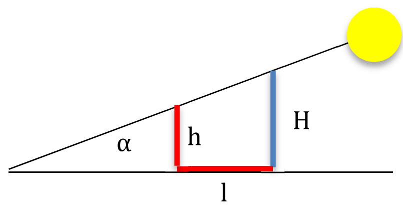
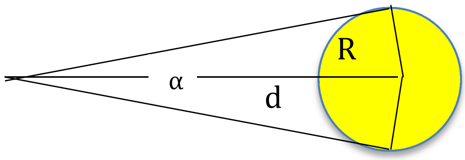
Solutions for the BOAA trial paper

Question	Answer	Mark
Section A		**20**
1.	C	2
2.	C	2
3.	D	2
4.	A	2
5.	B	2
6.	C	2
7.	B	2
8.	D	2
9.	B	2
10.	A	2
Section B		**10**
11.	a. 32° b. 2.9 m Solution:  a. Let H = 1.8 m, h = 0.8 m, l = 1.6 m. From the information in the question and the diagram above, the altitude of the Sun is: $\alpha = \tan^{-1}\left(\frac{H-h}{l}\right) = 32°$ b. The length of the shadow in the absence of the wall is: $d = \frac{H}{\tan\alpha} = 2.9\ \mathrm{m}$	3 2
12.	a. $6.00 \times 10^3\ \mathrm{km\ s^{-1}}$ b. 83 Mpc Solution: a. Doppler shift: $\frac{\lambda - \lambda_0}{\lambda_0} = \frac{v}{c}$ Where $\lambda_0 = 656.3$ nm – the rest wavelength and $\lambda = 669.4$ nm – the observed wavelength. Hence, $v = 5988\ \mathrm{km\ s^{-1}} \sim 6000\ \mathrm{km\ s^{-1}}$. b. Hubble law: $v = H_0\, r$ Using v and the given H_0, $r = 83.17$ Mpc ~ 83 Mpc .	3 2

<table>
<tr><th>Section C</th><th></th><th>20</th></tr>
<tr><td>13.</td><td>a.
Diagram


From the diagram, the apparent diameter of the object is:
$$\alpha = 2\sin^{-1}\left(\frac{R}{d}\right)$$
For small angles, $\sin x \approx \tan x \approx x$ (in radians), so any of these is acceptable.
Numerically:
Sun: $\alpha = 0.533°$
Moon: $\alpha = 0.545°$</td><td>3
1
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1</td></tr>
<tr><td></td><td>b.
The magnification of the telescope is:
$$M = \frac{f_{objective}}{f_{eyepiece}} = \frac{200 \text{ cm}}{2.5 \text{ cm}} = 80$$
The field of view of the telescope is thus:
$$FOV_{telescope} = \frac{FOV_{eyepiece}}{M} = \frac{52°}{80} = 0.65°$$
The FOV is larger than the apparent diameter of the Sun (0.533°), so it is possible to see the entire image of the Sun in the eyepiece.</td><td>2
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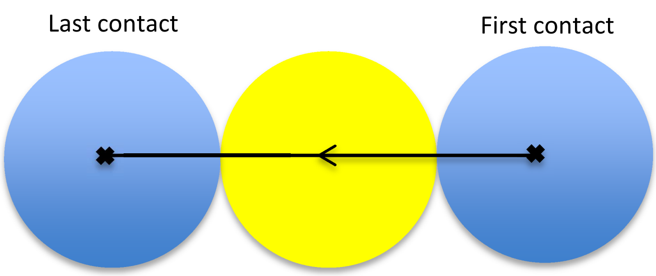
<tr><td></td><td>c.
At the North Pole the Earth is static. The reason why the eclipse occurs is that the Moon has its own motion around the Earth with a period of 29.5 days, relative to the Earth's motion around the Sun. The angular velocity of the Moon is thus:
$$\omega = \frac{2\pi}{T} = \frac{360°}{29.5 \text{ days}} = 12.2°/\text{day}$$


