Asteroid Belt (solutions)

a. *Given that the total mass of the asteroid belt is approximately $M_{belt} = 1.8 \times 10^{-9}\,M_\odot$, calculate the radius of the object that could be formed, assuming it has a density typical of rock ($\rho = 3.0$ g cm^{-3}). Compare this to the radius of the largest member of the asteroid belt, Ceres. ($R_{Ceres} = 473$ km)*

$M_{\text{belt}} = 1.8 \times 10^{-9}\,M_\odot = 3.6 \times 10^{21}$ kg

density = 3.0 g cm^{-3} = 3.0×10^{3} kg m^{-3}

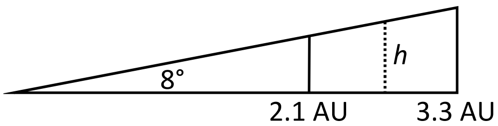
$$\frac{4}{3}\pi r^3 = \frac{M_{\text{belt}}}{\rho} \quad \therefore\ r = \sqrt[3]{\frac{3M_{\text{belt}}}{4\pi\rho}} = \sqrt[3]{\frac{3\times3.6\times10^{21}}{4\pi\times3.0\times10^{3}}} \qquad [1]$$

$$= 660\ \text{km} \qquad [1]$$

$$= 1.4\ R_{\text{Ceres}} \qquad [1]$$

[This means Ceres contains a sizeable fraction of the material in the whole belt, although it has a lower density than we have assumed, so less mass than implied here]

b. *The main part of the asteroid belt extends from 2.1 AU to 3.3 AU, and has an average angular width of 16.0°, as viewed from the Sun. Calculate the average thickness of the belt, and hence its total volume, V_{belt}.*



Average height above the orbital plane, $h = \left(\frac{2.1+3.3}{2}\right)\tan 8 = 0.38\ \text{AU}$ [0.5]

So total thickness = $2h$ = 2×0.38

= 0.76 AU [0.5]

Volume of belt = area of disk × total thickness

$$\therefore V_{\text{belt}} = \pi(3.3^2 - 2.1^2) \times 0.76 \qquad [1]$$

$$= 15.45\ \text{AU}^3$$

$$= 5.1 \times 10^{34}\ \text{m}^3 \qquad [1]$$

$$(= 5.1 \times 10^{25}\ \text{km}^3)$$

[This is a huge volume, which explains why each asteroid gets so much space to itself]

c. *Assuming this volume is uniformly filled by spherical rocky asteroids of average radius R_{av}, derive a relationship between the average distance between asteroids, d_{av}, and their radius R_{av}, remembering to keep the total mass equal to M_{belt}.*

Assuming there are N asteroids, each filling a cube with side length d_{av} then $N = \frac{V_{\text{belt}}}{d_{\text{av}}^3}$ [1]

Conserving the volume of rock in the belt means $N\left(\frac{4}{3}\pi R_{\text{av}}^3\right) = \frac{M_{\text{belt}}}{\rho}$ [1]

so we can cancel N to give

$$\frac{V_{\text{belt}}}{d_{\text{av}}^3}\left(\frac{4}{3}\pi R_{\text{av}}^3\right) = \frac{M_{\text{belt}}}{\rho}$$

$$\therefore\ \frac{R_{\text{av}}}{d_{\text{av}}} = \sqrt[3]{\frac{3M_{\text{belt}}}{4\pi\rho V_{\text{belt}}}} \qquad [1]$$

(a spherical approximation for the volume of belt allocated to each asteroid can also gain full credit so long as it is clear that d_{av} in that case is equal to double the radius of the spherical volume used)

i.e. $N = \frac{V_{\text{belt}}}{\frac{4}{3}\pi\left(\frac{d_{\text{av}}}{2}\right)^3}$ leading to $\frac{R_{\text{av}}}{d_{\text{av}}} = \sqrt[3]{\frac{M_{\text{belt}}}{8\rho V_{\text{belt}}}}$

d. *If R_{av} = 2.0 km, calculate d_{av}. How does this compare to the Earth-Moon distance? ($d_{E\rightarrow M}$ = 384,000 km)*

$$d_{\text{av}} = \frac{R_{\text{av}}}{\sqrt[3]{\frac{3M_{\text{belt}}}{4\pi\rho V_{\text{belt}}}}} = \frac{2.0\times10^3}{\sqrt[3]{\frac{3\times3.6\times10^{21}}{4\pi\times3.0\times10^3\times5.1\times10^{34}}}}$$

$$= 1.1\times10^9\ \text{m}\ \ (=1.1\times10^6\ \text{km}) \qquad [1]$$

$$= 2.9\ d_{\text{E}\rightarrow\text{M}} \qquad [1]$$

(using a spherical approximation in the previous question yields d_{av} = 1.4 × 10^9 m = 3.6 $d_{\text{E}\rightarrow\text{M}}$)

