BAAO
British Astronomy and Astrophysics Olympiad

British Astronomy and Astrophysics Olympiad 2017-2018

Astronomy & Astrophysics Competition Paper

Monday 22nd January 2018

This question paper must not be taken out of the exam room

Instructions

Time: 3 hours plus 15 minutes reading time (no writing permitted). Approx 45 minutes per question.

Questions: All four questions should be attempted.

Marks: The questions carry similar marks.

Solutions: Answers and calculations are to be written on loose paper or in examination booklets. Students should ensure their name and school is clearly written on all answer sheets and pages are numbered. A standard formula booklet with standard physical constants should be supplied.

Instructions: To accommodate students sitting the paper at different times, please **do not discuss** any aspect of the paper on the internet until 8 am Saturday 27th January.

Clarity: Solutions must be written legibly, in black pen (the papers are photocopied), and working down the page. Scribble will not be marked and overall clarity is an important aspect of this exam paper.

Eligibility: The International Olympiad will be held during September 2018; all sixth form students are eligible to participate, even if they will be attending university from October.

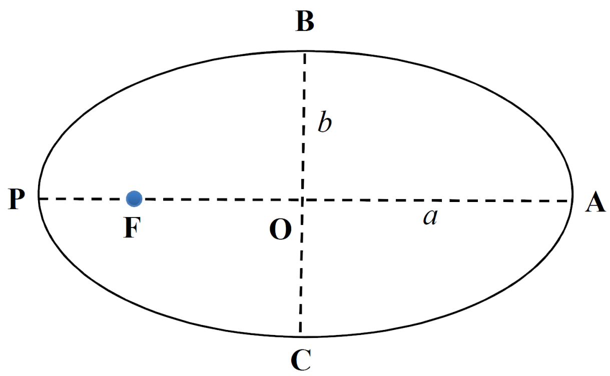
Training Dates and the International Astronomy and Astrophysics Olympiad (IOAA)

The team will be selected from sixth form students taking this paper **and** *Y12 students taking the AS Challenge in March. The best students eligible to represent the UK at the IOAA will be invited to attend the* **Training Camp** *to be held in the Physics Department at the University of Oxford,* **(Monday 9th April to Thursday 12th April 2018)**. *Astronomy material will be covered; problem solving skills and observational skills (telescope and naked eye observations) will be developed. At the Training Camp a data analysis exam and a short theory paper will be sat. Five students (plus one reserve) will be selected for further training. From May there will be mentoring by email to cover some topics and problems, followed by additional training camps in the summer.*

Important Constants

Constant	Symbol	Value
Speed of light	c	$3.00 \times 10^8\,\mathrm{m\,s^{-1}}$
Earth's rotation period	1 day	24 hours
Earth's orbital period	1 year	365.25 days
parsec	pc	$3.09 \times 10^{16}\,\mathrm{m}$
Astronomical Unit	au	$1.50 \times 10^{11}\,\mathrm{m}$
Radius of the Earth	$R_\oplus$	$6.37 \times 10^6\,\mathrm{m}$
Semi-major axis of the Earth's orbit		1 au
Radius of the Sun	$R_\odot$	$6.96 \times 10^8\,\mathrm{m}$
Mass of the Sun	$M_\odot$	$1.99 \times 10^{30}\,\mathrm{kg}$
Mass of the Earth	$M_\oplus$	$5.97 \times 10^{24}\,\mathrm{kg}$
Luminosity of the Sun	$L_\odot$	$3.85 \times 10^{26}\,\mathrm{W}$
Stephan-Boltzmann constant	σ	5.67×10^{-8} J m^{-2} K^{-4}
Gravitational constant	G	6.67×10^{-11} m^3 kg^{-1} s^{-2}

You might find the diagram of an elliptical orbit below useful in solving some of the questions:



Elements of an elliptic orbit:

$a = \mathbf{OA}\ (= \mathbf{OP})$	semi-major axis
$b = \mathbf{OB}\ (= \mathbf{OC})$	semi-minor axis
$e = \sqrt{1 - \frac{b^2}{a^2}}$	eccentricity
F	focus
P	periapsis (point nearest to F)
A	apoapsis (point furthest from F)

Keplers Third Law: For an elliptical orbit, the square of the period, T, of orbit of an object about the focus is proportional to the cube of the semi-major axis, a (the average of the minimum and maximum distances from the Sun). The constant of proportionality is $4\pi^2/GM$, where M is the mass of the central object and G is the universal gravitational constant.