As seen in the diagram above, the centre of the Moon covers an angular distance of $\Delta\alpha = \alpha_{Sun} + \alpha_{Moon} = 1.078°$ from first to last contact.
The time needed for the Moon to cover this distance is:
$$\Delta t = \frac{\Delta\alpha}{\omega} = \frac{1.078}{12.2} \text{ days} = 0.088 \text{ days} = 2.12 \text{ hours}$$
Thus, the duration of the eclipse seen from the North Pole is 2.12 hours.</td><td>3
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1</td></tr>
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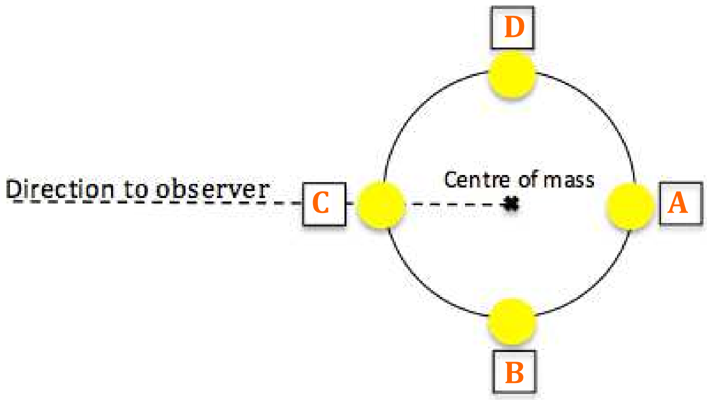
	d. Earth rotates about its axis from West to East (anticlockwise direction), so the Sun and Moon appear to move in the sky from East to West (clockwise). The Moon orbits the Earth, in an anticlockwise direction, from W to E, so the eclipse will begin on the W side of the Sun and will end in the E (as seen by an observer on Earth). Judging from the coordinates given in the image, this is the **beginning** of the solar eclipse.	**2**
	e. ***Optional*** In part (c) we calculated the duration of the eclipse in case of a static Earth, $\Delta t = 2.12$ hours. This is a special case that only occurs at the Poles of the Earth. At any other latitudes, we need to consider Earth's spin in the calculations, as the observer will be moving along Earth's surface. This will extend the duration of the eclipse, since the observer and the Moon will rotate in the same direction (anticlockwise). During the eclipse, the shadow of the Moon moves on the surface of the Earth with a linear velocity: $$v_{\text{Moon}} = \frac{2\pi a_{\text{Moon}}}{T_{\text{Moon}}} \approx 3{,}240 \text{ km h}^{-1}$$ The distance the shadow of the Moon travels on the static Earth (assuming that the Earth had a flat surface) during the $\Delta t = 2.12$ hours of eclipse is: $$d = v_{\text{Moon}}\, \Delta t = 6{,}870 \text{ km}$$ Now, consider the rotating Earth. Earth's rotational velocity at the Equator is: $$v_{\text{Earth}} = \frac{2\pi R_{\text{Earth}}}{T_{\text{Earth}}} \approx 1{,}670 \text{ km h}^{-1}$$ At London's latitude ($\varphi = 52.5°$), Earth's rotational velocity is: $$v_{\text{Earth},\varphi} = v_{\text{Earth}} \cos\varphi \approx 1{,}020 \text{ km h}^{-1}$$ The Earth-Moon relative velocity is thus: $$v_{\text{rel}} = v_{\text{Moon}} - v_{\text{Earth},\varphi} \approx 2220 \text{ km h}^{-1}$$ In this case, the duration of the eclipse will increase to: $$\Delta t' = \frac{d}{v_{\text{rel}}} = \Delta t \times \frac{v_{\text{Moon}}}{v_{\text{Moon}} - v_{\text{Earth}} \cos\varphi} \approx 3.1 \text{ hours}$$ In reality, the duration of the eclipse, as viewed from London, will be less than 3.1 hours as we need to consider that the Earth's surface is curved. So far, we considered that the shadow of the Moon moves on a projection of Earth's curved surface on a flat surface, but an accurate calculation is complicated. Also, in the question we considered the case of a total eclipse ("*the eclipse is central*"), while from London the eclipse on 20th March will be a partial one, with an obscuration of 85%. Therefore, the angular distance the Moon covers from first to last contact is smaller than the one we calculated. The partial eclipse on 20th March will last for $\mathbf{2^{h}16^{min}}$, with first contact at **08:45** UT and last contact at **10:41** UT.	

