British Astronomy and Astrophysics Olympiad 2016-2017

Astronomy & Astrophysics Competition Paper

Monday 23rd January 2017

Instructions

Time: 3 hours plus 15 minutes reading time (no writing permitted). Approx 35 minutes per question.

Questions: All five questions should be attempted.

Marks: The questions carry similar marks.

Solutions: Answers and calculations are to be written on loose paper or in examination booklets. Students should ensure their name and school is clearly written on all answer sheets and pages are numbered. A standard formula booklet with standard physical constants should be supplied.

Instructions: To accommodate students sitting the paper at different times, please **do not discuss** any aspect of the paper on the internet until 8 am Saturday 28th January.

Clarity: Solutions must be written legibly, in black pen (the papers are photocopied), and working down the page. Scribble will not be marked and overall clarity is an important aspect of this exam paper.

Eligibility: The International Olympiad will be held during November 2017; all sixth form students are eligible to participate, even if they will be attending university in November.

Training Dates and the International Astronomy and Astrophysics Olympiad (IOAA)

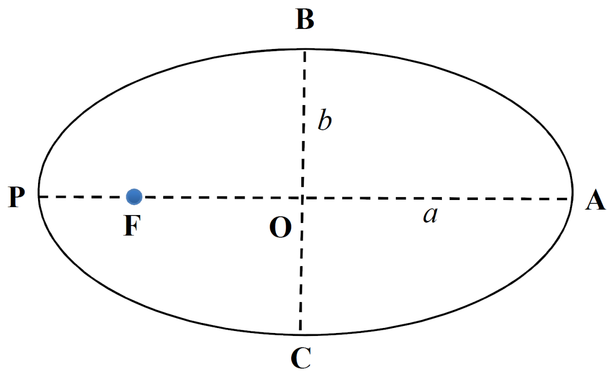
The IOAA this year will be held in Phuket, Thailand, from 12th to 21st November 2017.

The team will be selected from sixth form students taking this paper **and** *Y12 students taking the AS Challenge in March. The best students eligible to represent the UK at the IOAA will be invited to attend the* **Training Camp** *to be held in the Physics Department at the University of Oxford,* **(Tuesday 4th April to Friday 7th April 2017)**. *Astronomy material will be covered; problem solving skills and observational skills (telescope and naked eye observations) will be developed. At the Training Camp a data analysis exam and a short theory paper will be sat. Five students (plus one reserve) will be selected for further training. From May there will be mentoring by email to cover some topics and problems, followed by a training camp in the summer and also one in the autumn.*

Important Constants

Constant	Symbol	Value
Speed of light	c	$3.00 \times 10^8\,\mathrm{m\,s^{-1}}$
Earth's rotation period	1 day	24 hours
Earth's orbital period	1 year	365.25 days
parsec	pc	$3.09 \times 10^{16}\,\mathrm{m}$
Astronomical Unit	AU	$1.49 \times 10^{11}\,\mathrm{m}$
Radius of the Earth	$R_\oplus$	$6.37 \times 10^6\,\mathrm{m}$
Semi-major axis of the Earth's orbit		1 AU
Radius of the Sun	$R_\odot$	$6.96 \times 10^8\,\mathrm{m}$
Mass of the Sun	$M_\odot$	$1.99 \times 10^{30}\,\mathrm{kg}$
Mass of the Earth	$M_\oplus$	$5.97 \times 10^{24}\,\mathrm{kg}$
Luminosity of the Sun	$L_\odot$	$3.85 \times 10^{26}\,\mathrm{W}$
Gravitational constant	G	$6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$

You might find the diagram of an elliptical orbit below useful in solving some of the questions:



Elements of an elliptic orbit: $a = \mathbf{OA}\ (= \mathbf{OP})$ semi-major axis
$a = \mathbf{OB}\ (= \mathbf{OC})$ semi-minor axis
$e = \sqrt{1 - \frac{b^2}{a^2}}$ eccentricity
F focus
P periapsis (point nearest to F)
A apoapsis (point furthest from F)

Keplers Third Law: For an elliptical orbit, the square of the period, T, of orbit of an object about the focus is proportional to the cube of the semi-major axis, a (the average of the minimum and maximum distances from the Sun). The constant of proportionality is $4\pi^2/GM$, where M is the mass of the central object and G is the universal gravitational constant.