Magnitudes: The apparent magnitudes of two objects, m_1 and m_0, are related to their apparent brightnesses, b_1 and b_0, via the formula:

$$\frac{b_1}{b_0} = 10^{-0.4(m_1 - m_0)}$$

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Qu 1. Stellar Mass Loss

All stars lose mass during their lifetimes due to two main routes: particles escaping their surface (referred to as the stellar wind), and the mass defect of the nuclear reactions occurring in their cores.

a. Given that, at the distance of the Earth, the proton flux from the Sun's stellar wind (which is assumed to be radiated equally in all directions) is 3.0×10^{12} m^{-2} s^{-1}, and that the luminosity of the Sun is solely due to the fusion of hydrogen to helium:

   (i) Show that the rate at which the Sun is losing mass, $\dot{M} \equiv \frac{\mathrm{d}M}{\mathrm{d}t}$, due to its stellar wind is $\sim 10^{-14}\, M_\odot$ yr^{-1}. [Take the mass of a proton to be 1.67×10^{-27} kg.]

   (ii) Show that $\dot{M}$ due to nuclear fusion is greater than that from the solar wind.

   (iii) Estimate how much the Earth-Sun distance and the Earth's orbital period will have changed after the Sun has lost mass (via both routes) for one year. Assume the orbit remains circular throughout and ignore gravitational effects from all other bodies.

In practice, the mass loss rate can vary quite considerably during a star's lifetime, particularly once it has left the main sequence when the stellar wind can become much more substantial. Wolf-Rayet stars are massive stars near the end of their lives, presumed to be the in the stage just before a supernova, and are losing substantial amounts of mass due to very fast stellar winds. This deposits considerable energy into the surrounding interstellar medium (ISM) and can sweep up material into a thin bubble around the star, visible as a type of planetary nebula.

Figure 1: The nebula NGC 2359 around the Wolf-Rayet star WR7. The nebula is known as Thor's Helmet due to its resemblance to the helmet worn by the character from the Marvel Comics series.
Credit: Star Shadows Remote Observatory and PROMPT/UNC.

b. If every photon leaving a star is able to transfer all of its momentum (given by $p_{\mathrm{photon}} = \frac{E_{\mathrm{photon}}}{c}$) to drive a stellar wind (known as the single scattering limit):

   (i) Show that the maximum mass loss rate for a star can be written in terms of the luminosity of the star, L, the (terminal) velocity of the stellar wind far from the star, v_∞, and the speed of light, c, as

$$\dot{M}_{\mathrm{max}} = \frac{L}{v_\infty c}.$$

   (ii) Hence, derive an expression for the maximum kinetic energy deposited per second by the stellar wind as a function of the luminosity of the star. Comment on your answer.

c. The Wolf-Rayet star WR7 is in the constellation of Canis Major and its strong winds are responsible for the nebula known as Thor's Helmet (see Figure 1). The star has a mass of 16 $M_\odot$, a radius of 1.41 $R_\odot$, and a surface temperature of 112 000 K, with a measured v_∞ of 1545 $\mathrm{km\,s^{-1}}$.

(i) Predict the mass loss rate of WR7 based upon the properties of the star, assuming the single scattering limit. Give your answer in units of $M_\odot$ yr^{-1}.

(ii) The nebula is 5 arcmins in diameter ($1^\circ = 60$ arcmin) and 4.8 kpc away, and at its edge is a bright thin shell of swept up material expanding at a rate of 30 $\mathrm{km\,s^{-1}}$. The age of such a nebula, t, is related to the current values of radius, R, and expansion speed, v, by $t = 0.55R/v$. Using this model, determine the age of the nebula.

(iii) If the expansion is purely driven by the direct impact of the stellar winds, then the radius at time t, $R(t)$, can be related to $\dot{M}$ with the formula

$$R(t) = \left(\frac{3\dot{M} v_\infty t^2}{2\pi n_0 m_\mathrm{H}} \right)^{1/4}$$

where n_0 is the ambient hydrogen number density in the nebula. If $n_0 = 16$ cm^{-3} and $m_\mathrm{H} = 1.67 \times 10^{-27}$ kg, calculate the observed mass loss rate based upon the properties of the nebula. Compare it with the predicted one from earlier and comment on your answer.

