offset by an angle θ from the sightline to Earth, and that the real speed of the plasma knot, v, is less than c, and from it we can define the scaled speed $\beta \equiv v/c$.

a. Show, with use of an appropriate diagram, that the apparent value of the scaled transverse speed (for a jet coming towards us) is

$$\beta_{\text{app}} = \frac{\beta \sin\theta}{1 - \beta\cos\theta} .$$

b. Determine the relationship between β and θ that maximises β_{app} for a given value of β, and hence determine the minimum value of β needed to give rise to superluminal apparent speeds (i.e. when $\beta_{\text{app}}^{\text{max}} > 1$). [You are given that a graph of β_{app} against θ has only one turning point in the range $0 < \theta < \pi$, and that it is a maximum.]

Superluminal jets are not limited just to AGN, as they have also been observed from systems within our own galaxy. A particularly famous one is the 'microquasar' GRS 1915+105, which is a low mass X-ray binary consisting of a small star orbiting a black hole. A symmetrical jet with components approaching and receding from us is observed (as expected for jets coming from the poles of the black hole), and the apparent transverse motion of material in those jets has been measured using very high resolution radio imaging. Fender *et. al* (1999) measure these motions to be $\mu_a = 23.6$ mas day^{-1} and $\mu_r = 10.0$ mas day^{-1} for the approaching and receding jet respectively (1 mas = 1 milliarcsecond, a unit of angle, and there are 3600 arcseconds in a degree) and the distance to the system as 11 kpc.

c. Calculate β_{app} for both jets, and use your formula from part b. to calculate the minimum value of β to explain the apparent superluminal motion.

In practice, for a given $\beta_{\rm app}$ the values of β and θ are degenerate and it is unlikely that the orientation of the jet is such that $\beta_{\rm app}$ has been maximised, so the value in part c. is just a lower limit. However, since there are two jets then if we assume that they are from the same event (and so equal in speed but opposite in direction) we can break this degeneracy.

d. Given that μ_a and μ_r can be calculated (in radians $\rm s^{-1}$) as

$$\mu_a = \frac{\beta \sin\theta}{1-\beta\cos\theta}\frac{c}{D} \quad \text{and} \quad \mu_r = \frac{\beta \sin\theta}{1+\beta\cos\theta}\frac{c}{D} \,,$$

derive a formula for the distance, D, as a function of θ, μ_a and μ_r (i.e. independent of β), and hence calculate θ.

e. Show that $\beta\cos\theta$ can be expressed purely as a function of μ_a and μ_r, and hence use your value of θ to calculate the value of β.

Since it is a binary system, we can gain information about the masses of the objects by looking at their period and radial velocity. Formally, the relationship is

$$\frac{(M_{\rm BH}\sin i)^3}{(M_{\rm BH}+M_\star)^2} = \frac{P_{\rm orb}K_d^3}{2\pi G} \,,$$

where $M_{\rm BH}$ is the mass of the black hole, $M_\star$ is the mass of the orbiting star, i is the inclination of the orbit, $P_{\rm orb}$ is the orbital period, and K_d is the amplitude of the radial velocity curve. Normally the inclination can't be measured, however if we assume that the orbit is perpendicular to the jets then $i = \theta$ and we can measure the mass of the black hole.

f. Greiner *et. al* (2001) measure $K_d = 140$ km $\rm s^{-1}$, $P_{\rm orb} = 33.5$ days, and a mass ratio for the two objects of $M_{\rm BH}/M_\star = 12.3$. Using the assumption that $i = \theta$, calculate $M_{\rm BH}$. Give your answer in $M_\odot$.

END OF PAPER

Questions proposed by:
Dr Alex Calverley (Royal Grammar School, Guildford)
John Hayton (University of Cambridge)
Josh Brown (University of Cambridge)

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