Magnitudes: The apparent magnitudes of two objects, m_1 and m_0, are related to their apparent brightnesses, b_1 and b_0, via the formula:

$$\frac{b_1}{b_0} = 10^{-0.4(m_1 - m_0)}$$

British Physics Olympiad Sponsors

Qu 1. Martian GPS

On Earth the Global Positioning System (GPS) requires a minimum of 24 satellites in orbit at any one time (there are typically more than that to allow for redundancies, with the current constellation having more than 30) so that at least 4 are visible above the horizon from anywhere on Earth (necessary for an x, y, z and time co-ordinate). This is achieved by having 6 different orbital planes, separated by 60°, and each orbital plane has 4 satellites.

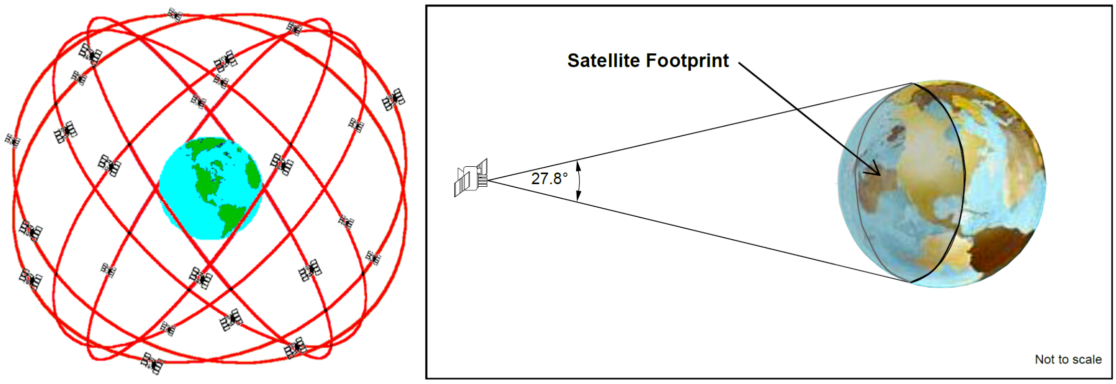


Figure 1: The current set up of the GPS system used on Earth.
Credits: *Left*: Peter H. Dana, University of Colorado;
Right: GPS Standard Positioning Service Specification, 4$^{\text{th}}$ edition

The orbits are essentially circular with an eccentricity < 0.02, an inclination of 55°, and an orbital period of exactly half a sidereal day (called a semi-synchronous orbit). The receiving angle of each satellite's antenna needs to be about 27.8°, and hence about 38% of the Earth's surface is within each satellite's footprint (see Figure 1), allowing the excellent coverage required.

a. Given that the Earth's sidereal day is 23h 56 mins, calculate the orbital radius of a GPS satellite. Express your answer in units of $\mathrm{R}_\oplus$.

b. How long would it take a radio signal to travel directly between a satellite and its closest neighbour in its orbital plane (assuming they're evenly spaced)? How far would a car on a motorway (with a speed of $30\,\mathrm{m\,s^{-1}}$) travel in that time? [This can be taken to be a very crude estimate of the positional accuracy of the system for that car.]

In the future we hope to colonise Mars, and so for navigation purposes it is likely that a type of GPS system will eventually be established on Mars too. Mars has a mass of $6.42 \times 10^{23}\,\mathrm{kg}$, a mean radius of $3390\,\mathrm{km}$, a sidereal day of 24h 37 mins, and two (low mass) moons with essentially circular orbits and semi-major axes of $9377\,\mathrm{km}$ (Phobos) and $23\,460\,\mathrm{km}$ (Deimos).

c. Using suitable calculations, explore the viability of a 24-satellite GPS constellation similar to the one used on Earth, in a semi-synchronous Martian orbit, by considering:

(i) Would the moons prevent such an orbit?

(ii) How would the GPS positional accuracy compare to Earth?

(iii) What would the receiving angle of each satellite's antenna need to be, and what would be the associated satellite footprint? By comparing these with the ones utilised by Earth's GPS, make a final comment on the viability of future Martian GPS.