[An average separation of about 1 million km is close to the real value for our asteroid belt and emphasises the vast space between asteroids, which is why many probes can travel through unharmed, although the real value of R_{av} is somewhat smaller – this is because the real asteroid belt contains far more small asteroids than big ones]

e. *Using the luminosity of the Sun, calculate the total power incident on an asteroid in the middle of the asteroid belt.*

Distance to middle of the belt, $r_{\text{mid}} = \frac{2.1+3.3}{2} = 2.7$ AU

Apparent brightness of the Sun in the middle of the belt:

$$b = \frac{L_\odot}{4\pi r_{\text{mid}}^2} = \frac{3.85\times10^{26}}{4\pi\times(2.7\times1.49\times10^{11})^2} \qquad [1]$$

$$= 190\ \text{W m}^{-2} \qquad [1]$$

Incident power, P_{i} = apparent brightness × cross-sectional area of an asteroid

$$\therefore\ P_{\text{i}} = b\pi R_{\text{av}}^2 = 190\times\pi(2.0\times10^3)^2 \qquad [1]$$

$$= 2.4\times10^9\ \text{W} \qquad [1]$$

f. Assuming only 30% of that is reflected by its rocky surface, calculate the apparent magnitude of the asteroid when viewed from its nearest neighbour. Given that objects with m > 6 are too faint for the naked eye, would it be visible to an astronaut stood on the asteroid surface?

Apparent brightness of one asteroid as viewed from another with only 30% reflectivity:

$$b = \frac{0.3P_i}{4\pi d_{av}^2} \qquad = \frac{0.3\times 2.4\times 10^9}{4\pi\times(1.1\times 10^9)^2} \qquad [1]$$

$$= 4.5\times 10^{-11}\ \text{W m}^{-2} \qquad [1]$$

$$\therefore m = -\frac{1}{0.4}\log\left(\frac{b}{b_0}\right) \qquad = -2.5\log\left(\frac{4.5\times 10^{-11}}{2.52\times 10^{-8}}\right) \qquad [1]$$

$$= 6.9 \qquad [1]$$

This means that the astronaut could **not** see the nearest asteroid with their naked eye [1]

(Using a spherical approximation gives $b = 2.9\times 10^{-11}$ W m^{-2} and $m = 7.3$, so the same conclusion)

[In practice even though asteroids come in a wide variety of sizes, and the spacing can vary quite far from our values of d_{av}, most asteroids only reflect about 10% of their light and so despite all the simplifying approximations we have made we still get the same result – that without binoculars you would be unable to tell you were flying through an asteroid belt, which is rather different to the picture painted by science fiction films!]

Figure 1: Image credit: John Gaughan / Pete Lardizabal / WJLA

Supermoons

The term"supermoon" was coined by astrologer Richard Nolle in 1979. He defined a supermoon as a new or full moon that occurs with the Moon at or near (within 90% of) its closest approach to Earth in a given orbit (perigee). The value of 90% was arbitrarily chosen, and other definitions are often used.

The media commonly associates supermoons with extreme brightness and size, sometimes implying that the Moon itself will become larger and have an impact on human behaviour, but just how different is a supermoon compared to the 'normal' Moon we see each month?

Lunar Data:

- Synodic Period = 29.530589 days - time between same phases *e.g. full moon to full moon, new moon to new moon.*
- Anomalistic Period = 27.554550 days - time between perigees *e.g. perigee to perigee.*
- Semimajor axis (a)= 3.844×10^5 km

- Orbit eccentricity (e)= 0.0549
- Radius of the Moon ($R_{☽}$)= 1738.1 km
- Mass of the Moon ($M_{☽}$)= 7.342×10^{22} kg

In this question, we will only consider a full moon that is at perigee to be a supermoon.

a) Calculate how many days separate a supermoon.

b). Show that the mean difference in the distance between the apogee and perigee is 4.22×10^4 km. *(The data given in this question allows the mean orbital parameters to be calculated. Note that in reality, perturbations in the lunar orbit mean that the perigee and apogee continually change over the course of the year).*

c). Determine the difference in the angular diameter of a supermoon and a full moon observed at apogee. Thus, determine the percentage difference in the brightness of a supermoon and a full moon observed at apogee. *(Ignore the effects of the Moon's orbital tilt with respect to the Earth).*

d). What change in magnitude does this brightness difference correspond to?

e). Suggest why it can be difficult to detect any differences in the brightness of a supermoon when observing with the naked eye?

f). Calculate the gravitational force of the supermoon on the Earth. What mass increase would a Moon at apogee need, to create the same gravitational force?

Solutions

a). $27.554550/(29.530589 - 27.554550) = 13.9443$ synodic periods between each supermoon,

so $13.9443 \times 29.530589 = 411.78$ days between each supermoon.

b). From equation of ellipse we have that: $R_a = a(1 + e)$ and $R_p = a(1 - e)$, where R_a is the lunar apogee, R_p is the lunar perigee, a is the semi-major axis and e is the eccentricity.