(iv) Using your new value for $\dot{M}$, calculate the total mass expelled from the star and hence the total kinetic energy the stellar wind has so far deposited into the ISM during this stage of the star's life.

Qu 2. Multi-Messenger Astronomy

GW170817 was the first gravitational wave event arising from a binary neutron star merger to have been detected by the LIGO & Virgo experiments, and careful localization of the source meant that the electromagnetic counterpart was quickly found in galaxy NGC 4993 (see Figure 2). Such a combination of two completely separate branches of astronomical observation begins a new era of 'multi-messenger astronomy'. Since the gravitational waves allow an independent measurement of the distance to the host galaxy and the light allows an independent measurement of the recessional speed, this observation allows us to determine a new, independent value of the Hubble constant H_0.

a. The galaxy NGC 4993 is measured to have a redshift of $z = 0.00980 \pm 0.00079$. Assuming it follows Hubble's Law, $v = H_0 d$, where $H_0 = 73.24 \pm 1.74$ km s^{-1} Mpc^{-1}, as determined by Hubble Space Telescope (HST) measurements of Cepheid variables, calculate the distance to the galaxy (in Mpc) and its (absolute) uncertainty. Give your distance to an appropriate number of significant figures.

Another way of measuring distances to galaxies is to use the Fundamental Plane (FP) relation, which relies on the assumption there is a fairly tight relation between radius, surface brightness, and velocity dispersion for bulge-dominated galaxies, and is widely used for galaxies like NGC 4993. It can be described by the relation

$$\log\left(\frac{D}{(1+z)^2}\right) = -\log R_e + \alpha \log \sigma - \beta \log \langle I_r \rangle_e + \gamma$$

where D is the distance in Mpc, R_e is the effective radius measured in arcseconds, σ is the velocity dispersion in $\mathrm{km\,s^{-1}}$, $\langle I_r \rangle_e$ is the mean intensity inside the effective radius measured in $L_\odot$ pc^{-2}, and γ is the distance-dependent zero point of the relation. Calibrating the zero point to the Leo I galaxy group, the constants in the FP relation become $\alpha = 1.24$, $\beta = 0.82$, and $\gamma = 2.194$.

b. For NGC 4993 we measure $R_e = 15.5$ arcseconds, $\sigma = 171\ \mathrm{km\,s^{-1}}$, and $\langle I_r \rangle_e = 407\, L_\odot$ pc^{-2}. Given that the scatter in the FP relation introduces an uncertainty in D of $\pm 17\%$, calculate the distance to the galaxy (in Mpc) and its (absolute) uncertainty using the FP relation.