Qu 2. Hohmann Transfer Orbits

In order to move a spacecraft between orbits we must apply a thrust using rockets, which changes the velocity of the spacecraft by Δv. In this question we will ignore changes in the mass of the spacecraft due to the burning of fuel.

For an object of mass m in a circular orbit of radius r around an object with mass M (where $m \ll M$) the orbital velocity, v_{orb}, is given by the formula $v_{\text{orb}} = \sqrt{\frac{GM}{r}}$.

a. Show that v_{orb} in low Earth orbit (LEO; about $200\,\text{km}$ above the surface), is about $8\,\text{km}\,\text{s}^{-1}$. This is an estimate of the Δv the rockets need to provide for the spacecraft to reach LEO.

An economical route to take when travelling between planets is called a Hohmann transfer orbit. This is an ellipse for which the perihelion coincides with the inner planetary orbit (with radius r_{A}) and the aphelion coincides with the outer planetary orbit (with radius r_{B}). It is achieved by increasing the velocity of the spacecraft at point A by Δv_{A} before then increasing it again at point B by Δv_{B}.

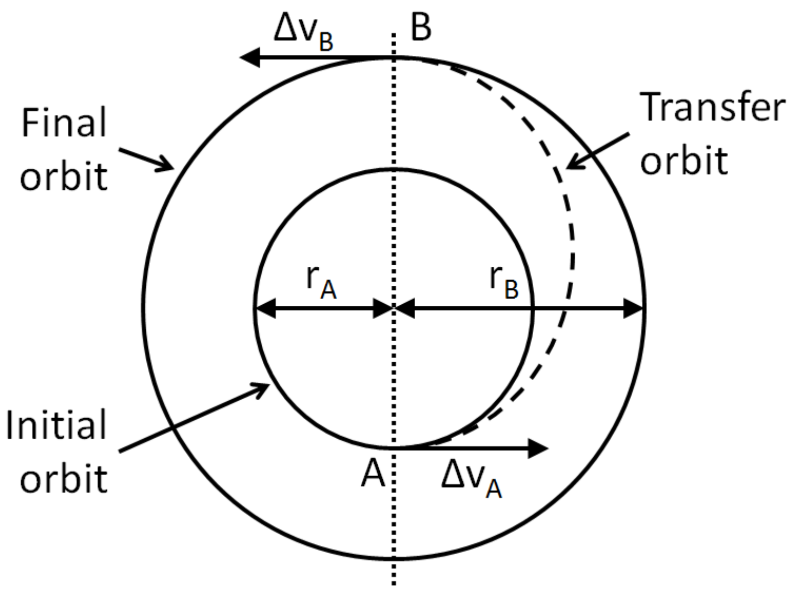


Figure 2: A diagram of a Hohmann transfer orbit between an inner and outer planet

For an ellipse with semi-major axis a it can be shown that the velocity v, at a distance r from mass M, can be written as:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

b. Derive expressions for Δv_{A} and Δv_{B} by comparing their circular orbital speeds with their transfer orbit speeds. Simplify your final expressions to include G, $\text{M}_{\odot}$, r_{A} and r_{B} only.

c. Approximating Mars' orbit as circular with a radius of 1.52 AU, calculate the Δv to go from Earth LEO to Mars i.e. $\Delta v = |\Delta v_{\text{A}}| + |\Delta v_{\text{B}}|$. Compare your answer to the Δv to reach Earth LEO.

d. Derive an expression for the total time spent on the transfer orbit, t_{H}, and calculate it for an Earth to Mars transfer. Give your answer in months. (Use 1 month = 30 days.)

e. Hence calculate the direct distance between Earth and Mars at the moment the spacecraft reaches Mars. How long would it take a radio message from the spacecraft to reach Earth?

f. How long would any astronauts on board the spacecraft need to wait until they could use a Hohmann transfer orbit to return to Earth? Hence calculate the total duration of the mission.

Qu 3. Starkiller Base

As part of their plan to rule the galaxy the First Order has created the Starkiller Base. Built within an ice planet and with a superweapon capable of destroying entire star systems, it is charged using the power of stars. The Starkiller Base has moved into the solar system and seeks to use the Sun to power its weapon to destroy the Earth.