14	a.  	**2** 0.5 marks each
	b. From Figure 2 the velocity of the star around the centre of mass is: $$v_{\mathrm{star}} = 60 \pm 10 \text{ m s}^{-1}$$ Accepted values for the velocity between $58 - 60$ m s^{-1} Accepted values for the uncertainty in velocity between $10 - 14$ m s^{-1}	**1** 0.5 marks each
	c. Use Kepler's third law: $$\frac{T^2}{a^3} = \frac{4\pi^2}{G(M_{\mathrm{star}} + m_{\mathrm{planet}})}$$ Neglect $m_{\mathrm{planet}} \ll M_{\mathrm{star}}$, and because the mass of 51 Pegasi is the same as of the Sun use: $$\frac{T^2}{a^3} = 1$$ Where T is in years and a in AU. The period of the planet is $T = 4.23$ days therefore the semi-major axis (same with radius in this case as the orbit is nearly circular) of the planet's orbit is: $$a = 0.05 \text{ AU}$$	**3** 1 1 1
	d. The position of the centre of mass, in the CM frame is zero. I.e. $$\boldsymbol{R} = \frac{m_{\mathrm{planet}}\, \boldsymbol{r}_{\mathbf{planet}} + M_{\mathrm{star}} \boldsymbol{r}_{\mathbf{star}}}{m_{\mathrm{planet}} + M_{\mathrm{star}}} = 0$$ Therefore: $$m_{\mathrm{planet}}\, r_{\mathrm{planet}} = M_{\mathrm{star}} r_{\mathrm{star}}$$ The star-CM distance: $$v_{\mathrm{star}} = \frac{2\pi r_{\mathrm{star}}}{T}$$ From where we get: $$m_{\mathrm{planet}} = \frac{M_{\mathrm{solar}}\, v_{\mathrm{star}}\, T}{2\pi r_{\mathrm{planet}}}$$ $$m_{\mathrm{planet}} = 9.3 \times 10^{26} \text{kg} \approx 0.5\, M_{\mathrm{Jupiter}}$$	**4** 1 1 1

	Different method For a closed system (like this planet-star system) the total linear momentum, in the centre of mass frame, is 0. Therefore, the momenta of the star and of the planet are equal. Conservation of momentum: $$m_{\text{planet}}\, v_{\text{planet}} = M_{\text{star}} v_{star}$$ Circular velocity: $$v_{\text{planet}} = \frac{2\pi r_{\text{planet}}}{T}$$ $$M_{star} = M_{\text{solar}}$$ Mass of the planet is: $$m_{\text{planet}} = \frac{M_{\text{solar}}\, v_{\text{star}}\, T}{2\pi r_{\text{planet}}} \approx 0.5\, M_{\text{Jupiter}}$$ This is a lower estimate of the mass of the planet because we are not given any information about the inclination of the orbit, which we assume to be edge-on. If the orbit were tilted by an angle i to the line of sight, the measured radial velocity would be $v \sin \text{i}$, and hence the true mass of the planet is: $M_{true} = M_{min}/sin\, i$. Unfortunately, using the radial velocity method we are not able to determine the inclination of the system, so all the masses we measure are lower estimates. [Any explanation about the tilt of the orbit is acceptable]	1
	e. ***Optional*** In part (c) we neglected the mass of the planet as $m_{\text{planet}} \ll M_{\text{star}}$. If we don't neglect it Kepler's 3rd law becomes: $$\frac{T^2 G\,(M_{\text{star}} + m_{\text{planet}})}{4\pi^2} = (r_{\text{planet}} + r_{\text{star}})^3$$ Using the hint in the question: $$m_p = M_s \frac{r_s}{r_p}$$ Replacing in Kepler's 3rd law: $$\frac{T^2 G\, M_s}{4\pi^2}\left(\frac{r_p + r_s}{r_p}\right) = (r_p + r_s)^3$$ Rearranging, $$\frac{T^2 G\, M_s}{4\pi^2\, r_s^3} = \frac{r_p}{r_s}\left(1 + \frac{r_p}{r_s}\right)^2$$ From Figure 3 we can get the radius of the orbit of 51 Pegasi: $$r_s = \frac{v_{\text{star}} T}{2\pi} = 3.49 \times 10^6 \text{ m}$$ Replacing numerically, $$1.057 \times 10^{10} = x(1+x)^2 \text{ where } x = {}^{r_p}/_{r_s}$$ Hence $x = 2195$ and $$m_{\text{planet}} = 0.48\, M_{\text{Jupiter}}$$	