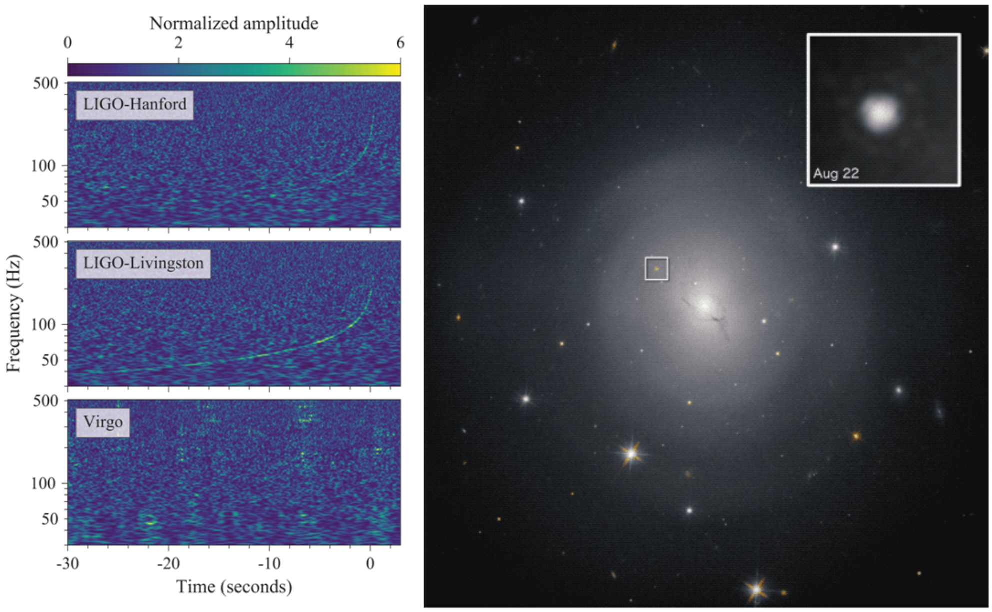

Figure 2: *Left*: The GW170817 signal as measured by the LIGO and Virgo gravitational wave detectors, taken from Abbott *et al.* (2017). The normalized amplitude (or strain) is in units of 10^{-21}. The signal is not visible in the Virgo data due to the direction of the source with respect to the detector's antenna pattern. *Right*: The optical counterpart of GW170817 in host galaxy NGC 4993, taken from Hjorth *et al.* (2017).

By measuring the amplitude (called strain, h) and the frequency of the gravitational waves, $f_{\rm GW}$, one can determine the distance to the source without having to rely on 'standard candles' like Cepheid variables or Type Ia supernovae. For two masses, m_1 and m_2, orbiting the centre of mass with separation a with orbital angular velocity ω, then the dimensionless strain parameter h is

$$h \simeq \frac{G}{c^4}\frac{1}{r}\mu a^2 \omega^2$$

where r is the luminosity distance, c is the speed of light, $\mu = m_1 m_2 / M_{\rm tot}$ is the reduced mass and $M_{\rm tot} = m_1 + m_2$ is the total mass.

c. Given that the gravitational frequency, $f_{\rm GW}$, is twice the orbital frequency (i.e. $f_{\rm GW} = \omega/\pi$) and the 'chirp mass', $\mathcal{M} = (\mu^3 M_{\rm tot}^2)^{1/5}$, express h in terms of only $\mathcal{M}$, r, $f_{\rm GW}$, and various fundamental constants.

The rate of change of frequency of the gravitational waves (called the 'chirp') from a merging binary can be written as

$$\dot{f}_{\rm GW} = \frac{96}{5}\pi^{8/3}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3} f_{\rm GW}^{11/3}$$

d. Combine your result from c. with the above equation to cancel out $\mathcal{M}$ and so express the distance to the gravitational wave source, r, as a function of fundamental constants and the measurables h, $\dot{f}_{\rm GW}$, and $f_{\rm GW}$ only.

e. Typically, you measure $\tau \equiv f_{\rm GW}/\dot{f}_{\rm GW}$, rather than $\dot{f}_{\rm GW}$ directly. Given that just as the merger began the detectors measured $\tau = 0.0023$ s, $f_{\rm GW} = 300$ Hz and $h = 6.0 \times 10^{-21}$, estimate the distance to GW170817 (in Mpc) and its absolute uncertainty (assuming a percentage uncertainty of $\pm 10\%$). How does this compare with your answers in parts a. and b.?

f. Using the redshift information from NGC 4993 and the gravitational wave distance you have just calculated, determine the Hubble constant H_0 in units of km s^{-1} Mpc^{-1}, along with its absolute uncertainty. Is this value consistent with the one derived by the HST using Cepheid variables (given in part a.)?

Qu 3. Great American Eclipse

On 21st August 2017 the continental United States experienced a total solar eclipse. Dubbed the 'Great American Eclipse', it was estimated to be one of the most watched eclipses in history.