Figure 3: The Starkiller Base charging its superweapon by draining energy from the local star. Credit: Star Wars: The Force Awakens, Lucasfilm.

For this question you will need that the gravitational binding energy, U, of a uniform density spherical object with mass M and radius R is given by

$$U = \frac{3GM^2}{5R}$$

and that the mass-luminosity relation of low-mass main sequence stars is given by $L \propto M^4$.

a. Assume the Sun was initially made of pure hydrogen, carries out nuclear fusion at a constant rate and will continue to do so until the hydrogen in its core is used up. If the mass of the core is 10% of the mass of the star, and 0.7% of the mass in each fusion reaction is converted into energy, show that the Sun's lifespan on the main sequence is approximately 10 billion years.

b. The Starkiller Base is able to stop nuclear fusion in the Sun's core.

   (i) At its current luminosity, how long would it take the Sun to radiate away all of its gravitational binding energy? (This is an estimate of how long it would take to drain a whole star when radiatively charging the superweapon.)

   (ii) How does your value compare to the main sequence lifetime of the Sun calculated in part a.?

   (iii) Comment on whether there were (or will be) any events in the life of the Sun with a timescale of this order of magnitude.

c. In practice, the gravitational binding energy of the Earth is much lower than that of the Sun, and so the First Order would not need to drain the whole star to get enough energy to destroy the Earth. Assuming the weapon is able to channel towards it all the energy being radiated from the Sun's entire surface, how long would it take them to charge the superweapon sufficiently to do this?

The First Order find that radiative charging of the weapon is too slow to satisfy their plans for galactic domination, and so instead the weapon charging process compresses and stores part of the star within the Starkiller Base (as shown in Figure 3). To avoid creating a black hole, the First Order cannot compress stellar matter below its Schwarzschild radius, $R_{\mathrm{S}} = 2GM/c^2$.

d. Taking the Starkiller Base's ice planet to have a diameter of 660 km, show that the Sun can be safely contained, even if it was fully drained.

e. The Starkiller Base wants to destroy all the planets in a stellar system on the far side of the galaxy and so drains $0.10\,\mathrm{M}_\odot$ from the Sun to charge its weapon. Assuming that the U per unit volume of the Sun stays approximately constant during this process, calculate:

   (i) The new luminosity of the Sun.

   (ii) The new radius of the Sun.

   (iii) The new temperature of the surface of the Sun (current $\mathrm{T}_\odot = 5780$ K), and suggest (with a suitable calculation) what change will be seen in terms of its colour.

The Resistance defeat the First Order and destroy the Starkiller Base when it was almost fully charged. Upon releasing the energy stored in the base it causes the planet to turn into a small star.

Figure 4: The Resistance fighters escaping the Starkiller Base as it turns into a star.
Credit: Star Wars: The Force Awakens, Lucasfilm.

f. Assume that at the moment of destruction of the Starkiller Base the mass of the new star formed is equal to the mass drained from the Sun ($0.10\,\mathrm{M}_\odot$). Derive an expression for the main sequence lifetime in terms of stellar mass, and hence calculate the main sequence lifetime of this new star.

Qu 4. Hanny's Voorwerp

Hanny's Voorwerp (Dutch for 'object') is a rare type of astronomical object discovered in 2007 by the school teacher Hanny van Arkel whilst participating as a volunteer in the Galaxy Zoo project. When inspecting the image of the galaxy IC 2497 in the constellation Leo Minor, she observed a bright green blob close to the galaxy.

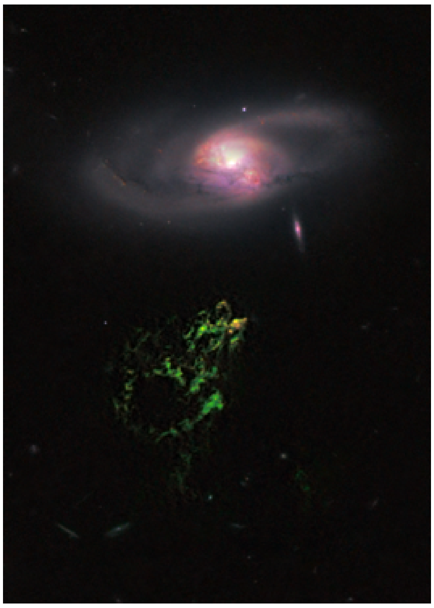


Figure 5: HST image of galaxy IC 2497 and the glowing Voorwerp below it.
Credit: Keel *et al.* (2012) & Galaxy Zoo.