Therefore the difference in apogee and perigee is: $R_a - R_p = a(1+e) - a(1-e) = 2ae = 4.22 \times 10^4$ km.

c). Angular diameter of a full moon at apogee: $R_☽/R_a = 4.286 \times 10^{-3}$

Angular diameter of a full moon at perigee: $R_☽/R_p = 4.784 \times 10^{-3}$

So the Moon appears $4.784 \times 10^{-3}/4.286 \times 10^{-3} = 1.116$ times or 11.6% larger at perigee.

Since light will follow an inverse square law, the brightness will decrease according to the distance^2. Hence, in terms of brightness, a Moon appears $1.11^2 = 1.246$ times or 25% brighter at perigee.

d). Difference in magnitude = $\log_{10}$(difference in brightness)/0.4
$= \log_{10}(1.246)/0.4 = 0.239$ magnitudes.

e). Typically, a difference of 0.239 magnitudes should be detectable with the naked eye. However, several months separate these two extremes of the brightest and dimmest full moons. In addition, the Moon is passing through different phases (e.g. crescent, quarter, gibbous) as it reaches full Moon, all of which have different brightnesses. Therefore, it is unlikely that a naked eye observer will notice, and furthermore remember, the difference in brightness between a supermoon and a full moon at apogee.

f). Gravitational force of Moon at perigee: $F_{grav,p} = \frac{G\ \mathrm{M}_\odot M_☽}{R_p^2}$,

and gravitational force of Moon at apogee: $F_{grav,a} = \frac{G\ \mathrm{M}_\odot M_☽}{R_a^2}$

or write as: $F_{grav,a} = \frac{G\ \mathrm{M}_\odot M_☽ + m}{(R_p + r)^2}$.

But from part b), know that $r = 4.22 \times 10^4$ km.

If $F_{grav,p} = F_{grav,a}$, the increase in lunar mass, $M_☽ + m = 1.246 M_☽$, or $m = 0.246 M_☽$. (since gravity also follows the inverse square law, this is analagous to the answer in part c).

Qu. 3 Interstellar   Given $g = 1.3 g_E$, tidal height $= 1000\,m$, $t_f = 7$ years, $t_0 = 1$ hour

(a) R = planet radius
r = radius of orbit
r_B = Schwarzschild radius
M_B = Mass of BH

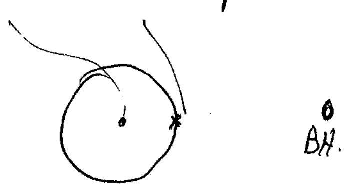
$$g = \frac{GM}{R^2} = G\rho \tfrac{4}{3}\pi R$$

density of planet = density of Earth.

$$\therefore \; R = 1.3 R_E = 8280\text{ km}$$

$$M = 1.3^3 M_E = 1.31\times10^{25}\text{ kg}$$

(b) Calculate the difference in grav. field strength between centre and near side of planet due to Black Hole.



$$\Delta g = \frac{GM_B}{(r-R)^2} - \frac{GM_B}{r^2}$$

$$= \frac{GM_B}{r^2}\left(\left[1-\frac{R}{r}\right]^{-2} - 1\right)$$

$$= \frac{GM_B}{r^2}\left(1 + \frac{2R}{r} - 1\right) = GM_B \cdot \frac{2R}{r^3}$$

But we may write $\Longrightarrow \Delta g = \quad \Delta g = \frac{gh}{R}$

$$\rho g R = \rho(g-\Delta g)(R+h)$$
$$= \rho g R - \rho \Delta g R + \rho g h + O(\Delta^2)$$
i.e. $\rho g h = \rho \Delta g R$.

So $\frac{gh}{R} = GM_B \cdot \frac{2R}{r^3}$

$$\frac{M_B}{r^3} = \frac{gh}{2GR^2}$$

$$= \frac{1.3\times9.81\times10^3}{2\times6.67\times10^{-11}\times(8280\times10^3)^2}$$

$$= 1.39\text{ kg m}^{-3}$$

From time dilation

$$\left(\frac{t_0}{t_f}\right)^2 = 1 - \frac{2GM_B}{rc^2}$$

$$2.66\times10^{-10} = 1 - \frac{2G}{c^2}\frac{M_B}{r}$$

$$\Longrightarrow \frac{M_B}{r} = 6.75\times10^{26}\text{ kg m}^{-1}$$

$\frac{mV^2}{r} = \frac{GM_B m}{r^2} \Rightarrow V^2 = \frac{G M_B}{r}$ so that $V = 2.1\times10^8$ m/s
So motion is relativistic.

Hence $r = 2.20\times10^{13}\text{ m} = 147\text{ AU}$

$M_B = 1.49\times10^{40}\text{ kg}$
$= 7.48\times10^9 M_\odot$

(c) $$T^2 = \frac{4\pi^2}{GM_B}\cdot r^3 \Longrightarrow T = 6.52\times10^5\text{ s}$$
$$= 181\text{ hours.}$$

(d) $r_{BH} = 2.20\times10^{13}$ m and $1 = \frac{2GM_B}{c^2 r_B}$ c.f. $(1-2.66\times10^{-10}) = \frac{2GM_B}{c^2 r}$

So $r - r_B = 5840\text{ m} = 5.84\text{ km} \ll R$