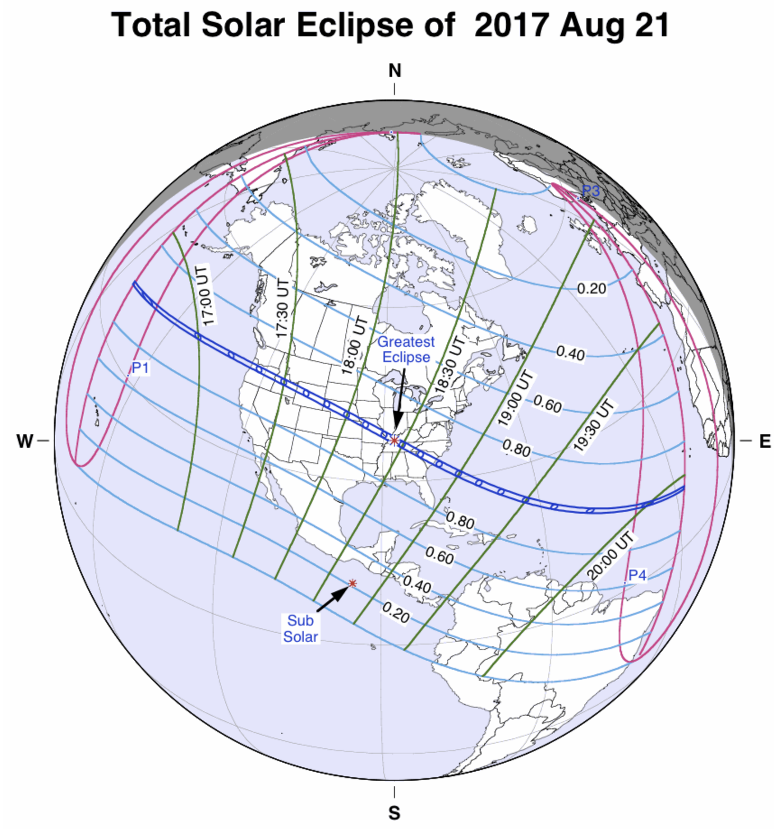

Figure 3: The path of totality for the Great American Eclipse. The narrow dimension is its width. Credit: Fred Espenak, NASA's GSFC.

The path of totality (where the Moon completely obscures the Sun) is shown in Figure 3, and the point of greatest eclipse ("GE"; where the path was widest since the axis of the cone of the Moon's shadow passed closest to the centre of the Earth) was near the village of Cerulean, Kentucky. The following data can be used for this question:

- The angular radii of the Sun and the Moon (if observed from the centre of the Earth) at the moment of GE are $15'48.7''$ and $16'03.4''$, respectively, where the notation $xx'yy.y''$ corresponds to xx arcminutes and $yy.y$ arcseconds (60 arcminutes = 1 degree, and 60 arcseconds = 1 arcminute)
- The latitude and longitude of the location of GE are $36°58.0'$N and $87°40.3'$W, respectively
- Take the mean radii of the Sun, Earth and Moon to be respectively $R_\odot = 695\,700$ km, $R_\oplus = 6371$ km and $R_{\text{Moon}} = 1737$ km, and a day to be 24 hours
- Take the semi-major axes of the Sun-Earth and Earth-Moon systems to be $149\,600\,000$ km and $384\,400$ km, respectively
- As viewed from a location far above the North Pole, the Moon orbits in an anticlockwise direction around the Earth, and the Earth spins in an anticlockwise direction

For an ellipse with semi-major axis a it can be shown that the velocity v, at a distance r from mass M, can be written as:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

a. Calculate the width of the path of totality at GE (in km). You may use the approximation that the Moon is directly overhead so that the shadow is circular, and small enough that the curvature of the Earth can be neglected.

b. Show that at this point the shadow is moving across the Earth's surface at approximately $0.68\ \mathrm{km\,s^{-1}}$. You may use the simplifying approximation that for short time intervals it can be considered as travelling with a constant latitude, and that the apparent movement of the Sun due to the Earth's orbit can be neglected.

c. Hence calculate the duration of the eclipse. Give your answer to the nearest 0.1 s.

The point of greatest eclipse and greatest duration do not generally coincide, as a more elliptical shadow with a major axis aligned with the path of maximum totality (and thinner path width, equal to the minor axis) can compensate for the shadow moving faster at higher latitudes. For this eclipse the point of greatest duration ("GD") was at co-ordinates of $37°35'$N latitude and $89°07'$W longitude, reached about 4 minutes before GE, and where totality lasted 0.1 s longer than the value calculated in part c.