Subsequent observations have shown that the galaxy IC 2497 is at a redshift of $z = 0.05$, with the Voorwerp at a similar distance and with a projected angular separation of 20 arcseconds from the centre of the galaxy (3600 arcseconds = $1°$). Radio observations suggest that the Voorwerp is a massive cloud of gas, made of ionized hydrogen, with a size of 10 kpc and a mass of $10^{11}\ \mathrm{M}_\odot$. It is probably a cloud of gas that was stripped from the galaxy during a merger with another nearby galaxy.

In this question you will explore the cause of the 'glow' of the Voorwerp and will learn about a new type of an astronomical object; a quasar.

a. Given that Hubble's constant is measured as $H_0 = 70$ km s^{-1} Mpc^{-1}, calculate the distance to the galaxy (in Mpc).

The rate of ionizing photons from a source (in photons per second) can be expressed as:

$$S_* = Vn^2\alpha$$

where V is the volume of the ionized region, n is the number density of the ionized gas and α is the ionization coefficient, $\alpha = 2.6 \times 10^{-13}$ cm^3 s^{-1}.

b. Calculate the power (luminosity) of the source required to completely ionize the Voorwerp (assumed to be spherical), given that the mass of a hydrogen atom is 1.67×10^{-27} kg and the ionization energy of hydrogen is 13.6 eV, where 1 eV = 1.60×10^{-19} J.

One possible source of ionizing radiation is the jet arising from the accretion of material onto the supermassive black hole (SMBH) situated in the centre of the galaxy. This produces an enormous amount of energy, greatly brightening the galaxy; a galaxy shining due to this process is known as a quasar.

c. The gravitational potential energy of the material falling to radius R, which in this case is a black hole with radius equal to the Schwarzschild radius, $R_{\mathrm{S}} = 2GM/c^2$, at a mass accretion rate $\dot{m} \equiv \delta m/\delta t$, is converted into radiation with an efficiency of η. Show that the power (luminosity) output of the SMBH is given by:
$$L = \frac{1}{2}\eta\dot{m}c^2 \ .$$

d. The typical mass accretion rate onto an active SMBH is $\sim 2\,\mathrm{M}_\odot$ yr^{-1} and the typical efficiency is $\eta = 0.1$. Calculate the typical luminosity of a quasar. Compare the luminosity of the quasar with the power needed to ionize the Voorwerp.

Detailed astronomical observations have shown than the nucleus of the galaxy has a modest luminosity of $L < 10^{33}$ W, thus the black hole in IC 2497 is not currently active (i.e. the accretion rate is very low). Quasars are thought to ignite every time the black hole starts accreting a fresh source of matter, and switch off once that supply is exhausted. Therefore, this might be the first evidence of a quasar switching off recently (by astronomical standards), with the Voorwerp reflecting the light emitted by the quasar whilst it was still active. This would make the Voorwerp a 'quasar ionization echo' and IC 2497 the nearest galaxy to us to host a quasar.

e. Calculate the projected physical separation, r_{p}, between the galaxy and the Voorwerp.

f. Derive an expression for the difference in the light travel time between photons travelling directly to Earth from the galaxy and photons reflected off the Voorwerp first. Give your formula as a function of r_{p} and θ, where θ is the angle between the lines of sight to the Earth and to the centre of the Voorwerp as measured by an observer at the centre of IC 2497. (For example $\theta = 90°$ would correspond to the galaxy and Voorwerp both being the exact same distance from the Earth, and so the projected distance r_{p} is therefore also the true distance between them.)

g. High precision measurements showed that the Voorwerp is slightly further away than the galaxy, and so $\theta = 125°$. Use this with your expression from the previous part of the question to estimate an upper limit for the number of years that have passed since the quasar was last active.

Qu 5. Imaging an Exoplanet

Recently a group of researchers announced that they had discovered an Earth-sized exoplanet around our nearest star, Proxima Centauri. Its closeness raises an intriguing possibility about whether or not we might be able to image it directly using telescopes. The difficulty comes from the small angular scales that need to be resolved and the extreme differences in brightness between the reflected light from the planet and the light given out by the star.