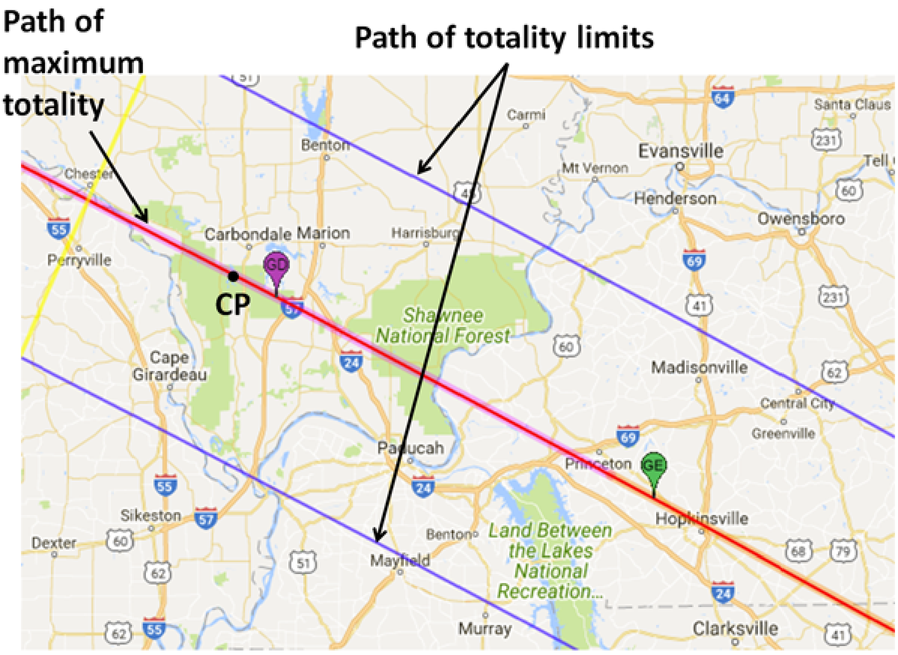

Figure 4: The route of the Moon's shadow in the vicinity of the points of greatest duration (GD, near Carbondale) and greatest eclipse (GE, near Hopkinsville). Any places between the two limits on the path of totality will experience at least a very short period of totality - outside that region will only be a partial eclipse - and the perpendicular distance between them is the path width. The longest duration of totality at that point of the shadow's journey is indicated as the path of maximum totality, which both GE and GD sit on. The closest part of that path to Carbondale is indicated as CP.
Credit: Fred Espenak & Google Maps.

d. The town of Carbondale, Illinois, is the closest big town to the point of GD, with co-ordinates $37°44'$N latitude and $89°13'$W longitude. Assuming the path of maximum totality can be treated as linear as it passes through the region around GD and GE:

(i) Calculate the co-ordinates of the closest point ("CP") to Carbondale on the path of maximum totality.

(ii) Calculate the distance (in km) between Carbondale and CP.

(iii) Hence calculate (to the nearest 0.1 s) how much shorter totality was for residents in Carbondale compared with CP. Take the duration at CP to be the same as GD, the width of the path to be 0.5 km less than for GE (so the Moon's shadow is elliptical), and the speed of the Moon's shadow to have only been affected by the change in latitude.

Qu 4. Super-Earths and Planet Nine

Recent years have seen an explosion in the discovery of new exoplanets. About 85% of transiting exoplanets discovered by the NASA Kepler telescope have radii less than Neptune ($\sim 4\,R_\oplus$), meaning we are improving our understanding of what the transition between rocky Earth-size planets and gaseous Neptune-size planets looks like.

Given how common these "super-Earths" and "gas dwarfs" seem to be, it was odd that we didn't have any in our own Solar System. However, Batygin & Brown (2016) suggested that a hypothetical ninth planet (called 'Planet Nine') could explain some of the unusual properties of the orbits of objects in the Kuiper Belt. This planet is inferred to have a mass of $10\,M_\oplus$, and so would be an example of a super-Earth.

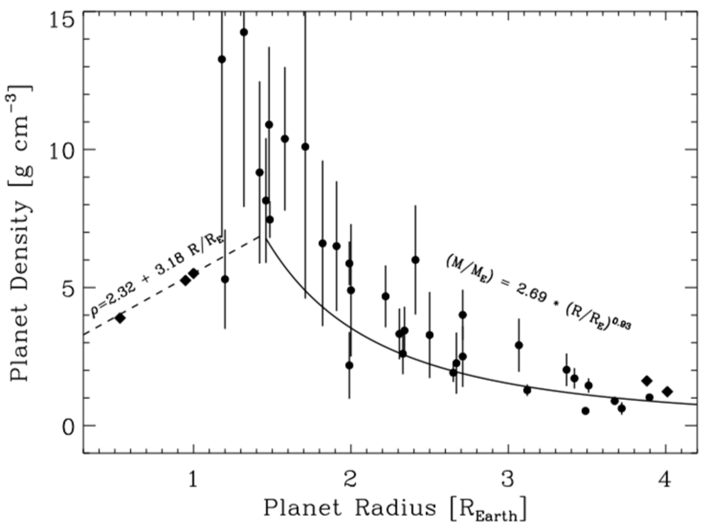

Figure 5: A plot of planet density versus radius for 33 extrasolar planets (circles) and the planets in our solar system (diamonds).
Credit: Marcy *et al.* (2014).