Figure 6: Artist's impression of the view from the surface of Proxima Centauri b.
Credit: ESO / M. Kornmesser

Data about the star and the planet are summarised below:

Proxima Centauri (star)		**Proxima Centauri b (planet)**	
Distance	$1.295\,\mathrm{pc}$	Orbital period	11.186 days
Mass	$0.123\,\mathrm{M}_\odot$	Mass (min)	$\approx 1.27\,\mathrm{M}_\oplus$
Radius	$0.141\,\mathrm{R}_\odot$	Radius (min)	$\approx 1.1\,\mathrm{R}_\oplus$
Surface temperature	3042 K		
Apparent magnitude	11.13		

The following formulae may also be helpful:

$$m - \mathcal{M} = 5\log\left(\frac{d}{10}\right) \qquad \mathcal{M} - \mathcal{M}_\odot = -2.5\log\left(\frac{L}{\mathrm{L}_\odot}\right) \qquad \Delta m = 2.5\log CR$$

where m is the apparent magnitude, $\mathcal{M}$ is the absolute magnitude, d is the distance in parsecs, and the contrast ratio (CR) is defined as the ratio of fluxes from the star and planet, $CR = \frac{f_{\mathrm{star}}}{f_{\mathrm{planet}}}$.

a. Calculate the maximum angular separation between the star and the planet, assuming a circular orbit. Give your answer in arcseconds (3600 arcseconds = 1°).

b. Determine the luminosity of the star and hence calculate the flux received on the Earth (in W m^{-2}) from both the star and the planet. Use them to work out the contrast ratio and thus the apparent magnitude of the planet. Assume the planet reflects half of the incident light and that $\mathcal{M}_\odot = 4.83$.

The resolving power of a diffraction limited telescope is given by

$$\theta_{\text{min}} = 1.22\frac{\lambda}{D}$$

where λ is the wavelength being observed at, D is the diameter of the telescope aperture, and θ_{min} is the smallest angular separation (in radians) the telescope can distinguish.

Data about some current and planned telescopes are summarised below:

Telescope	Diameter (m)	Faintest m detectable
Hubble Space Telescope (HST)	2.4	31
Keck II (based in Hawaii)	10.0	(variable)
James Webb Space Telescope (JWST)	6.5	34

c. Verify that the HST (which is diffraction limited since it's in space) would be sensitive enough to image the planet in the visible, but is unable to resolve it from its host star (take $\lambda = 550$ nm).

Ground-based telescopes have bigger mirrors than the HST, but are not diffraction limited due to movements in the atmosphere and so need to be fitted with 'adaptive optics' (AO) to compensate for this effect. However, even with perfect AO the faintest object the telescope can detect is limited by the brightness of the atmosphere.

The signal-to-noise ratio (SNR) can be approximated as:

$$SNR \approx \frac{fA\epsilon t}{\sqrt{fA\epsilon t + b\epsilon t}}$$

where f is the flux from the object (in photons m^{-2} s^{-1}), A is the area of the telescope mirror, ϵ is the overall efficiency of the telescope and detector, b is the flux from the sky (in photons s^{-1}), and t is the length of the exposure.

d. Calculate the exposure time needed for a Keck II image of the exoplanet to have an SNR of 3 (i.e. barely detectable). Assume that the telescope has perfect AO (so it is diffraction limited), is observed at the longest wavelength for which the planet can still be resolved from the star, all the received flux from the planet consists of photons of that longest wavelength, $\epsilon = 0.1$ and $b = 10^9$ photons s^{-1} (so $b \gg f$). Comment on your answer.

The James Webb Space Telescope (JWST) is the successor to the HST and is due to launch in 2018. It should be able to both resolve the system and cope with the contrast ratio. Since it is in space it is diffraction limited, and the SNR should be dominated by the flux from the planet (i.e. $f \gg b$).

e. How long an exposure would JWST need in order to get the same SNR as Keck II, again if observed at the longest wavelength for which the planet can still be resolved from the star by the telescope? (Make similar assumptions about the received flux and use the same value of ϵ.)

END OF PAPER

Questions proposed by:
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