Analysing exoplanets discovered by Kepler, Marcy *et al.* (2014) used a piecewise function to describe their planetary density data such that:

For $R_\mathrm{P} \le 1.5\,R_\oplus$ $\qquad$ $$\rho = 2.32 + 3.18\frac{R_\mathrm{P}}{R_\oplus}\ [\mathrm{g\ cm^{-3}}]$$

For $1.5\,R_\oplus < R_\mathrm{P} \le 4.2\,R_\oplus$ $\qquad$ $$\frac{M_\mathrm{P}}{M_\oplus} = 2.69\left(\frac{R_\mathrm{P}}{R_\oplus}\right)^{0.93}$$

where R_P is the radius of the planet, M_P is the mass of the planet, and the model's transition between rocky super-Earth and non-rocky gas dwarf occurs at $R_\mathrm{P} = 1.5\,R_\oplus$.

a. Based on the Marcy *et al.* (2014) model, Planet Nine is most likely to be a gas dwarf with a thick gaseous envelope. Calculate R_P (in units of $R_\oplus$) for Planet Nine using this model.

b. Planetary formation models suggest such a gas dwarf would have a solid rocky core the size of the Earth, with similar composition and density. Calculate a simple estimate of the atmospheric pressure on the rocky surface. Compare your answer to the atmospheric pressure at sea level on the Earth, $p_\oplus = 100$ kPa.

c. In practice the transition between rocky and gaseous planets is not likely to be sharply at 1.5 $R_\oplus$, with some planets of both types existing above and below the limit. Calculate R_P (in units of $R_\oplus$) for Planet Nine if it was a rocky super-Earth. How does its average density compare to an Earth-sized rocky exoplanet?

The minimum speed necessary to fully escape a planet's gravity (rather than be put into an elliptical orbit) is called the escape velocity and is calculated as

$$v_{\rm esc} = \sqrt{\frac{2GM_{\rm P}}{R_{\rm P}}}$$

where G is the universal gravitational constant.

In contrast, the maximum speed a rocket can provide is determined by the ejection velocity, $v_{\rm e}$, of the gas used, as determined by the chemical energy stored in the bonds of the fuel used, and the fraction of the rocket that is fuel. Since the rocket gets lighter as it burns its fuel the final speed can be bigger than $v_{\rm e}$. The rocket equation is

$$v_{\rm max} = v_{\rm e} \ln \frac{m_0}{m_1}$$

where m_0 is the mass at launch and m_1 is the mass once all the fuel has been burnt. The most energetic chemical reaction we can use in a rocket is hydrogen-oxygen, which gives $v_{\rm e} = 4.46\ {\rm km\,s^{-1}}$, and engineering limits us to a rocket design with a maximum of 96% of launch mass being fuel (as was used with the solid rockets that launched the space shuttle).

d. Verify that the above rocket design is sufficient to escape from Earth but **not** sufficient to escape from the surface of a rocky Planet Nine.

e. Calculate the maximum value of $R_{\rm P}$ (for a rocky exoplanet) above which any alien civilization would be unable to escape their planet's gravity using simple chemical rocket propulsion systems. (They could of course still have orbital satellites, since the speeds for planetary orbit are lower.)

f. In the Marcy *et al.* (2014) model, above $R_{\rm P} = 1.5\,R_\oplus$ the density of gas dwarfs rapidly decreases with radius. By looking at the piecewise function explain why this does **not** improve the situation for any alien civilization living on such a gas dwarf hoping to explore their solar system. You do not need to calculate any new escape velocities.

END OF PAPER

Questions proposed by:
Dr Alex Calverley (Royal Grammar School, Guildford)
Dr Emile Doran (The Langley Academy)
Sandor Kruk (University of Oxford)